Extensions of $D(−1)$-pairs in some imaginary quadratic fields

Mirela Jukić Bokun and Ivan Soldo

ABSTRACT. In this paper, we discuss the extensibility of Diophantine $D(−1)$ pairs $\{a, b\}$, where a, b are positive integers in the ring $\mathbb{Z}[\sqrt{−k}]$, $k > 0$. We prove that families of such $D(−1)$-pairs with $b = p^i q^j$, where p, q are different odd primes and i, j are positive integers cannot be extended to quadruples in certain rings $\mathbb{Z}[\sqrt{−k}]$, where k depends on p^i, q^i and a. Further, we present the result on non-existence of $D(−1)$-quintuples of a specific form in certain imaginary quadratic rings.

CONTENTS

1. Introduction

A set $\{a_1, a_2, \dots, a_m\}$ of m distinct positive integers is called a Diophantine m-tuple with the property $D(n)$ (or simply a $D(n)$-m-tuple) for a given non-zero integer n, if $a_i a_j + n$ is a perfect square, for all $1 \le i < j \le m$. The problem of the existence of such sets has been dealt with by many distinguished mathematicians over the past years and centuries. It was Fermat who first found a $D(1)$-quadruple $\{1, 3, 8, 120\}$ in integers. In [5], Dujella proved that there is no $D(1)$-sextuple in integers, setting the conjecture that there could be only finitely many integer $D(1)$-quintuples. The nonexistence of such quintuples was handled by He, Togbé and Ziegler in [19]. On the other hand, in [2], Bonciocat, Cipu and Mignotte proved a conjecture of Dujella from [3], which states that there are no integer $D(−1)$-quadruples. Moreover, due to Trebješanin and Filipin it is known that there do not exist integer $D(4)$-quintuples (see [1]). On regularly

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updated Dujella's web page [7], interested readers can find an extended list of references on $D(n)$-m-tuples in integers and rationals.

The problem of Diophantus was also considered in a commutative ring with unity. The authors with collaborators mostly studied it over imaginary quadratic fields and the corresponding rings. In [4, 12], it is considered for Gaussian integers and integers of $\mathbb{Q}(\sqrt{-3})$, respectively. In [10, 23], we have almost completely characterized elements z of $\mathbb{Z}[\sqrt{-2}]$ for which a Diophantine quadruple with the property $D(z)$ exists. As a sporadic possible exception appears the case $z = -1$. That motivated us to consider the existence of $D(-1)$-m-tuples in the ring $\mathbb{Z}[\sqrt{-k}]$, with a positive integer k. Due to the result of Dujella for Gaussian integers [4, Theorem 3] and [9, Proposition 2] it is possible to find a $D(-1)$-quadruple in $\mathbb{Z}[\sqrt{-k}]$ in case k is a square. Further, in [14], the authors proved that there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$, $k \geq 2$ with positive integers $a < b \leq 8a - 3$ and negative integers c and d. They applied the obtained result to show that such a $D(-1)$-pair $\{a, b\}$ cannot be extended to a $D(-1)$-quintuple $\{a, b, c, d, e\}$ in $\mathbb{Z}[\sqrt{-k}]$ with integers c, d and e. Very recently, in [15], the authors proved a general result on the nonexistence of a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$, $k \geq 2$, with integers a, b, c, d, but in the range $a < b \leq 8a - 3$.

Using a computer program, if q is an odd prime and if we take $a = 2, b \in \{5q, 13q\}$, we found out that the set of $D(-1)$-pairs $\{a, b\}$ is not at all rare. Namely, odd primes of the form $(x^2 + 1)/10$ and $(x^2 + 1)/26$, $x \in \mathbb{Z}$, are members of the sequence A207337 and A208292 in [22], respectively. That motivated us to consider the extendibility of more general $D(-1)$-pairs $\{a, p^i q^j\}$, where a, i, j are positive integers and p, q are different odd primes, to quadruples in the ring $\mathbb{Z}[\sqrt{-k}]$, with an integer k. Thus, the main result of this paper is as follows:

Theorem 1.1. *Suppose that p, q are different odd primes, a, i, j, k positive integers such that $\gcd(a, pq) = 1$, $a < p^i q^j$ and $a p^i q^j$ is not a square. Moreover, let $k \geq \max\{p^i, q^j\}$ and $m_k = \dfrac{p^i q^j - a}{k}$. Let us further assume that either of the following holds:*

1) *$m_k = 1$ and $a \neq 1$;*
2) *$m_k = 2$ with $a \neq 1, 5$ and $\gcd(a, 2) = 1$;*
3) *m_k is a prime and $r \geq a m_k$ or $r \geq 2a^2$ with a square-free a, where $a p^i q^j - 1 = r^2$;*
4) *m_k is not an integer.*

Then, there does not exist a $D(-1)$-quadruple of the form $\{a, p^i q^j, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$.

Now, we specify the following known result on $D(-1)$-quadruples in integers:

Theorem 1.2 (see [2, Theorem 1.4.]). *There is no Diophantine $D(-1)$-quadruple.*

It is our intention to transform the whole problem in the ring of integers. In this respect, we will adapt the strategy used in [24, 14, 16], and we will mark the main steps, and describe the new ideas in detail in Theorem 2.3 and Theorem 2.8. In combination with Theorem 1.2 and other results from the next section, we will be able to prove our main result Theorem 1.1. Furthermore, we also prove the result on $D(-1)$-quintuples of the specific form (see Theorem 2.15 and its example).

2. Results on $D(-1)$-m-tuples

First, we have the following general result:

Theorem 2.1. *Let k be a positive integer and $\{a, b\}$ a $D(-1)$-pair in the ring $\mathbb{Z}[\sqrt{-k}]$ with positive integers a, b. If $k \nmid b - a$, then there does not exists a $D(-1)$-triple $\{a, b, c\}$ in $\mathbb{Z}[\sqrt{-k}]$ with a negative integer c. In that case, there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with an integer d.*

Proof. Suppose that the set $\{a, b, c\}$ is such a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-k}]$. Then we will obtain $c < 0$ if the following holds:

$$ac - 1 = -kx^2, \quad bc - 1 = -ky^2, \quad x, y \in \mathbb{Z}.$$

That is equivalent to

$$bx^2 - ay^2 = \frac{b - a}{k}. \tag{1}$$

Therefore, it must be satisfied $k | b - a$ and the equation (1) is solvable in integers (x, y). Since by the assumption it holds $k \nmid b - a$, we have only $c \in \mathbb{N}$. Thus, if we suppose that we have a $D(-1)$-quadruple $\{a, b, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$, then this contradicts with Theorem 1.2. $\square$

As a direct consequence of Theorem 2.1 we obtain the result as follows:

Corollary 2.2. *Let k be a positive integer and $\{a, b, c\}$ a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$ with positive integers a, b, c.*
   1) *If $k \nmid b - a$ or $k \nmid b - c$ or $k \nmid c - a$, then there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with an integer d.*
   2) *If $k | b - a$, $k | b - c$ and $k | c - a$, then the existence of a semi-regular $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with an integer d is possible if the condition $k | 4 - 3a^2$ holds.*

Proof. 1) It directly follows from Theorem 2.1.
   2) Let $ab - 1 = r^2$ and $c = a + b \pm 2r$. From $k | c - a$ and $k | c - b$ we obtain $k | 4a(c - a) \mp (2r \pm 3a)(c - b)$, i.e., $k | 4 - 3a^2$. $\square$

Now, we prove the next theorem:

Theorem 2.3. *Suppose that p, q are different odd primes, i, j, a, k positive integers such that $k \geq \max\{p^i, q^j\}$, $\gcd(a, pq) = 1$ and $a < p^i q^j$. If $\{a, p^i q^j, c\}$ is a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$, then $c \in \mathbb{Z}$.*

Proof. In the most cases, the proof follows the same strategy as in [24, Theorem 2.2] and [14, Theorem 5.4.]. Therefore, we will mark only the main steps, while in the case where the strategy is different from the used one, we will mark the details.

Let the set $\{a, p^i q^j, c\}$ be such a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-k}]$. Then there exist integers c_1, d_1, x, y, u, v such that $c = c_1 + d_1\sqrt{-k}$ and

$$ac - 1 = (x + y\sqrt{-k})^2,$$

$$p^i q^j c - 1 = (u + v\sqrt{-k})^2.$$

It follows that we have to consider the system of equations

$$p^i q^j xy = auv, \tag{2}$$

$$p^i q^j (x^2 - ky^2 + 1) = a(u^2 - kv^2 + 1). \tag{3}$$

First, consider the case where $xy \neq 0$. From (2) we conclude that $p^i q^j | auv$, so $p^i q^j | uv$.

(I) Let $u = p^i q^j w$ for some $w \in \mathbb{Z}$. From (2) we obtain $xy = awv$, so there exist nonzero integers p_1, q_1, r_1, s_1 such that $x = p_1 q_1, y = r_1 s_1, aw = p_1 r_1, v = q_1 s_1$. Now, from (3) we obtain

$$k = \frac{a(p^i q^j - a) - p^i q^j p_1^2 h}{a s_1^2 h}, \tag{4}$$

$$p^i q^j = \frac{a(a + h k s_1^2)}{a - h p_1^2}, \tag{5}$$

where $h = p^i q^j r_1^2 - a q_1^2$. From (5) we conclude that $a | h p_1^2$. It is easy to see that in each one of the cases $h \leq 0$ and $h > 0$ we obtain a contradiction with the fact that k and $p^i q^j$ are positive integers.

(II) In the second case, set $v = p^i q^j w$ for some $w \in \mathbb{Z}$. From (2) we obtain $xy = auw$. By following the same lines as in (I) we arrive to a contradiction.

(III) If $u = p^i w_1, v = q^j w_2$, where $w_1, w_2 \in \mathbb{Z}$. From (2) we obtain $xy = aw_1 w_2$, so there exist nonzero integers p_1, q_1, r_1, s_1 such that $x = p_1 q_1, y = r_1 s_1, aw_1 = p_1 r_1, w_2 = q_1 s_1$. Now, from (3) we obtain

$$k = \frac{a(a - p^i q^j) - p^i p_1^2 h}{a q^j s_1^2 h}, \tag{6}$$

$$q^j = \frac{a^2 - h p^i p_1^2}{a(p^i + k h s_1^2)}, \tag{7}$$

where $h = a q^j q_1^2 - p^i r_1^2$. Since $k \geq \max\{p^i, q^j\}$ from (6) and (7) we reach a contradiction in both cases $h \leq 0$ and $h > 0$.

(IV) Similarly as in (III), if $u = q^j w_1$, $v = p^i w_2$, where $w_1, w_2 \in \mathbb{Z}$, from (2) we obtain $xy = aw_1 w_2$, so there exist nonzero integers p_1, q_1, r_1, s_1 such that $x = p_1 q_1$, $y = r_1 s_1$, $aw_1 = p_1 r_1$, $w_2 = q_1 s_1$. Now, from (3) we obtain

$$k = \frac{a(p^i q^j - a) - q^j p_1^2 h}{a p^i s_1^2 h}, \tag{8}$$

$$p^i = \frac{a^2 + h p_1^2 q^j}{a(q^j - h k s_1^2)}, \tag{9}$$

where $h = q^j r_1^2 - a p^i q_1^2$. Since $k \geq \max\{p^i, q^j\}$ from (8) and (9) we obtain a contradiction in both cases $h \leq 0$ and $h > 0$.

(V) Suppose that $p^{i_1} q^{j_1} | \gcd(u, v)$, with $0 < i_1 + j_1 < i + j$. By (3) we have $p^i q^j | (u^2 - kv^2 + 1)$, which yields a contradiction.

Thus, the system of equations (2) and (3) has no solutions in case of $xy \neq 0$. Therefore, $x = 0$ or $y = 0$ or $x = y = 0$. In any case we have $d_1 = 0$ and $c = c_1$, which is an integer. $\qquad\square$

Suppose that p, q are different odd primes, $a, i, j, k \geq \max\{p^i, q^j\}$ positive integers and $\gcd(a, pq) = 1$. We consider the extendibility of the $D(-1)$-pairs $\{a, p^i q^j\}$ with $a < p^i q^j$ to quadruples of the form $\{a, p^j q^j, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$. By Theorem 2.3, if such a quadruple exists, then $c, d \in \mathbb{Z}$.

Now we will use the approach for instance from [20, Chapter 11.5] and consider the equation

$$ax^2 - dy^2 = \delta s, \quad \delta = \pm 1, \tag{10}$$

with a positive non-square integer d and positive integers a, s.

Assume that (x, y) is a primitive solution to (10). Then there exist integers u, v such that

$$xu - yv = \delta. \tag{11}$$

Multiplying (10) by $av^2 - du^2$ we obtain

$$(axv - dyu)^2 - ad = \delta s(av^2 - du^2).$$

If we set

$$|axv - dyu| = l,$$

we conclude that the equation

$$av^2 - du^2 = \frac{l^2 - ad}{\delta s} \tag{12}$$

has a solution in integers.

If (u_0, v_0) is a solution of (11), then the following parametric equations

$$v = v_0 + tx, \quad u = u_0 + ty, \quad t \in \mathbb{Z}$$

generate all solutions to (11). Thus,

$$l = |axv_0 - dyu_0 + \delta st|,$$

which implies that there exists an integer t such that $l \leq \frac{s}{2}$.

These facts can be summarized in the following lemma:

Lemma 2.4. *Let d be positive non-square integer and a, s positive integers. If there exists a primitive solution (x, y) of the equation (10), then there exists an integer l satisfying $l^2 \equiv ad \pmod{s}$ and $0 \leq l \leq \frac{s}{2}$ such that the equation (12) has s a primitive solution.*

Moreover, the next lemma will be useful.

Lemma 2.5 (see [13, Lemma 2.3]). *Let N and K be integers with $1 < |N| \leq K$. Then the Pellian equation*

$$x^2 - (K^2 + 1)y^2 = N$$

has no primitive integer solution (x, y), i.e., there does not exist integers x, y such that $\gcd(x, y) = 1$.

Also, for the equation $Ux^2 - Vy^2 = c, c \in \{1, 2\}$ we use the following solvability criteria in integers x, y.

Lemma 2.6 (see [17, Criterion 1.]). *Let $U > 1$, V be positive integers such that $\gcd(U, V) = 1$ and $D = UV$ is not a square of a positive integer. Moreover, let (u_0, w_0) denote the least positive integer solution of Pell's equation $u^2 - Dw^2 = 1$. Then equation $Ux^2 - Vy^2 = 1$ has a solution in positive integers x, y if and only if*

$$2U | u_0 + 1 \quad and \quad 2V | u_0 - 1.$$

Lemma 2.7 (see [17, Criterion 2.]). *Let U, V be positive integers such that*

$$\gcd(U, V) = \gcd(U, 2) = \gcd(V, 2) = 1$$

and $D = UV$ is not a square of a positive integer. Let (u_0, w_0) denote the least positive integer solution of Pell's equation $u^2 - Dw^2 = 1$. Then equation $Ux^2 - Vy^2 = 2$ has a solution in positive integers x, y if and only if

$$U | u_0 + 1 \quad and \quad V | w_0 - 1.$$

Now, by using the above arguments we were able to prove the following:

Theorem 2.8. *Let p, q be different odd primes, a, i, j, k positive integers such that $\gcd(a, pq) = 1$, $a < p^i q^j$ and $ap^i q^j$ is not a square. Further, let $k \geq \max\{p^i, q^j\}$ and $m_k = \dfrac{p^i q^j - a}{k}$. Suppose that either of the following holds:*

1) *$m_k = 1$ and $a \neq 1$;*
2) *$m_k = 2$ with $a \neq 1, 5$ and $\gcd(a, 2) = 1$;*
3) *m_k is a prime and $r \geq am_k$ or $r \geq 2a^2$ with a square-free a, where $ap^i q^j - 1 = r^2$.*

If $\{a, p^i q^j, c\}$ is a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-k}]$, then $c \in \mathbb{N}$.

Proof. By Theorem 2.3, we know that in such a $D(-1)$-triple we have $c \in \mathbb{Z}$.

Now we will use the argument from the proof of Theorem 2.1, i.e., we will consider the equation (1) which is equivalent to

$$p^i q^j x^2 - ay^2 = m_k, \tag{13}$$

and prove that it is not solvable in integers x, y. In that way we will immediately obtain that c cannot be a negative integer.

1) Let $m_k = 1$ and $a \neq 1$. In that case the fundamental solution of equation $u^2 - ap^i q^j w^2 = 1$ is $u_0 + w_0 \sqrt{ap^i q^j}$, where $u_0 = 2ap^i q^j - 1$, $w_0 = 2\sqrt{ap^i q^j - 1}$, One can check it very easily by using for instance [6, Proposition 10.13.]. By applying Lemma 2.6, the condition $2a|2(ap^i q^j - 1)$ implies that $a = 1$, which is not possible.

2) If we take $m_k = 2$ with $a \neq 1, 5$, then we obtain the equation $p^i q^j x^2 - ay^2 = 2$. Since the condition $a|2\sqrt{ap^i q^j - 1} - 1$ from Lemma 2.7 leads to $a = 1, 5$, we arrive to a contradiction.

3) Since $\gcd(a, p^i q^j) = 1$ and m_k is a prime, we know that the equation (13) can only have primitive integer solution.

The equation (13) can be transformed to

$$y_1^2 - ap^i q^j x^2 = -am_k, \tag{14}$$

with $y_1 = ay$. Since $ap^i q^j = r^2 + 1$, by Lemma 2.5 the equation (14) has no primitive solution (y_1, x), so neither does (13) for $r \geq am_k$. If $r < am_k$, then for $r \geq 2a^2$ we obtain

$$a < \frac{m_k}{2}. \tag{15}$$

Note that $ap^i q^j \equiv a^2 \pmod{m_k}$. Since m_k is a prime and inequality (15) holds, the congruence

$$l^2 \equiv ap^i q^j \pmod{m_k}, \tag{16}$$

has only the solution $l = a$ satisfying the condition $0 \leq l \leq m_k/2$. Now, from Lemma 2.4 it follows that if (13) has a primitive integer solution (x, y), then

$$p^i q^j x^2 - ay^2 = \frac{a^2 - ap^i q^j}{m_k} = -ak \tag{17}$$

also has a primitive integer solution. Since we know that a is square-free and $\gcd(a, p^i q^j)=1$, we can set $x = ax_0$, where x_0 is a positive integer.

Then from (17) we obtain

$$y^2 - ap^i q^j x_0^2 = k. \tag{18}$$

Since $r < am_k$ implies $1 < k \leq r$, it follows from Lemma 2.5 that (18) has no primitive integer solution (y, x_0), and neither does (13). $\square$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. If we suppose that there exists such a $D(-1)$-quadruple in the ring $\mathbb{Z}[\sqrt{-k}]$, then by Theorem 2.3, Theorem 2.1 and Theorem 2.8 we know that c, d are positive integers. This contradicts with Corollary 2.2 and Theorem 1.2. $\qquad\square$

Remark 2.9. *From* [15]*, it is known that there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$, $k \geq 2$, with integers a, b, c, d satisfying $a < b \leq 8a - 3$ (see* [15]*, Theorem 1.6.]). That implies, if we take $b = p^i q^j$ from Theorem 1.1, we can conclude the nonexistence of a $D(-1)$-quadruple $\{a, p^i q^j, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$ with c, d integers and $1 \leq a < p^i q^j \leq 8a - 3$.*

Further, if we consider our previously mentioned results in terms of Corollary 2.2, we are able to construct the following example:

Example 2.10. *By Corollary 2.2 the $D(-1)$-quadruple of the form $\{a, b, c, d\}$, with positive integers $a, b, c = a + b \pm 2r$ and $ab - 1 = r^2$ could exists in the ring $\mathbb{Z}[\sqrt{-k}]$ if the condition $k \mid 4 - 3a^2$ is satisfied.*

We recall the results on $D(-1)$-triples in the ring $\mathbb{Z}[\sqrt{-k}]$ with $k > 0$.

Lemma 2.11 (see [14, Theorem 5.4])**.** *Let i, j, k be positive integers and p, q different primes with $p^i < q^j$. If $\{p^i, q^j, c\}$ is a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$, then $c \in \mathbb{Z}$.*

Lemma 2.12 (see [24, Theorem 2.2], [9, Theorem 4])**.** *Let i, k be positive integers. If b is a prime or $b = 2p^i$, with an odd prime p and $\{1, b, c\}$ is a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$, then $c \in \mathbb{Z}$. Moreover, for every k there exists $c > 0$, while the case of $c < 0$ is possible if and only if $k \mid b - 1$ and the equation*

$$x^2 - by^2 = \frac{1-b}{k} \tag{19}$$

has an integer solution.

Let

$$b_1 = \begin{cases} p, & p \text{ is an odd prime}, k \geq 2, \\ 2p^j, & p \text{ is an odd prime}, j \in \mathbb{N}, k \geq 2, \\ p^i q^j, & p, q, \text{ are different odd primes and }, i, j \in \mathbb{N}, k \geq \max\{p^i, q^j\} \end{cases}$$

and

$$b_2 = \begin{cases} p^j, & p \text{ is an odd prime}, j \in \mathbb{N} \text{ and } k \geq 2, \\ p^i q^j, & p, q \text{ are different odd primes and }, i, j \in \mathbb{N}, k \geq \max\{p^i, q^j\}. \end{cases}$$

For $a = 1$ and $b = b_1$, by Lemma 2.12, Theorem 2.3 we have that $d \in \mathbb{Z}$. Now, the Corollary 2.2 implies that such $D(-1)$-quadruple could exists if $k = 1$.

Similarly, if $a = 2$ and $b = b_2$, by Lemma 2.11 and Theorem 2.3 we know that $d \in \mathbb{Z}$. The condition $k \mid 4 - 3a^2$ implies $k \mid 8$. By combining that with $k \mid b_2 - 2$ we conclude the existence of a qaudruple only in the case $k = 1$.

Note that this coincides with the following result of Dujella on Gaussian integers:

Theorem 2.13 (see [4, Theorem 3]). *Let l be a Gaussian integer and suppose that the set $\{a, b\} \subset \mathbb{Z}[i]$ has the property $D(l^2)$. If the number ab is not a square of a Gaussian integer, then there exist an infinite number of complex Diophantine quadruples of the form $\{a, b, c, d\}$ with the property $D(l^2)$.*

Namely, if $l^2 = i^2 = -1$, considering the above $D(-1)$-pairs $\{a, b\}$, since ab is not a square by Theorem 2.13 we know that in the case $k = 1$ there exist infinitely many $D(-1)$-quadruples of the form $\{a, b, c, d\}$ in Gaussian integers.

Further, in [9], we proved the following:

Lemma 2.14 (see [9, Proposition 2]). *Let $m, n > 0$ and $b = n^2 + 1$. If $m|n$ and $k = m^2$, then there exist infinitely many $D(-1)$-quadruples of the form $\{1, b, -c, d\}$, $c, d > 0$ in $\mathbb{Z}[\sqrt{-k}]$.*

In [24], one can also find some special cases of such quadruples. We wondered if $D(-1)$-quadruples of such form can be extended to quintuples in similar imaginary quadratic rings. We were able to obtain the following result:

Theorem 2.15. *Let m, n, k be a positive integers and $b = n^2 + 1$. Suppose that either of the following holds:*
 1) $k = m^2$;
 2) $k \nmid n^2$.
There does not exists a $D(-1)$-quintuple of the form $\{1, b, -c, d, e\}$ with integers $c, d > 0, e$ in $\mathbb{Z}[\sqrt{-k}]$.

Proof. Let us suppose that under the above conditions there exists a $D(-1)$-quintuple of the form $\{1, b, -c, d, e\}$ in $\mathbb{Z}[\sqrt{-k}]$.

1) $e > 0$ and $\{1, b, d, e\}$ is a $D(-1)$-quadruple contradicts with Theorem 1.2.

Thus we have to consider the existence of a $D(-1)$-quintuple of the form $\{1, b, -c, d, -e\}$, where $b = n^2+1, c, d, e$ are positive integers. Since $k = m^2, m \in \mathbb{N}$ and $\mathbb{Z}[\sqrt{-k}] = \mathbb{Z}[mi] \leq \mathbb{Z}[i]$ it is sufficient to prove the nonexistence of such quintuple in $\mathbb{Z}[i]$. Therefore, we have to consider if there exists a positive integer e such that

$$-e - 1 = -\alpha^2, -be - 1 = -\beta^2, ce - 1 = \gamma^2, -de - 1 = \delta^2,$$

for some integers $\alpha, \beta, \gamma, \delta$. Eliminating e we obtain

$$c\delta^2 + d\gamma^2 \;\;=\;\; -(c + d).$$

It is clear that the above equation is not solvable in integers δ, γ.

2) According to Corollary 2.2 1) we conclude that in $\mathbb{Z}[\sqrt{-k}]$ there does not exist a $D(-1)$-quadruple $\{1, b, -c, d\}$, with positive integers b, c, d. $\qquad\square$

Example 2.16. *In the proof of Lemma 2.14 we obtain that*

$$\begin{aligned} b &= n^2 + 1, \quad k = m^2, m \mid n, \\ c &= n^2 x^2 - 1, \\ bc &= n^2 y^2 - 1, \end{aligned}$$

and x, y are positive solutions of the Pellian equation $y^2 - (n^2 + 1)x^2 = -1$. Moreover,

$$d = d_{\pm} = \pm 2n^3 xy + (2n^2 + 1)c + n^2 + 2.$$

If the fifth element e in $D(-1)$-quintuple of the form $\{1, b, -c, d, e\}$ in $\mathbb{Z}[\sqrt{-k}]$ is necessarily integer (for instance, like in the case $b = b_1$ of Example 2.10), by applying Theorem 2.15 1) we conclude that $D(-1)$-quadruples of the form $\{1, b, -c, d\}$ cannot be extended to quintuples in such rings.

Acknowledgement

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(Mirela Jukić Bokun) SCHOOL OF APPLIED MATHEMATICS AND INFORMATICS, UNIVERSITY OF OSIJEK, TRG LJUDEVITA GAJA 6, HR-31000 OSIJEK, CROATIA
mirela@mathos.hr

(Ivan Soldo) SCHOOL OF APPLIED MATHEMATICS AND INFORMATICS, UNIVERSITY OF OSIJEK, TRG LJUDEVITA GAJA 6, HR-31000 OSIJEK, CROATIA
isoldo@mathos.hr