МАТЕМАТИЧКИ ВЕСНИК
68, 4 (2016), 277–286
December 2016

оригинални научни рад
research paper

TOPOLOGICAL PROPERTIES OF TRIPLED FIXED POINTS SET OF MULTIFUNCTIONS

Hojjat Afshari

Abstract. In 1970, Schirmer provided some results about topological properties of the fixed points set of multifunctions. Later, some authors continued this review by providing different conditions. In 2008, 2009 Sintamarian proved some results on absolute retractivity of the common fixed points set of two multivalued operators. In this paper we shall present some new results on absolute retractivity of tripled fixed points set for multifunctions of the form $F : X \times X \times X \to P_{b,cl}(X)$.

1. Introduction

Let X and Y be two nonempty sets and $P(Y)$ the set of all nonempty subsets of Y, also let $F : X \to P(Y)$ be a multifunction. A mapping $\varphi : X \to Y$ is called a selection of F whenever $\varphi(x) \in F(x)$ for all $x \in X$. Throughout the paper, for a topological space X we denote the set of all nonempty closed subsets of X by $P_{cl}(X)$ and the set of all nonempty closed and bounded subsets of X by $P_{b,cl}(X)$ when X is a metric space.

Let (X, d) be a metric space, $B(x_0, r) = \{x \in X \mid d(x_0, x) < r\}$. For $x \in X$ and $A, B \subseteq X$, set $D(x, A) = \inf_{y\in A} d(x, y)$ and

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}.$$

It is known that, H is a metric on closed bounded subsets of X which is called the Hausdorff metric.

We say that a topological space X is an absolute retract for metric spaces whenever for each metric space Y, $A \in P_{cl}(Y)$ and continuous function $\psi : A \to X$, there exists a continuous function $\varphi : Y \to X$ such that $\varphi|_A = \psi$. Let $\mathcal{M}$ be the set of all metric spaces, $X \in \mathcal{M}$, $\mathcal{D} \in P(\mathcal{M})$ and $F : X \to P_{b,cl}(X)$ a lower semi-continuous multifunction. We say that F has the selection property with respect to $\mathcal{D}$ if for each $Y \in \mathcal{D}$, continuous function $f : Y \to X$ and continuous functional

2010 Mathematics Subject Classification: 47H10, 54H25
Keywords and phrases: Absolute retract; tripled fixed points set; multifunction.

$g : Y \to (0,\infty)$ such that $G(y) := \overline{F(f(y)) \cap B(f(y), g(y))} \neq \emptyset$ for all $y \in Y$, $A \in P_{cl}(Y)$, every continuous selection $\psi : A \to X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X$, which is a selection of G. If $\mathcal{D} = \mathcal{M}$, then we say that F has the selection property and we denote this by $F \in SP(X)$ [9].

An interesting problem in fixed point theory of multivalued operators is to investigate under what conditions the properties of the values of a multifunction are inherited by its fixed point set. For some multifunctions, this problem was studied by Schirmer in 1970 [8], by Alicu and Mark in 1980 [4] and by Ricceri in 1987 [7]. For example, Schirmer proved that if the values of a contractive multifunction $F : \mathbb{R} \to P(\mathbb{R})$ are closed, bounded and convex, then the fixed points set of F is compact and convex. In 2008, 2009, Sintamarian proved some results on absolute retractivity of the common fixed points set of two multivalued operators under some conditions [9, 10]. Recently Afshari, Rezapour and Shahzad in [1, 2] have obtained new results on absolute retractivity of fixed points set for multifunctions and two variable multifunctions by providing some different conditions.

In 2012, Berinde and Borcut introduced in [5] the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained its existence.

Definition 1.1. [6] An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $F : X \times X \times X \to X$ if $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$.

In this paper, we shall present some new results on absolute retractivity of tripled fixed points set for multifunctions of the form $F : X \times X \times X \to P_{b,cl}(X)$.

2. Main results

Definition 2.1. Let $\mathcal{M}$ be the set of all metric spaces, $X \in \mathcal{M}$, $\mathcal{D} \in P(\mathcal{M})$ and $F : X \times X \times X \to P_{b,cl}(X)$ a lower semi-continuous multifunction. We say that F has the selection property with respect to $\mathcal{D}$ if for each $Y \in \mathcal{D}$, continuous function $f : Y \to X \times X \times X$ and continuous functional $g : Y \to (0,\infty)$ such that $G(y) := \overline{F(f(y)) \cap B(f(y), g(y))} \neq \emptyset$ for all $y \in Y$, $A \in P_{cl}(Y)$, every continuous selection $\psi : A \to X \times X \times X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X \times X \times X$, which is a selection of G. If $\mathcal{D} = \mathcal{M}$, then we say that F has the selection property and we denote this by $F \in SP(X \times X \times X)$.

Theorem 2.2. *Let (X, d) be a complete metric space and absolute retract for metric spaces and $F : X \times X \times X \to P_{b,cl}(X)$ a lower semicontinuous multifunction such that $F \in SP(X \times X \times X)$, also F satisfies the following condition: there exist $a_1, a_2, \dots, a_5 \in (0,\infty)$ with $a_1 + a_3 + 2a_4 + a_5 < 1$ and*

$$H(F(x,y,z), F(u,v,w)) \leq a_1 d(x,u) + a_2 D(u, F(x,y,z)) + a_3 D(u, F(u,v,w)) + a_4 D(x, F(u,v,w)) + a_5 D(x, F(x,y,z)),$$

for all $x, y, z, u, v, w \in X$. Then the set

$$\mathcal{B} = \{(x,y,z) \in X \times X \times X : x \in F(x,y,z), y \in F(y,x,y), z \in F(z,y,x)\},$$

is an absolute retract for metric spaces.

Proof. It is easy to see that $X \times X \times X$ is an absolute retract for metric spaces. Let

$$1 < q < (a_1 + a_3 + 2a_4 + a_5)^{-1}, \quad \text{and} \quad l := \frac{a_1 + a_4 + a_5}{1 - (a_3 + a_4)}.$$

It is not difficult to verify that $ql < 1$. Let Y be a metric space, $A \in P_{cl}(Y)$ and $\psi : A \to \mathcal{B}$ a continuous function. Since $X \times X \times X$ is an absolute retract for metric spaces, there exists a continuous function $\varphi_0 : Y \to X \times X \times X$ such that $\varphi_0|_A = \psi$. Let $\varphi_0 = (\varphi_0^1, \varphi_0^2, \varphi_0^3)$. Consider the functions $g_0 : Y \to (0,\infty) \times (0,\infty) \times (0,\infty)$ defined by $g_0 = (g_0^1, g_0^2, g_0^3)$, where $g_0^1,\ g_0^2,\ g_0^3$ are defined by

$$\begin{aligned}
g_0^1(y) &= \sup\{d(\varphi_0^1(y), z) : z \in F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y))\} + 1, \\
g_0^2(y) &= \sup\{d(\varphi_0^2(y), z) : z \in F(\varphi_0^2(y), \varphi_0^1(y), \varphi_0^2(y))\} + 1, \\
g_0^3(y) &= \sup\{d(\varphi_0^3(y), z) : z \in F(\varphi_0^3(y), \varphi_0^2(y), \varphi_0^1(y))\} + 1,
\end{aligned}$$

for all $y \in Y$. Regarding the values of F are bounded and closed, it is easy to see that the functions g_0^i, $(i = 1,2,3)$ are continuous. Define

$$\begin{aligned}
G_1^1(y) &:= F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y)) \cap B(\varphi_0^1(y), g_0^1(y)) = F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y)), \\
G_1^2(y) &:= F(\varphi_0^2(y), \varphi_0^1(y), \varphi_0^2(y)) \cap B(\varphi_0^2(y), g_0^2(y)) = F(\varphi_0^2(y), \varphi_0^1(y), \varphi_0^2(y)), \\
G_1^3(y) &:= F(\varphi_0^3(y), \varphi_0^2(y), \varphi_0^1(y)) \cap B(\varphi_0^3(y), g_0^3(y)) = F(\varphi_0^3(y), \varphi_0^2(y), \varphi_0^1(y)),
\end{aligned}$$

and set

$$\begin{aligned}
G_1(y) &= (G_1^1(y), G_1^2(y), G_1^3(y)) \\
&= (F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y)), F(\varphi_0^2(y), \varphi_0^1(y), \varphi_0^2(y)), F(\varphi_0^3(y), \varphi_0^2(y), \varphi_0^1(y))),
\end{aligned}$$

for all $y \in Y$. For $G_1(y)$, note that the function ψ is a continuous selection of the multivalued mapping $A \ni y \mapsto G_1(y)$. Since $F \in SP(X \times X \times X)$, there exists a continuous function $\varphi_1 : Y \to X \times X \times X$ such that

$$\begin{aligned}
&\varphi_1|_A = \psi, \\
&\varphi_1^1(y) \in F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y)), \\
&\varphi_1^2(y) \in F(\varphi_0^2(y), \varphi_0^1(y), \varphi_0^2(y)), \\
&\varphi_1^3(y) \in F(\varphi_0^3(y), \varphi_0^2(y), \varphi_0^1(y)),
\end{aligned}$$

where $\varphi_1 = (\varphi_1^1, \varphi_1^2, \varphi_1^3)$. Thus, we obtain

$$\begin{aligned}
D(\varphi_1^1(y), F(\varphi_1(y))) &= D(\varphi_1^1(y), F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y)) \\
&\le H(F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y)), F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y)) \\
&\le a_1 d(\varphi_0^1(y), \varphi_1^1(y)) + a_2 D(\varphi_1^1(y), F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y))) \\
&\quad + a_3 D(\varphi_1^1(y), F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y))) \\
&\quad + a_4 D(\varphi_0^1(y), F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y))) \\
&\quad + a_5 D(\varphi_0^1(y), F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y)))
\end{aligned}$$

$$\begin{aligned}&\le a_1 d(\varphi_0^1(y),\varphi_1^1(y)) + a_3 D(\varphi_1^1(y), F(\varphi_1^1(y),\varphi_1^2(y),\varphi_1^3(y))) + a_4 d(\varphi_0^1(y),\varphi_1^1(y))\\ &\quad + a_4 D(\varphi_1^1(y), F(\varphi_1^1(y),\varphi_1^2(y),\varphi_1^3(y))) + a_5 d(\varphi_0^1(y),\varphi_1^1(y)),\end{aligned}$$

for all $y \in X$. Hence,

$$(1-a_3-a_4)D(\varphi_1^1(y), F(\varphi_1^1(y),\varphi_1^2(y),\varphi_1^3(y))) \le (a_1+a_4+a_5)d(\varphi_0^1(y),\varphi_1^1(y)).$$

Thus,

$$\begin{aligned}D(\varphi_1^1(y), F(\varphi_1^1(y),\varphi_1^2(y),\varphi_1^3(y))) &\le \frac{a_1+a_4+a_5}{1-(a_3+a_4)} d(\varphi_0^1(y),\varphi_1^1(y))\\ &= l d(\varphi_0^1(y),\varphi_1^1(y)) < l d(\varphi_0^1(y),\varphi_1^1(y)) + q^{-1},\end{aligned}$$

for all $y \in X$. Hence,

$$G_2^1(y) := F(\varphi_1^1(y),\varphi_1^2(y),\varphi_1^3(y)) \cap B(\varphi_1^1(y), l d(\varphi_0^1(y),\varphi_1^1(y)) + q^{-1}) \ne \emptyset.$$

Also,

$$\begin{aligned}&D(\varphi_1^2(y), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y))\\ &\quad\le H(F(\varphi_0^2(y),\varphi_0^1(y),\varphi_0^2(y)), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y))\\ &\quad\le a_1 d(\varphi_0^2(y),\varphi_1^2(y)) + a_2 D(\varphi_1^2(y), F(\varphi_0^2(y),\varphi_0^1(y),\varphi_0^2(y)))\\ &\qquad + a_3 D(\varphi_1^2(y), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y)))\\ &\qquad + a_4 D(\varphi_0^2(y), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y)))\\ &\qquad + a_5 D(\varphi_0^2(y), F(\varphi_0^2(y),\varphi_0^1(y),\varphi_0^2(y))\\ &\quad\le a_1 d(\varphi_0^2(y),\varphi_1^2(y)) + a_3 D(\varphi_1^2(y), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y)))\\ &\qquad + a_4 d(\varphi_0^2(y),\varphi_1^2(y)) + a_4 D(\varphi_1^2(y), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y)))\\ &\qquad + a_5 d(\varphi_0^2(y),\varphi_1^2(y)),\end{aligned}$$

for all $y \in Y$. Hence,

$$(1-a_3-a_4)D(\varphi_1^2(y), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y))) \le (a_1+a_4+a_5)d(\varphi_0^2(y),\varphi_1^2(y)).$$

Thus,

$$\begin{aligned}D(\varphi_1^2(y), F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y))) &\le \frac{a_1+a_4+a_5}{1-a_3-a_4)} d(\varphi_0^2(y),\varphi_1^2(y))\\ &= l d(\varphi_0^2(y),\varphi_1^2(y)) < l d(\varphi_0^2(y),\varphi_1^2(y)) + q^{-1},\end{aligned}$$

for all $y \in Y$. Therefore,

$$G_2^2(y) := F(\varphi_1^2(y),\varphi_1^1(y),\varphi_1^2(y)) \cap B(\varphi_1^2(y), l d(\varphi_0^2(y),\varphi_1^2(y)) + q^{-1}) \ne \emptyset.$$

Again we have

$$\begin{aligned}&D(\varphi_1^3(y), F(\varphi_1^3(y),\varphi_1^2(y),\varphi_1^1(y))\\ &\quad\le H(F(\varphi_0^3(y),\varphi_0^2(y),\varphi_0^1(y)), F(\varphi_1^3(y),\varphi_1^2(y)),\varphi_1^1(y))\end{aligned}$$

$$
\begin{aligned}
&\le a_1 d(\varphi_0^3(y), \varphi_1^3(y)) + a_2 D(\varphi_1^3(y), F(\varphi_0^3(y), \varphi_0^2(y), \varphi_0^1(y)))\\
&\quad + a_3 D(\varphi_1^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)))\\
&\quad + a_4 D(\varphi_0^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)))\\
&\quad + a_5 D(\varphi_0^3(y), F(\varphi_0^3(y), \varphi_0^2(y), \varphi_0^1(y))\\
&\le a_1 d(\varphi_0^3(y), \varphi_1^3(y)) + a_3 D(\varphi_1^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)))\\
&\quad + a_4 d(\varphi_0^3(y), \varphi_1^3(y)) + a_4 D(\varphi_1^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)))\\
&\quad + a_5 d(\varphi_0^3(y), \varphi_1^3(y)),
\end{aligned}
$$

for all $y \in Y$. Hence,

$$(1 - a_3 - a_4) D(\varphi_1^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y))) \le (a_1 + a_4 + a_5) d(\varphi_0^3(y), \varphi_1^3(y)).$$

Thus,

$$
\begin{aligned}
D(\varphi_1^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y))) &\le \frac{a_1 + a_4 + a_5}{1 - a_3 - a_4)} d(\varphi_0^3(y), \varphi_1^3(y))\\
&= l d(\varphi_0^3(y), \varphi_1^3(y)) < l d(\varphi_0^3(y), \varphi_1^3(y)) + q^{-1},
\end{aligned}
$$

for all $y \in Y$. Hence,

$$G_2^3(y) := F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)) \cap B(\varphi_1^3(y), l d(\varphi_0^3(y), \varphi_1^3(y)) + q^{-1}) \neq \emptyset.$$

Set

$$
\begin{aligned}
G_2(y) &= (G_2^1(y), G_2^2(y), G_2^3(y))\\
&= (F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y)), F(\varphi_1^2(y), \varphi_1^1(y), \varphi_1^2(y)), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y))),
\end{aligned}
$$

for all $y \in Y$. For $G_2(y)$ note that the function ψ is a continuous selection of the multivalued mapping $A \ni y \mapsto G_2(y)$. Since $F \in SP(X \times X \times X)$, there exists a continuous function $\varphi_2 : Y \to X \times X \times X$ such that

$$
\begin{aligned}
&\varphi_2|_A = \psi,\\
&{\varphi_2}^1(y) \in F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y)),\\
&{\varphi_2}^2(y) \in F(\varphi_1^2(y), \varphi_1^1(y), \varphi_1^2(y)),\\
&{\varphi_2}^3(y) \in F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)),
\end{aligned}
$$

where $\varphi_2(y) = (\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y))$ for all $y \in Y$. Thus, we obtain

$$
\begin{aligned}
&D(\varphi_2^1(y), F(\varphi_2(y))) = D(\varphi_2^1(y), F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y))\\
&\quad \le H(F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y)), F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y))\\
&\quad \le a_1 d(\varphi_1^1(y), \varphi_2^1(y)) + a_2 D(\varphi_2^1(y), F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y)))\\
&\qquad + a_3 D(\varphi_2^1(y), F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y)))\\
&\qquad + a_4 D(\varphi_1^1(y), F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y)))
\end{aligned}
$$

$$
\begin{aligned}
&+ a_5 D(\varphi_1^1(y), F(\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y))) \\
&\le a_1 d(\varphi_1^1(y), \varphi_2^1(y)) + a_3 D(\varphi_2^1(y), F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y))) + a_4 d(\varphi_1^1(y), \varphi_2^1(y)) \\
&\quad + a_4 D(\varphi_2^1(y), F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y))) + a_5 d(\varphi_1^1(y), \varphi_2^1(y)),
\end{aligned}
$$

for all $y \in X$. Hence,

$$(1 - a_3 - a_4) D(\varphi_2^1(y), F(\varphi^1(y), \varphi_2^2(y), \varphi_2^3(y))) \le (a_1 + a_4 + a_5) d(\varphi_1^1(y), \varphi_2^1(y)).$$

Thus,

$$
\begin{aligned}
D(\varphi_2^1(y), F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y))) &\le \frac{a_1 + a_4 + a_5}{1 - (a_3 + a_4)} d(\varphi_1^1(y), \varphi_2^1(y)) \\
&= l d(\varphi_1^1(y), \varphi_2^1(y)) < l d(\varphi_1^1(y), \varphi_2^1(y)) + q^{-1}, \\
&< l^2 d(\varphi_0^1(y), \varphi_1^1(y)) + q^{-2},
\end{aligned}
$$

for all $y \in X$. Thus,

$$G_3{}^1(y) := F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y)) \cap B(\varphi_2^1(y), l^2 d(\varphi_0^1(y), \varphi_1^1(y)) + q^{-2} \neq \emptyset.$$

Also,

$$
\begin{aligned}
&D(\varphi_2^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y)) \\
&\quad \le H(F(\varphi_1^2(y), \varphi_1^1(y), \varphi_1^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y)) \\
&\quad \le a_1 d(\varphi_1^2(y), \varphi_2^2(y)) + a_2 D(\varphi_2^2(y), F(\varphi_1^2(y), \varphi_1^1(y), \varphi_1^2(y))) \\
&\qquad + a_3 D(\varphi_2^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y))) \\
&\qquad + a_4 D(\varphi_1^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y))) \\
&\qquad + a_5 D(\varphi_1^2(y), F(\varphi_1^2(y), \varphi_1^1(y), \varphi_1^2(y)) \\
&\quad \le a_1 d(\varphi_1^2(y), \varphi_2^2(y)) + a_3 D(\varphi_2^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y))) \\
&\qquad + a_4 d(\varphi_1^2(y), \varphi_2^2(y)) + a_4 D(\varphi_2^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y))) \\
&\qquad + a_5 d(\varphi_1^2(y), \varphi_2^2(y)),
\end{aligned}
$$

for all $y \in Y$. Hence,

$$(1 - a_3 - a_4) D(\varphi_2^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y))) \le (a_1 + a_4 + a_5) d(\varphi_1^2(y), \varphi_2^2(y)).$$

Thus,

$$
\begin{aligned}
D(\varphi_2^2(y), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y))) &\le \frac{a_1 + a_4 + a_5}{1 - a_3 - a_4)} d(\varphi_1^2(y), \varphi_1^2(y)) \\
&= l d(\varphi_1^2(y), \varphi_2^2(y)) < l d(\varphi_1^2(y), \varphi_2^2(y)) + q^{-1} \\
&< l^2 d(\varphi_0^2(y), \varphi_1^2(y)) + q^{-2},
\end{aligned}
$$

for all $y \in Y$. Thus,

$$G_3{}^2(y) := F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y)) \cap B(\varphi_2^2(y), l^2 d(\varphi_0^2(y), \varphi_1^2(y)) + q^{-2} \neq \emptyset.$$

Again we have

$$
\begin{aligned}
&D(\varphi_2^3(y), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y))\\
&\quad \le H(F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y))\\
&\quad \le a_1 d(\varphi_1^3(y), \varphi_2^3(y)) + a_2 D(\varphi_2^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y)))\\
&\qquad + a_3 D(\varphi_2^3(y), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y)))\\
&\qquad + a_4 D(\varphi_1^3(y), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y)))\\
&\qquad + a_5 D(\varphi_1^3(y), F(\varphi_1^3(y), \varphi_1^2(y), \varphi_1^1(y))\\
&\quad \le a_1 d(\varphi_1^3(y), \varphi_2^3(y)) + a_3 D(\varphi_2^3(y), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y)))\\
&\qquad + a_4 d(\varphi_1^3(y), \varphi_2^3(y)) + a_4 D(\varphi_2^3(y), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y)))\\
&\qquad + a_5 d(\varphi_1^3(y), \varphi_2^3(y)),
\end{aligned}
$$

for all $y \in Y$. Hence,

$$
(1 - a_3 - a_4) D(\varphi_2^3(y), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y))) \le (a_1 + a_4 + a_5) d(\varphi_1^3(y), \varphi_2^3(y)).
$$

Thus,

$$
\begin{aligned}
D(\varphi_2^3(y), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y))) &\le \frac{a_1 + a_4 + a_5}{1 - a_3 - a_4)} d(\varphi_1^3(y), \varphi_2^3(y))\\
&= l d(\varphi_1^3(y), \varphi_2^3(y)) < l d(\varphi_1^3(y), \varphi_2^3(y)) + q^{-1},\\
&< l^2 d(\varphi_0^3(y), \varphi_1^3(y)) + q^{-2},
\end{aligned}
$$

for all $y \in Y$. Thus,

$$
{G_3}^3(y) := F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y)) \cap B(\varphi_2^3(y), l^2 d(\varphi_0^3(y), \varphi_1^3(y)) + q^{-2} \neq \emptyset.
$$

Set

$$
\begin{aligned}
G_3(y) &= (G_3^1(y), G_3^2(y), G_3^3(y))\\
&= (F(\varphi_2^1(y), \varphi_2^2(y), \varphi_2^3(y)), F(\varphi_2^2(y), \varphi_2^1(y), \varphi_2^2(y)), F(\varphi_2^3(y), \varphi_2^2(y), \varphi_2^1(y))).
\end{aligned}
$$

By continuing this process, we obtain a sequence of continuous functions $\{\varphi_n : Y \to X \times X \times X\}_{n \ge 0}$ such that

$$
\begin{aligned}
&\varphi_n|_A = \psi,\\
&\varphi_n^1(y) \in F(\varphi_{n-1}^1(y), \varphi_{n-1}^2(y), \varphi_{n-1}^3(y)),\\
&\varphi_n^2(y) \in F(\varphi_{n-1}^2(y), \varphi_{n-1}^1(y), \varphi_{n-1}^2(y)),\\
&\varphi_n^3(y) \in F(\varphi_{n-1}^3(y), \varphi_{n-1}^2(y), \varphi_{n-1}^1(y)),\\
&d(\varphi_{n-1}^i(y), \varphi_n^i(y)) \le l^{n-1} d(\varphi_0^i(y), \varphi_1^i(y)) + q^{-(n-1)},\ (i = 1, 2, 3),
\end{aligned}
$$

for all $n \ge 1$ and $y \in Y$. Define $Y_\lambda := \{y \in Y : d(\varphi_0^i(y), \varphi_1^i(y)) < \lambda,\ i = 1, 2, 3\}$ for all $\lambda > 0$. Now, we prove that the family $\{Y_\lambda : \lambda > 0\}$ is an open covering of Y. Note that, for each $y \in Y$

$$
\varphi_1^1(y) \in F(\varphi_0^1(y), \varphi_0^2(y), \varphi_0^3(y)),
$$

$$F(\varphi_0^1(y),\varphi_0^2(y),\varphi_0^3(y)) \cap B(\varphi_0^1(y),g_0^1(y)) = F(\varphi_0^1(y),\varphi_0^2(y),\varphi_0^3(y)),$$
$$\varphi_1^2(y) \in F(\varphi_0^2(y),\varphi_0^1(y),\varphi_0^2(y)),$$
$$F(\varphi_0^2(y),\varphi_0^1(y),\varphi_0^2(y)) \cap B(\varphi_0^2(y),g_0^2(y)) = F(\varphi_0^2(y),\varphi_0^1(y),\varphi_0^2(y)),$$
$$\varphi_1^3(y) \in F(\varphi_0^3(y),\varphi_0^2(y),\varphi_0^1(y)),$$
$$F(\varphi_0^3(y),\varphi_0^2(y),\varphi_0^1(y)) \cap B(\varphi_0^3(y),g_0^3(y)) = F(\varphi_0^3(y),\varphi_0^2(y),\varphi_0^1(y)),$$

and we have

$$\varphi_1^1(y) \in B(\varphi_0^1(y),g_0^1(y)),\varphi_1^2(y) \in B(\varphi_0^2(y),g_0^2(y)),\varphi_1^3(y) \in B(\varphi_0^3(y),g_0^3(y)).$$

Hence, if $\lambda = \max\{g_0^1(y),g_0^2(y),g_0^3(y)\}$, then we get that $d(\varphi_0^i(y),\varphi_1^i(y)) < \lambda$, $(i = 1,2,3)$, thus $y \in Y_\lambda$. Since Y_λ is open for each $\lambda > 0$, the family of sets $\{Y_\lambda \mid \lambda > 0\}$ is an open covering of Y, and we have

$$d(\varphi_{n-1}^i(y),\varphi_n^i(y)) \le l^{n-1} d(\varphi_0^i(y)), f(\varphi_1^i(y))) + q^{-(n-1)}$$

for all $n \ge 1$ and $y \in Y, i = 1,2,3$.

Since $l < 1$, $q > 1$ and X is complete, the sequence $\{\varphi_n^i\}_{n\ge 0}$, $(i = 1,2,3)$ converges uniformly on Y_λ for all $\lambda > 0$. Then, the functions φ^i, $(i = 1,2,3)$ are continuous. Since $\varphi_n|_A = \psi$ for all $n \ge 1$, $\varphi|_A = \psi$. Note that

$$\varphi_n^1(y) \in F(\varphi_{n-1}^1(y),\varphi_{n-1}^2(y),\varphi_{n-1}^3(y)),$$
$$\varphi_n^2(y) \in F(\varphi_{n-1}^2(y),\varphi_{n-1}^1(y),\varphi_{n-1}^2(y)),$$
$$\varphi_{n-1}^3(y) \in F(\varphi_{n-1}^3(y),\varphi_{n-1}^2(y),\varphi_{n-1}^1(y)),$$

for all $y \in Y$ and $n \ge 1$. If $n \to \infty$, then

$$\varphi^1(y) \in F(\varphi^1(y),\varphi^2(y),\varphi^3(y)),$$
$$\varphi^2(y) \in F(\varphi^2(y),\varphi^1(y),\varphi^2(y)),$$
$$\varphi^3(y) \in F(\varphi^3(y),\varphi^2(y),\varphi^1(y)),$$

for all $y \in Y$. Therefore, $\varphi : Y \to \mathcal{B}$ is a continuous extension of ψ, that is $\mathcal{B}$ is an absolute retract for metric spaces. ∎

COROLLARY 2.3. *Let (X,d) be a complete metric space and absolute retract for metric spaces and $F : X \times X \to P_{b,cl}(X)$ a lower semicontinuous multifunction such that $F \in SP(X \times X)$; also let F satisfy the following condition: there exist $a_1,a_2,\dots,a_5 \in (0,\infty)$ with $a_1 + a_3 + 2a_4 + a_5 < 1$ and*

$$d(F(x,y),F(u,v)) \le a_1 d(x,u) + a_2 D(u,F(x,y)) + a_3 D(u,F(u,v)) + a_4 D(x,F(u,v)) + a_5 D(x,F(x,y)),$$

for all $x,y,u,v \in X$. Then the set

$$\mathcal{B} = \{(x,y) \in X \times X : x \in F(x,y), y \in F(y,x)\}$$

is an absolute retract for metric spaces.

EXAMPLE 2.4. Let $X = [-1, \infty)$, $d(x, y) = |x - y|$ and for any $A, B \subset X$,

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\};$$

also, let $F : X \times X \times X \to P_{b,cl}(X)$ be defined by

$$F(x, y, z) = \begin{cases} [x, 0], & \text{if } x \le 0,\ y, z \in [-1, \infty) \\ 0, & \text{if } x > 0,\ y, z \in [-1, \infty). \end{cases}$$

It is easy to see that (X, d) is a complete metric space. We have

$$H(F(x, y, z), F(u, v, w)) = \begin{cases} |x - u|, & \text{if } x, u < 0 \\ u, & \text{if } x \le 0, u > 0 \\ x, & \text{if } u \le 0, x > 0 \\ 0, & \text{if } u > 0, x > 0; \end{cases}$$

$$D(u, F(x, y, z)) = \begin{cases} |x - u|, & \text{if } u < x \le 0 \\ u, & \text{if } x \le 0, u > 0 \\ |u|, & \text{if } u < 0, x > 0 \\ 0, & \text{otherwise}; \end{cases}$$

$$D(x, F(u, v, w)) = \begin{cases} |x - u|, & \text{if } x < u \le 0 \\ x, & \text{if } u \le 0, x > 0 \\ |x|, & \text{if } u > 0, x < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Also, $D(x, F(x, y, z)) = D(u, F(u, v, w)) = 0$. It is straightforward that there exists $a_1, a_2, \dots, a_5 \in (0, \infty)$ with $a_1 + a_3 + 2a_4 + a_5 < 1$ and

$$\begin{aligned} H(F(x, y, z), F(u, v, w)) \le a_1 d(x, u) + a_2 D(u, F(x, y, z)) + a_3 D(u, F(u, v, w)) \\ + a_4 D(x, F(u, v, w)) + a_5 D(x, F(x, y, z)), \end{aligned}$$

for all $x, y, z, u, v, w \in X$. In this example we have

$$\begin{aligned} \mathcal{B} = \{&(x, y, z) \in [-1, \infty) \times [-1, \infty) \times [-1, \infty) : \\ &x \in F(x, y, z), y \in F(y, x, y), z \in F(z, y, x)\} \\ = \{&(x, y, z) \in [-1, \infty) \times [-1, \infty) \times [-1, \infty) : x \in [x, 0], y \in [y, 0], z \in [z, 0]\} \\ = [&-1, 0] \times [-1, 0] \times [-1, 0]. \end{aligned}$$

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(received 11.03.2016; in revised form 20.04.2016; available online 20.05.2016)

Department of Mathematics, University of Bonab, Bonab 5551761167, Iran

E-mail: hojat.afshari@yahoo.com, hojat.afshari@bonabu.ac.ir