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ALMOST PARA-HERMITIAN SUBMERSIONS

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Abstract. In this paper, we introduce the concept of almost para-Hermitian submersions between almost para-Hermitian manifolds. We investigate the influence of a given structure defined on the total manifold on the determination of the corresponding structure on the base manifold. Moreover, we provide an example, investigate various properties of the O'Neill's tensors for such submersions, find the integrability of the horizontal distribution and obtain necessary and sufficient conditions for the fibres of an almost para-Hermitian submersion to be totally geodesic. We also obtain curvature relations between the base manifold and the total manifold.

1. Introduction

The theory of Riemannian submersions was introduced by O'Neill and Gray in [18] and [10], respectively. Later, Riemannian submersions were considered between almost Hermitian manifolds by Watson in [21] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. For instances, Riemannian submersions between almost contact manifolds were studied by Chinea in [5] under the name of almost contact submersions. (Semi-)Riemannian submersions have been also considered for quaternionic Kähler manifolds [15,16], para-quaternionic Kähler manifolds [4] and paracontact manifolds [11,12].

On the other hand, para-complex manifolds, almost para-Hermitian manifolds and para-Kähler manifolds were defined by Libermann [17] in 1952. In fact, such manifolds arose in [20]. Indeed, Rashevskij introduced the properties of para-Kähler manifolds when he considered a metric of signature (n, n) defined from a potential function, the so-called scalar field on a $2n$-dimensional locally product manifold called by him stratified space.

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Semi-Riemannian submersions were introduced by O'Neill in his book [19]. It is known that such submersions have their applications in Kaluza-Klein theories, Yang-Mills equations, strings, supergravity. For applications of semi-Riemannian submersions, see [8]. Since almost para-Hermitian manifolds are semi-Riemannian manifolds, one should consider semi-Riemannian submersions between such manifolds.

In this paper, we define almost para-Hermitian submersions between almost para-Hermitian manifolds, and study the geometry of such submersions. We observe that almost para-Hermitian submersions have also rich geometric properties.

The paper is organized as follows: In Section 2 we collect basic definitions, some formulas and results for later use. In Section 3 we introduce the notion of almost para-Hermitian submersions and give an example of almost para-Hermitian submersion. Moreover, we investigate properties of O'Neill's tensors and show that such tensors have nice algebraic properties for almost para-Hermitian submersions. We find the integrability of the horizontal distribution. We also find necessary and sufficient conditions for the fibres of an almost para-Hermitian submersion to be totally geodesic and examine the influence of a given type of almost para-Hermitian structure of the total manifold on the determination of the corresponding structure on the base manifold. In Section 4, we obtain relations between bisectional curvatures and sectional curvatures of the base manifold, the total manifold and the fibres of an almost para-Hermitian submersion.

2. Preliminaries

In this section we are going to recall main definitions and properties of almost para-Hermitian manifolds and semi-Riemannian submersions.

2.1. Almost para-Hermitian manifolds

A $(1,1)$-tensor field P on a $2n$-dimensional smooth manifold M is said to be an almost product structure if $P^2 = I$. In this case the pair (M, P) is called almost product manifold. An almost para-complex manifold is an almost product manifold (M, P) such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues ± 1 of P have the same rank.

An almost para-Hermitian manifold (M, g, P) is a smooth manifold endowed with an almost para-complex structure P and a pseudo-Riemannian metric g compatible in the sense that

$$g(PX, Y) + g(X, PY) = 0, \qquad X, Y \in \chi(M). \tag{1}$$

It follows that the metric g is neutral, i.e., it has signature (n, n) and the eigenbundles $TM^{\pm}$ are totally isotropic with respect to g. Let $e_1, \ldots, e_n, e_{n+1} = Pe_1, \ldots, e_{2n} = Pe_n$ be an orthonormal basis and denote $\epsilon_i = \operatorname{sign}(g(e_i, e_i)) = \pm 1, \epsilon_i = 1, i = 1, \ldots, n, \epsilon_j = -1, j = n+1, \ldots, 2n$.

The fundamental 2-form of the almost para-Hermitian manifold is defined by

$$\Phi(X, Y) = g(X, PY). \tag{2}$$

By (1), it immediately follows that Φ is skew-symmetric [6,14].

An almost para-Hermitian manifold, with Levi-Civita connection ∇, is called

(i) *para-Kähler*, if $\nabla P = 0$;

(ii) *para-Hermitian* , if $N = 0 \Leftrightarrow (\nabla_{PX}P)PY + (\nabla_X P)Y = 0$, where N is the Nijenhuis torsion of P;

(iii) *nearly para-Kähler*, if $(\nabla_X P)X = 0$;

(iv) *almost para-Kähler,* if $d\Phi = 0$ [6,13,14].

2.2. Semi-Riemannian submersions

Let (M, g) and (B, g') be two connected semi-Riemannian manifolds of index s $(0 \leq s \leq \dim M)$ and s' $(0 \leq s' \leq \dim B)$ respectively, with $s > s'$. A semi-Riemannian submersion is a smooth map $\pi : M \to B$ which is onto and satisfies the following conditions:

(i) $\pi_{*p} : T_pM \to T_{\pi(p)}B$ is onto for all $p \in M$;

(ii) The fibres $\pi^{-1}(p'), p' \in B$, are semi-Riemannian submanifolds of M;

(iii) π_* preserves scalar products of vectors normal to fibres.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. An horizontal vector field X on M is said to be basic if X is π-related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A semi-Riemannian submersion $\pi : M \to B$ determines two $(1, 2)$ tensor fields T and A on M, by the formulas:

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{3}$$

and

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{4}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and the horizontal projections (see [2,7]). From (3) and (4), one can obtain

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{5}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{6}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{7}$$

for any $X, Y \in \Gamma(\mathcal{H})$, $U \in \Gamma(\mathcal{V})$. Moreover, if X is basic then $h(\nabla_U X) = h(\nabla_X U) = A_X U$.

We note that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2} v[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for

$X, Y \in \Gamma(\mathcal{H})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(\mathcal{V})$.

We now recall the following result which will be useful for later.

Lemma 2.1. (see [7,19]) *If $\pi : M \to B$ is a semi-Riemannian submersion and X, Y basic vector fields on M, π-related to X' and Y' on B, then we have the following properties:*

1. *$h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;*
2. *$h(\nabla_X Y)$ is a basic vector field π-related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;*
3. *$[E, U] \in \Gamma(\mathcal{V})$, $\forall U \in \Gamma(\mathcal{V})$ and $\forall E \in \chi(M)$.*

3. Almost para-Hermitian submersions

In this section, we define the notion of almost para-Hermitian submersion, give an example and study the geometry of such submersions. We now define a (P, P')-paraholomorphic map which is similar to the notion of a (P, P')-holomorphic map between almost Hermitian manifolds.

Definition 3.1. Let M^{2m} and B^{2n} be almost para-Hermitian manifolds with almost para-complex structures P and P', respectively. A mapping $\pi : M^{2m} \to B^{2n}$ is said to be almost para-complex map if $\pi_* \circ P = P' \circ \pi_*$.

By using the above definition, we are ready to give the following notion.

Definition 3.2. Let (M, P, g) and (B, P', g') be almost para-Hermitian manifolds. A semi-Riemannian submersion $\pi : M \to B$ is called an almost para-Hermitian submersion if π is an almost para-complex map.

Note that given a semi-Euclidean space R^{2n}_n with coordinates $(x_1, \ldots, x_{2n})$ on R^{2n}_n, we can naturally choose an almost para-complex structure P on R^{2n}_n as follows:

$$P\Big(\frac{\partial}{\partial x_{2i}}\Big) = \frac{\partial}{\partial x_{2i-1}}, \ \ P\Big(\frac{\partial}{\partial x_{2i-1}}\Big) = \frac{\partial}{\partial x_{2i}},$$

where $i = 1, \ldots, n$. Let R^{2n}_n be a semi-Euclidean space of signature (+,-,+,-, ...) with respect to the canonical basis $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}})$.

We give an example of an almost para-Hermitian submersion.

Example 3.1. Consider the following submersion defined by

$$\pi : R^4_2 \to R^2_1$$
$$(x_1, x_2, x_3, x_4) \to \Big(\frac{x_1 + x_3}{\sqrt{2}}, \frac{x_2 + x_4}{\sqrt{2}}\Big).$$

Then, the kernel of π_* is

$$\mathcal{V} = \operatorname{Ker} \pi_* = \operatorname{Span}\Big\{V_1 = -\frac{\partial}{\partial\, x_1} + \frac{\partial}{\partial\, x_3}\,, V_2 = -\frac{\partial}{\partial\, x_2} + \frac{\partial}{\partial\, x_4}\Big\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (\operatorname{Ker}\pi_*)^{\perp} = \operatorname{Span}\Big\{X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, Y = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\Big\}.$$

Hence, we have

$$g(X,X) = g'(\pi_* X, \pi_* X) = 2, \quad g(Y,Y) = g'(\pi_* Y, \pi_* Y) = -2.$$

Thus, π is a semi-Riemannnian submersion. Moreover, we can easily obtain that π satisfies

$$\pi_* PX = P'\pi_* X \quad \text{and} \quad \pi_* PY = P'\pi_* Y.$$

Thus, π is an almost para-Hermitian submersion. △

By using Definition 3.1, we have the following result.

PROPOSITION 3.1. *Let $\pi : M \to B$ be a para-Hermitian submersion from an almost para-Hermitian manifold M onto an almost para-Hermitian manifold B, and let X be a basic vector field on M, π-related to X' on B. Then, PX is also a basic vector field π-related to $P'X'$.*

The following result can be proved in a standard way.

PROPOSITION 3.2. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from an almost para-Hermitian manifold M onto an almost para-Hermitian manifold B. If X, Y are basic vector fields on M, π-related to X', Y' on B, Then, we have:*

(i) $h(\nabla_X P)Y$ *is the basic vector field π-related to $(\nabla'_{X'} P')Y'$;*

(ii) $h[X,Y]$ *is the basic vector field π-related to $[X', Y']$.*

Next proposition shows that an almost para-Hermitian submersion puts some restrictions on the distributions $\mathcal{V}$ and $\mathcal{H}$.

PROPOSITION 3.3. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from an almost para-Hermitian manifold M onto an almost para-Hermitian manifold B. Then, the horizontal and the vertical distributions are P-invariant.*

Proof. Consider a vertical vector field U. It is known that $\pi_*(PU) = P'(\pi_* U)$. Since U is vertical and π is a semi-Riemannian submersion, we have $\pi_* U = 0$ from which $\pi_*(PU) = 0$ follows and implies that PU is vertical, being in the kernel of π_*. As concerns the horizontal distribution, let X be a horizontal vector field. We have $g(PX, U) = -g(X, PU) = 0$ because PU is vertical and X is horizontal. From $g(PX, U) = 0$ we deduce that PX is orthogonal to U and then PX is horizontal. ∎

PROPOSITION 3.4. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from an almost para-Hermitian manifold M onto an almost para-Hermitian manifold B. Then, we have:*

(i) $\pi^* \Phi' = \Phi$;

(ii) *The fibres are almost para-Hermitian manifolds.*

Proof. We prove only statement (i), the other assertion can be obtained in a similar way. If X and Y are basic vector fields on M, π-related to X', Y' on B, then using the definition of an almost para-Hermitian submersion, we have

$$\begin{aligned}\pi^*\Phi'(X,Y) &= \Phi'(\pi_*X,\pi_*Y) = g'(\pi_*X,P'\pi_*Y) = g'(\pi_*X,\pi_*PY)\\ &= \pi^*g'(X,PY) = g(X,PY) = \Phi(X,Y),\end{aligned}$$

which gives the proof of assertion (i). ■

In the sequel, we show that base space is a para-Hermitian manifold if the total space is a para-Hermitian manifold.

Proposition 3.5. *Let $\pi : M \to B$ be an almost para-Hermitian submersion. If the total space M is a para-Hermitian manifold, then the base space B is a para-Hermitian manifold.*

Proof. Let X and Y be basic. Then, we have

$$\pi_*N(X,Y) = \pi_*[P,P](X,Y),$$

where $N = [P,P]$ is the Nijenhuis torsion of the almost para-complex structure P. On the other hand, $\pi_*P = P'\pi_*$ implies that

$$\begin{aligned}\pi_*[P,P](X,Y) &= \pi_*(P^2[X,Y] + [PX,PY] - P[PX,Y] - P[X,PY])\\ &= [\pi_*X,\pi_*Y] + [\pi_*PX,\pi_*PY] - P'\pi_*[PX,Y] - P'\pi_*[X,PY]\\ &= [X',Y'] + [P'X',P'Y'] - P'[P'X',Y'] - P'[X',P'Y'].\end{aligned}$$

Then, we have

$$\pi_*[P,P](X,Y) = N'(X',Y') = 0. \quad ■$$

Proposition 3.6. *Let $\pi : M \to B$ be an almost para-Hermitian submersion. If the total space M is a para-Kähler manifold, then the base space B is a para-Kähler manifold.*

Proof. For $X', Y' \in \Gamma(TB)$ such that $\pi_*X = X'$, $\pi_*Y = Y'$, where $X, Y \in \Gamma(TM)$, since M is a para-Kähler manifold, for $X, Y \in \Gamma(\mathcal{H})$, we have

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y = 0.$$

Then, by using $\pi_*P = P'\pi_*$, we get

$$\pi_*((\nabla_X P)Y) = \pi_*(\nabla_X PY) - P'\pi_*(\nabla_X Y) = 0.$$

On the other hand, from Proposition 3.1, we know that if X is π-related to X', then PX is π-related to $P'X'$. Also, from lemma 2.1(ii), it follows $\nabla_X PY$ and $\nabla_X Y$ are π-related to $\nabla'_{X'}P'Y'$ and $\nabla'_{X'}Y'$. Thus, we have

$$\pi_*((\nabla_X P)Y) = \nabla'_{X'}P'Y' - P'\nabla'_{X'}Y' = 0.$$

Hence
$$\pi_*((\nabla_X P)Y) = (\nabla'_{X'} P')Y' = 0,$$
which proves the assertion. ∎

In a similar way, we have the following result.

Proposition 3.7. *Let $\pi : M \to B$ be an almost para-Hermitian submersion. If the total space M is a nearly para-Kähler manifold, then the base space B is a nearly para-Kähler manifold.*

Proposition 3.8. *Let $\pi : M \to B$ be an almost para-Hermitian submersion. If the total space M is an almost para-Kähler manifold, then the base space B is an almost para-Kähler manifold.*

Proof. Let X, Y, Z be basic vector fields. Since M is an almost para-Kähler manifold it implies $d\Phi(X, Y, Z) = 0$. Then, we have
$$\begin{aligned} &X(\Phi(Y, Z)) - Y(\Phi(X, Z)) + Z(\Phi(X, Y)) \\ &\qquad - \Phi([X, Y], Z) + \Phi([X, Z], Y) - \Phi([Y, Z], X) = 0. \end{aligned}$$

On the other hand, by direct calculations, we obtain
$$\begin{aligned} 0 &= g(\nabla_X Y, PZ) + g(Y, \nabla_X PZ) - g(\nabla_Y X, PZ) - g(X, \nabla_Y PZ) + g(\nabla_X Z, PY) \\ &\quad + g(X, \nabla_Z PY) - g([X, Y], PZ) + g([X, Z], PY) - g([Y, Z], PX). \end{aligned}$$

Then, by using $\pi_* P = P' \pi_*$, we get
$$\begin{aligned} 0 &= g'(\nabla'_{X'} Y', P'Z') + g'(Y', \nabla'_{X'} P'Z') - g'(\nabla'_{Y'} X', P'Z') - g'(X', \nabla'_{Y'} P'Z') \\ &\quad + g'(\nabla'_{X'} Z', P'Y') + g'(X', \nabla'_{Z'} P'Y') - g'([X', Y'], P'Z') \\ &\quad + g'([X', Z'], P'Y') - g'([Y', Z'], P'X') \\ 0 &= X'(\Phi'(Y', Z')) - Y'(\Phi'(X', Z')) + Z'(\Phi'(X', Y')) \\ &\quad - \Phi'([X', Y'], Z') + \Phi'([X', Z'], Y') - \Phi'([Y', Z'], X'). \end{aligned}$$

Hence $0 = d\Phi'(X', Y', X')$, which proves the assertion. ∎

We now check the properties of the tensor fields T and A for an almost para-Hermitian submersion; we will see that such tensors have extra properties for such submersions.

Lemma 3.1. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from a para-Kähler manifold M onto an almost para-Hermitian, hence para-Kähler, manifold B, and let X and Y be horizontal vector fields, and U vertical. Then, we have*

(i) $A_X PY = PA_X Y$,

(ii) $A_{PX} Y = PA_X Y$,

(iii) $A_X PU = PA_X U$.

Proof. (i) Let X and Y be horizontal vector fields. Para-Kähler manifold M implies that $\nabla_X PY = P\nabla_X Y$. Then using (7), we have

$$A_X PY + h\nabla_X PY = P\{A_X Y + h\nabla_X Y\}.$$

Thus, taking the vertical components of this equation we get

$$A_X PY = PA_X Y.$$

(ii) In a similar way, by using (i) we have

$$A_{PX} Y = -A_Y PX = -PA_Y X.$$

Hence, we obtain $A_{PX} Y = PA_X Y$.

(iii) is obtained in a similar way. ∎

For the tensor field T we have the following.

Lemma 3.2. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from a para-Kähler manifold M onto an almost para-Hermitian manifold B, and let U and V be vertical vector fields, and X horizontal. Then, we have:*

(i) $T_U PV = PT_U V$,

(ii) $T_{PU} V = PT_U V$,

(iii) $T_U PX = PT_U X$.

Since for a nearly para-Kähler manifold $\nabla P = 0$, we have the following result.

Lemma 3.3. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from a nearly para-Kähler manifold M onto an almost para-Hermitian manifold B, X a horizontal vector field on M, U vertical. Then, we have:*

(i) $T_U PU = PT_U U$,

(ii) $A_{PX} X = PA_X X$.

We now investigate the integrability of the horizontal distribution $\mathcal{H}$.

Theorem 3.1. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from an almost para-Kähler manifold M onto an almost para-Hermitian manifold B. Then, the horizontal distribution is integrable.*

Proof. Let X and Y be basic vector fields. It suffices to prove that $v([X,Y]) = 0$, for basic vector fields on M. Since M is an almost para-Kähler manifold, it implies $d\Phi(X,Y,V) = 0$, for any vertical vector V. Then, one obtains

$$X(\Phi(Y,V)) - Y(\Phi(X,V)) + V(\Phi(X,Y)) \\ - \Phi([X,Y],V) + \Phi([X,V],Y) - \Phi([Y,V],X) = 0.$$

Since $[X,V],[Y,V]$ are vertical and the two distributions are P-invariant, the last two and the first two terms vanish. Thus, one gets

$$g([X,Y],PV) = V(g(X,PY)).$$

On the other hand, if X is basic then $h(\nabla_V X) = h(\nabla_X V) = A_X V$, thus we have

$$\begin{aligned} V(g(X, PY)) &= g(\nabla_V X, PY) + g(\nabla_V PY, X) \\ &= g(A_X V, PY) + g(A_{PY} V, X). \end{aligned}$$

Since A is skew-symmetric and alternating operator, we get $V(g(X, PY)) = 0$. This proves the assertion. ∎

Since for a para-Kähler manifold $\nabla P = 0$, we have the following result.

THEOREM 3.2. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from a para-Kähler manifold M onto an almost para-Hermitian manifold B. Then, the horizontal distribution is integrable.*

Proof. Let X be basic vector field on M, a and U vertical and Y horizontal. By using Lemma 3.1, we have

$$g(A_{PX} Y, U) = g(A_X PY, U) = -g(A_X U, PY).$$

On the other hand, if X is a basic then $h\nabla_U X = h\nabla_X U = A_X U$, thus we obtain

$$\begin{aligned} g(A_{PX} Y, U) &= -g(PY, h\nabla_U X) = g(Y, P\nabla_U X) = g(Y, \nabla_U PX) \\ &= g(Y, \nabla_{PX} U) = g(Y, A_{PX} U). \end{aligned}$$

Since A is skew-symmetric operator, we get $2g(A_{PX} Y, U) = O$. Hence, we have $A_{PX} Y = 0$, that is $A = 0$. ∎

THEOREM 3.3. *Let $\pi : M \to B$ be an almost para-Hermitian submersion from an almost para-Kähler manifold M onto an almost para-Hermitian manifold B. Then, fibres are totally geodesic if and only if $L_X P = 0$ for any X horizontal vector field.*

Proof. Let W and V be vertical vector fields on M, X horizontal. Since M is an almost para-Kähler manifold, it implies $d\Phi = 0$. Then, we obtain:

$$\begin{aligned} d\Phi(W, PV, X) &= W(\Phi(PV, X)) - PV(\Phi(W, X)) + X(\Phi(W, PV)) \\ &\quad - \Phi([W, PV], X) + \Phi([W, X], PV) - \Phi([PV, X], W) = 0. \end{aligned}$$

Since $[W, PV]$ is vertical and the two distributions are P-invariant, the first two terms and the fourth term vanish. Thus, one gets

$$X(\Phi(W, PV)) + \Phi([W, X], PV) - \Phi([PV, X], W) = 0.$$

Thus, by direct computations, we have

$$0 = g([X, V] + \nabla_V X, W) + g(\nabla_W X, V) + g([X, PV], PW).$$

From (1) and (5), we obtain

$$\begin{aligned} 0 &= -2g(T_V W, X) - g(P[X, V], PW) + g([X, PV], PW) \\ 0 &= -2g(T_V W, X) + g((L_X P)V, PW), \end{aligned}$$

which proves the assertion. ∎

4. Curvature relations for almost para-Hermitian submersions

We begin this section relating the P-paraholomorphic bisectional and sectional curvatures of the total space, the base and the fibres of an almost para-Hermitian submersions.

Let us recall the sectional curvature of semi-Riemannian manifolds for non-degenerate planes. Let M be a semi-Riemannian manifold and S a non-degenerate tangent plane to M at p. The number

$$K(U,V)=\frac{g(R(U,V)U,V)}{g(U,U)g(V,V)-g(U,V)^2}$$

is independent on the choice of basis U,V for S and is called the sectional curvature [9].

Let π be an almost para-Hermitian submersion from an almost para-Hermitian manifold (M,P,g) onto an almost para-Hermitian manifold $(N,P^{'},g^{'})$. We denote the Riemannian curvatures of M,N and any fibre $\pi^{-1}(x)$ by R,R' and $\hat{R}$, respectively.

Let $\pi : M \to N$ be an almost para-Hermitian submersion, where M and N are almost para-Hermitian manifolds with structures (P,g) and (P',g'), respectively. We denote by B the *P-paraholomorphic* bisectional curvature, defined for any pair of nonzero nonlightlike vectors X and Y on M by the formula

$$B(X,Y)=\frac{R(X,PX,Y,PY)}{\|X\|^2\|Y\|^2}.$$

We note that if X is a nonlightlike vector field, the PX is also a nonlightlike vector field.

The P-paraholomorphic sectional curvature is $H(X)=B(X,X)$ for any nonzero nonlightlike vector X. We denote by B' and H' the P-paraholomorphic bisectional and P-paraholomorphic sectional curvature of B [3]. Similarly, $\hat{B}$ and $\hat{H}$ denote the bisectional and the sectional paraholomorphic curvatures of a fibre.

The following is a translation of the results of Gray [10] and O'Neill [18] to the present situation:

Proposition 4.1. *Let $\pi : M \to N$ an almost para-Hermitian submersion from an almost para-Hermitian manifold M onto an almost para-Hermitian manifold N. Let U and V be nonzero nonlightlike unit vertical vectors, and X and Y nonzero nonlightlike unit horizontal vectors. Then, we have*

$$\begin{aligned}
(a)\quad & B(U,V)=\hat{B}(U,V)-\epsilon_U\epsilon_V[g(T_UV,T_{PU}PV)-g(T_{PU}V,T_UPV)];\\
(b)\quad & B(X,U)=\epsilon_U\epsilon_X[g((\nabla_U A)_X PX,PU)-g((\nabla_{PU}A)_X PX,U)\\
&\qquad +g(A_XU,A_{PX}PU)-g(A_XPU,A_{PX}U)\\
&\qquad -g(T_UX,T_{PU}PX)+g(T_{PU}X,T_UPX)];
\end{aligned}$$

$$(c) \quad B(X,Y) = B'(\pi_*X,\pi_*Y)\circ\pi - \epsilon_X\epsilon_Y[2g(A_XPX,A_YPY) - g(A_{PX}Y,A_XPY) + g(A_XY,A_{PX}PY)],$$

where $\epsilon_U = g(U,U) \in \{\pm1\}$, $\epsilon_V = g(V,V) \in \{\pm1\}$, $\epsilon_X = g(X,X) \in \{\pm1\}$ *and* $\epsilon_Y = g(Y,Y) \in \{\pm1\}$.

Using Proposition 4.1, we have the following result.

Proposition 4.2. *Let* $\pi : M \to N$ *be an almost para-Hermitian submersion from an almost para-Hermitian manifold* M *onto an almost para-Hermitian manifold* N. *Let* U *be nonzero nonlightlike unit vertical vector, and* X *nonzero nonlightlike unit horizontal vector. Then, one has:*

(a) $H(U) = \hat{H}(U) + \|T_UPU\|^2 - g(T_{PU}PU,T_UU)$;

(b) $H(X) = H'(\pi_*X)\circ\pi - 3\|A_XPX\|^2$.

If the total manifold is a para-Kähler manifold, then we have the following result for curvature relations between M, N and $\pi^{-1}(x)$.

Theorem 4.1. *Let* $\pi : M \to N$ *be an almost para-Hermitian submersion from a para-Kähler manifold* M *onto an almost para-Hermitian manifold* N. *Let* U *and* V *be nonzero nonlightlike unit vertical vectors, and* X *and* Y *nonzero nonlightlike unit horizontal vectors. Then, we have:*

(a) $B(U,V) = \hat{B}(U,V) - \epsilon_U\epsilon_V 2\|T_UV\|^2$;

(b) $B(X,Y) = B'(\pi_*X,\pi_*Y)\circ\pi + \epsilon_X\epsilon_Y[2g(A_XX,A_YY) - 2\|A_XY\|^2]$.

Proof. (a) From Proposition 4.1(a), we have

$$B(U,V) = \hat{B}(U,V) + g(T_UPV,T_{PU}V) - g(T_{PU}PV,T_UV).$$

Using Lemma 3.2, we get

$$g(T_UPV,T_{PU}V) = g(PT_UV,PT_UV) = -g(T_UV,T_UV) = -\|T_UV\|^2. \tag{8}$$

Using again Lemma 3.2, we get

$$g(T_{PU}PV,T_UV) = g(P^2T_UV,T_UV) = g(T_UV,T_UV) = \|T_UV\|^2. \tag{9}$$

From (8) and (9), we have (a).

(b) From Proposition 4.1(c), we have

$$B(X,Y) = B'(\pi_*X,\pi_*Y)\circ\pi - 2g(A_XPX,A_YPY) + g(A_{PX}Y,A_XPY) - g(A_XY,A_{PX}PY).$$

Using Lemma 3.1, we get

$$g(A_XPX,A_YPY) = -g(A_XX,A_YY). \tag{10}$$

Using again Lemma 3.1, we get

$$g(A_{PX}Y, A_X PY) = -\|A_X Y\|^2, \tag{11}$$

$$g(A_X Y, A_{PX} PY) = \|A_X Y\|^2. \tag{12}$$

From (10)–(12), we have (b). ■

As a result of Theorem 4.1, we have the following results.

COROLLARY 4.1. *Let $\pi : M \to N$ be an almost para-Hermitian submersion from a para-Kähler manifold M onto an almost para-Hermitian manifold N. Let U be unit vertical vector, and X a unit horizontal vector field. Then, one has*

(a) $H(U) = \hat{H}(U) - 2\|T_U U\|^2$;

(b) $H(X) = H'(\pi_* X) \circ \pi$.

Since M is an almost para-Kähler manifold and the distribution $\mathcal{H}$ is integrable we have $A = 0$, then we have the following relations

THEOREM 4.2. *Let $\pi : M \to N$ be an almost para-Hermitian submersion from an almost para-Kähler manifold M onto an almost para-Hermitian manifold N. Let U be unit vertical vector, and X and Y unit horizontal vectors. Then, we have:*

(a) $B(X, Y) = B'(\pi_* X, \pi_* Y) \circ \pi$;

(b) $B(X, U) = -g(T_U X, T_{PU} PX) + g(T_{PU} X, T_U PX)$.

Using Lemma 3.3, we have following results.

Corollary 4.2. *Let $\pi : M \to N$ be an almost para-Hermitian submersion from a nearly para-Kähler manifold M onto an almost para-Hermitian manifold N. Let U be unit vertical vector, and X unit horizontal vector. Then, we have:*

(a) $H(U) = \hat{H}(U) - 2\|T_U U\|^2$;

(b) $H(X) = H'(\pi_* X) \circ \pi$.

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