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Two strand twisting

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ABSTRACT. We prove that fibred knots cannot be untied with $\bar{t}_{2k}$-moves, for all $k \geq 2$. More generally, we give an upper bound on the number of two strand twist operations that allow us to untie a knot with non-trivial HOMFLY polynomial, in terms of the minimal crossing number, and the braid index. As a by-product, we prove that the braid index of a two-bridge knot cannot be lowered by applying t_{2k}-moves, for all but finitely many $k \in \mathbb{N}$.

CONTENTS

1. Introduction

Twisting is a family of local operations on oriented links in S^3, introduced by Ralph Fox in the late fifties [4]. The special case of two strand twisting comes in two families called t_m-moves and $\bar{t}_m$-moves. The effect of a t_m-move (resp. $\bar{t}_m$-move) is best described on oriented link diagrams, where it inserts m consecutive right-handed half-twists into two parallel strands with equal orientations (resp. opposite orientations), as in the two strand braid $\sigma_1^m \in B_2$, see Figure 1. We will restrict our attention to even numbers $m \in \mathbb{N}$, since we want all moves to preserve the number of link components. The two simplest moves, t_2 and $\bar{t}_2$, are also known as crossing changes, since they can be expressed as a Reidemeister II move followed by a crossing change. While every knot can be untied by a finite sequence of crossing changes, there exist obstructions for knots to be unknotted by higher order t_{2k}- and $\bar{t}_{2k}$-moves. In particular, the Alexander–Conway polynomial of knots with coefficients reduced modulo k, $\nabla_K(z) \in \mathbb{Z}/k\mathbb{Z}[z]$, is invariant under $\bar{t}_{2k}$-moves [4, 14]. This has a remarkable consequence for fibred knots, since these have monic Alexander-Conway polynomial [12].

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FIGURE 1. A t_k-move and a $\bar{t}_k$-move. The right hand sides contain k many half-twists.

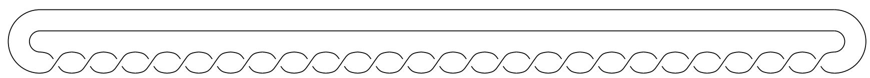

FIGURE 2. Closure of the braid σ_1^{25}, which can be unknotted by $(t_{2k})^{-1}$-moves, for $k = 1, 2, 3, 4, 6, 12, 13$.

Theorem 1.1. *Let K be a non-trivial fibred knot. For all $k \geq 2$, the knot K is not related to the trivial knot by a finite sequence of $(\bar{t}_{2k})^{\pm 1}$-moves.*

As far as the authors are aware, this statement never found its way into the literature, most likely since Neuwirth's theory of fibred knots was developed after Fox's note on congruence classes of knots.

In contrast with Theorem 1.1, the situation is very different with t_{2k}-moves. For all $n \in \mathbb{N}$, there exists a fibred knot which, for any $k \leq n$, can be unknotted by $(t_{2k})^{\pm 1}$-moves, for example the closure of the braid

$$\sigma_1^{1+\mathrm{lcm}(2,4,6\ldots 2n)} \in B_2,$$

as shown in Figure 2 for $n = 4$.

Nevertheless, the two families of two strand twisting operations share a common property: most knots can be unknotted by finitely many different types of t_{2k}-moves and $\bar{t}_{2k}$-moves only. This was derived by Lackenby in one of his early papers (Corollary 2.4 in [8]). Our main result provides independent quantitative bounds, in terms of well-known link invariants. Let $c(K) \in \mathbb{N}$ and $b(K) \in \mathbb{N}$ be the minimal crossing number and the braid index of a link K, respectively. The latter is defined as the minimal number of strands among all braids whose closure represents the link K. Furthermore, let $P_K(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ be the HOMFLY polynomial of K, defined in the next section.

Theorem 1.2. *Let K be a knot with $P_K(a, z) \neq 1$.*

(1) *The set $\{k \geq 3 \mid K$ can be unknotted by $(t_{2k})^{\pm 1}$-moves$\}$ has at most $c(K) - 1$ elements.*
(2) *The set $\{k \geq 2 \mid K$ can be unknotted by $(\bar{t}_{2k})^{\pm 1}$-moves$\}$ has at most $b(K) - 1$ elements.*

A version of the first statement for the Jones polynomial was derived by Lackenby (Corollary 2.11 in [8]). The condition $P_K(a, z) \neq 1$ seems quite generic, and is possibly even satisfied by all non-trivial knots. Nevertheless, it would be great to derive bounds of the above kind for all non-trivial knots, for example by using Khovanov homology, which is known to detect the trivial knot [7]. The mere existence of finite upper bounds in Theorem 1.2 was conjectured by Lackenby in the paper cited above. It might even admit a geometric proof, due to its resemblance with Thurston's hyperbolic Dehn surgery theorem [16]. However, the latter deals with fixed twist regions, which provides an a priori weaker statement.

The technique used in our proof allows a precise determination of the set of unknotting moves for certain classes of knots, such as two strand torus knots and twist knots, see Proposition 4.2. More importantly, we obtain the following refined result for two-bridge knots.

Proposition 1.3. *Let K be a two-bridge knot. For all but finitely many $k \in \mathbb{N}$, all knots K' that are related to K by a finite sequence of $(t_{2k})^{\pm 1}$-moves satisfy*

$$b(K') \geq b(K).$$

Results providing a lower bound for the braid index within an equivalence class of knots are not so common; an interesting one was recently derived by Feller and Hubbard: closures of quasipositive braids with sufficiently many full twists are not concordant to quasipositive knots with a strictly smaller braid index [3].

The proofs of Theorem 1.2 and Proposition 1.3 are presented in Sections 3 and 4; the next section contains the necessary fundamentals about the HOMFLY polynomial. We would like to emphasise that most of the statements included here are applications of Przytycki's theory on two strand twisting [14]. We would like to thank Marc Lackenby for informing us about similar finiteness results on the Jones polynomial and Fox congruence in his early work [8].

2. HOMFLY polynomial and twisting

The HOMFLY polynomial $P_K(a,z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ of links K is defined by the following skein relation, together with the normalisation $P_O(a,z) = 1$ for the trivial knot O [6]:

$$a^{-1}P_{L_+}(a,z) - aP_{L_-}(a,z) = zP_{L_\infty}(a,z).$$

Here we use the standard notation L_+, L_-, L_∞ for link diagrams that coincide except in a disc that intersects these diagrams in a positive crossing, a negative crossing, and two parallel strands, respectively. As observed by various authors, the skein relation is well-suited to compute the effect of t_m-moves on links, see for example [13, 14]. In fact, the following Proposition is basically a reformulation of Corollaries 1.2 and 1.8 in [14], except for the case $k = 2$.

Proposition 2.1. *Let K be a knot, and let $\zeta_{2k} \in \mathbb{C}$ be a primitive $2k$-th root of unity.*

(i) *If K' is a knot obtained from K by a t_{2k}-move with $k \geq 3$, then*

$$P_{K'}(a, \zeta_{2k} - \zeta_{2k}^{-1}) = a^{2k} P_K(a, \zeta_{2k} - \zeta_{2k}^{-1}).$$

(ii) *If K' is a knot obtained from K by a t_4-move, then*

$$P_{K'}(a, 0) = a^4 P_K(a, 0) \in \mathbb{F}_2[a^{\pm 1}].$$

(iii) *If K' is a knot obtained from K by a $\bar{t}_{2k}$-move with $k \geq 2$, then*

$$P_{K'}(\zeta_{2k}, z) = P_K(\zeta_{2k}, z).$$

Proof. For the first two statements, let $L_0, L_1, L_2, \ldots$ be a family of oriented link diagrams, which coincide except in a single twist region consisting of two parallel oriented strands with a certain number of positive crossings – n for the link diagram L_n – again as in the two strand braid $\sigma_1^n \in B_2$. The skein relation for the HOMFLY polynomial translates into the recursion

$$P_{L_{n+1}}(a, z) = a^2 P_{L_{n-1}}(a, z) + az P_{L_n}(a, z),$$

which admits the following matrix representation:

$$\begin{pmatrix} P_{L_n} \\ P_{L_{n+1}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a^2 & az \end{pmatrix} \begin{pmatrix} P_{L_{n-1}} \\ P_{L_n} \end{pmatrix}.$$

Let $\zeta_{2k} \in \mathbb{C}$ be a primitive $2k$-th root of unity with $k \geq 3$. The specialisation $z = \zeta_{2k} - \zeta_{2k}^{-1} \neq 0$ gives rise to a recursion matrix

$$M = \begin{pmatrix} 0 & 1 \\ a^2 & a(\zeta_{2k} - \zeta_{2k}^{-1}) \end{pmatrix}$$

with $\operatorname{tr}(M) = a(\zeta_{2k} - \zeta_{2k}^{-1})$ and $\det(M) = -a^2$, hence

$$M^{2k} = \begin{pmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{pmatrix}.$$

The last step requires M to have two distinct eigenvalues $a\zeta_{2k}$, $-a\zeta_{2k}^{-1}$, which is the case for $k \geq 3$, but not for $k = 2$. This implies the first statement of the proposition.

In the case $k = 2$, i.e. for $z = 2i$, the recursion matrix M has a double eigenvalue ia and is not diagonalisable. However, we still have

$$M^4 = \begin{pmatrix} -3a^4 & -4ia^3 \\ -4ia^5 & 5a^4 \end{pmatrix} \equiv \begin{pmatrix} a^4 & 0 \\ 0 & a^4 \end{pmatrix} \pmod{4\mathbb{Z}[i, a^{\pm 1}]}.$$

Now suppose that L_1 is a knot. Then L_0 is a two-component link, thus $P_{L_0}(a, z)$ contains a simple pole in z. Indeed, the lowest degree in z of the HOMFLY polynomial of a link with c components is $1 - c$, see [9]. Consequently, we find that that

$$P_{L_0}(a, 2i) \in \frac{1}{2i} \mathbb{Z}[i, a^{\pm 1}].$$

Multiplying with $-4ia^5$, we see that $-4ia^5 P_{L_0}(a, 2i)$ lies in $2\mathbb{Z}[i, a^{\pm 1}]$. Reducing

$$P_{L_5}(a, 2i) = -4ia^5 P_{L_0}(a, 2i) + 5a^4 P_{L_1}(a, 2i)$$

modulo 2 leaves us with the congruence

$$P_{L_5}(a,0) \equiv a^4 P_{L_1}(a,0) \pmod{2\mathbb{Z}[i,a^{\pm 1}]},$$

i.e. with the second statement of the proposition.

The proof of the third statement is a recursion similar to the above one. Taking K' a knot obtained from K by a $\bar{t}_{2k}$-move with $k \geq 2$, and denoting by L_∞ the link obtained by smoothing one crossing in the twist region, we obtain

$$P_{K'}(a,z) = a^{-2k} P_K(a,z) - (a^{-1} + a^{-3} + \cdots + a^{-2k+1}) z P_{L_\infty}(a,z),$$

and the result immediately follows. For the details, we refer the reader to Przytycki's proof of Theorem 1.7 in [14], with the caveat that the sign convention for the skein relation of the HOMFLY polynomial is different in [14], leading to an overall sign $(-1)^k$ in the formula there.

□

Remark. Reductions of the form $P_K(a,N) \in \mathbb{F}_p[a^{\pm 1}]$ have been studied in [1]. Proposition 2.1 provides infinitely many reductions of the HOMFLY polynomial, invariant under t_{2k}-moves, up to multiplication with powers of a^{2k}. Indeed, let $k \geq 3$, and let $p \in 2k\mathbb{N}+1$ be a prime number. Then the finite field $\mathbb{F}_p$ has a primitive $2k$-th root of unity ζ_{2k}, since the multiplicative group of $\mathbb{F}_p$ is cyclic of order $p-1$. Let $N = \zeta_{2k} - \zeta_{2k}^{-1} \in \mathbb{F}_p$. The statement of Proposition 2.1 carries over to the reduction $P_K(a,N) \in \mathbb{F}_p[a^{\pm 1}]$: let K' be a link obtained from a link K by a positive t_{2k}-move with $k \geq 3$. For every prime number $p \in 2k\mathbb{N}+1$, and $N = \zeta_{2k} - \zeta_{2k}^{-1} \in \mathbb{F}_p$, the equality

$$P_{K'}(a,N) = a^{2k} P_K(a,N)$$

holds in $\mathbb{F}_p[a^{\pm 1}]$. Thanks to Dirichlet's theorem on arithmetic progressions [2], the number of primes p in $2k\mathbb{N}+1$ is infinite, for each fixed $k \geq 2$. As a consequence, we obtain infinitely many invariant reductions $P_K(a,N) \in \mathbb{F}_p[a^{\pm 1}]$ under t_{2k}-moves, provided $k \geq 3$. Similarly, there exist infinitely many reductions of the form $P_K(M,z) \in \mathbb{F}_p[z^{\pm 1}]$ invariant under $\bar{t}_{2k}$-moves, for all $k \geq 2$.

3. Bounding the order of untwisting

The skein relation of the HOMFLY polynomial implies that the specialisation $P_K(a, a^{-1} - a)$ is constantly one. As a consequence, the polynomial $P_K(a,z)$ cannot be of the form $a^m f(z)$, unless $P_K(a,z) = 1$. Moreover, recall that $P_K(a,z) \in \mathbb{Z}[a^{\pm 1}, z] \subset \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ if K is a knot.

Proof of Theorem 1.2. For the first statement, let K be a knot with $P_K(a,z) \neq 1$ and let $d = \deg_z(P_K)$. Write the terms of lowest and highest a-degree in $P_K(a,z)$ as $a^m f(z)$ and $a^n g(z)$, respectively, with $m,n \in \mathbb{Z}$, $m < n$, and $f(z), g(z) \in \mathbb{Z}[z]$. Suppose that K is related to the trivial knot by a finite sequence of $(t_{2k})^{\pm 1}$-moves, for some $k \geq 3$. Then, by Proposition 2.1, either $f(\zeta_{2k} - \zeta_{2k}^{-1})$ or $g(\zeta_{2k} - \zeta_{2k}^{-1})$ must be zero, since the trivial knot O satisfies

$P_O(a,z) = 1$. Therefore, the product $f(z)g(z)$ vanishes at $z = \zeta_{2k} - \zeta_{2k}^{-1}$. The degree bound $\deg(f(z)g(z)) \leq 2d$ implies that the set

$$\{k \geq 3 \mid K \text{ can be unknotted by } (t_{2k})^{\pm 1}\text{-moves}\}$$

has at most d elements, since the minimal polynomial of the purely imaginary number $\zeta_{2k} - \zeta_{2k}^{-1}$ has degree at least two. This yields the first statement, thanks to Franks–Williams and Morton's upper bound for the z-degree of the HOMFLY polynomial [5, 10]:

$$\deg_z(P_K) \leq c(K) - 1.$$

For the second statement, write

$$P_K(a,z) = h_0(a) + h_1(a)z^2 + h_2(a)z^4 + \ldots + h_l(a)z^{2l}$$

with $l \geq 1$ and $h_l(a) \neq 0$, since $P_K(a,z) \neq 1$. This is possible since the HOMFLY polynomial of a link with an odd (resp. even) number of components has only even (resp. odd) powers in z, see again [9]. Suppose that K is related to the trivial knot by a finite sequence of $(\bar{t}_{2k})^{\pm 1}$-moves, for some $k \geq 2$. Then, by Proposition 2.1, $h_l(\zeta_{2k}) = 0$. Therefore, the set

$$\{k \geq 2 \mid K \text{ can be unknotted by } (\bar{t}_{2k})^{\pm 1}\text{-moves}\}$$

has at most as many elements as half the number of roots of the Laurent polynomial $h_l(a)$, again since the minimal polynomial of the number ζ_{2k} has degree at least two. As a consequence, the above set has at most $\frac{1}{2}a\text{-span}(P_K(a,z))$ elements, where $a\text{-span}(P_K(a,z))$ is the difference of the highest and lowest a-degree in $P_K(a,z)$. This yields the second statement, thanks to another inequality by Franks–Williams and Morton [5, 10]:

$$2b(K) \geq a\text{-span}(P_K(a,z)) + 2.$$

□

4. Parallel twisting and braid index

The inequality of Franks–Williams and Morton [5, 10], used at the end of the last section, remains true under the specialisation $z = \zeta_{2k} - \zeta_{2k}^{-1}$. We observe that for all but finitely many $k \in \mathbb{N}$,

$$a\text{-span}(P_K(a, \zeta_{2k} - \zeta_{2k}^{-1})) = a\text{-span}(P_K(a,z)).$$

Furthermore, the a-span of $P_K(a, \zeta_{2k} - \zeta_{2k}^{-1})$ is certainly invariant under multiplication with powers of a^{2k}. Therefore, the first item of Proposition 2.1 implies the following statement.

Proposition 4.1. *Let K be a knot and $k \geq 3$. Then every knot K' that is related to K by a finite sequence of $(t_{2k})^{\pm 1}$-moves satisfies*

$$2b(K') \geq a\text{-}span(P_K(a, \zeta_{2k} - \zeta_{2k}^{-1})) + 2.$$

In the case of two-bridge knots K, Murasugi [11] showed that there is even an equality $2b(K) = a\text{-span}(P_K(a,z)) + 2$. This implies Proposition 1.3.

For two special families of two-bridge knots, we obtain even better results: let K_n be the family of twist knots with 2 negative crossings, and $2n$ positive crossings. In Rolfsen's notation [15], the first four knots of this sequence are $4_1, 6_1, 8_1, 10_1$, see Figure 3 for $n = 2$. For convenience, we add the trivial knot $K_0 = O$. Furthermore, let $T(2, 2n+1)$ be the two strand torus knot with $2n+1$ positive crossings.

Proposition 4.2.

(i) *For all $n \geq 1$, the set*

$$\{k \geq 1 \mid K_n \textit{ can be unknotted by } (\bar{t}_{2k})^{\pm 1}\textit{-moves}\}$$

coincides with the set of divisors of n. Moreover, every knot K' that is related to K_n by a finite sequence of $(t_{2k})^{\pm}$-moves with $k \geq 2$ satisfies

$$b(K') \geq b(K_n).$$

In particular, the knot K_n cannot be unknotted by $(t_{2k})^{\pm 1}$-moves, except for $k = 1$.

(ii) *For all $n \geq 1$, the set*

$$\{k \geq 1 \mid T(2, 2n+1) \textit{ can be unknotted by } (t_{2k})^{\pm 1}\textit{-moves}\}$$

coincides with the set of divisors of n and $n+1$. Moreover, the knot $T(2, 2n+1)$ cannot be unknotted by $(\bar{t}_{2k})^{\pm 1}$-moves, except for $k = 1$.

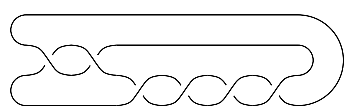

FIGURE 3. Twist knot $K_2 = 6_1$.

Proof. For the first statement, suppose that k is a divisor of n. Then there are n/k consecutive $\bar{t}_{2k}$-moves which transform K_n into the trivial knot. Next, suppose that K_n can be unknotted by a finite sequence of $(\bar{t}_{2k})^{\pm 1}$-moves, for $k \geq 2$. By Fox's congruence statement mentioned in the introduction, the Alexander–Conway polynomial of K_n must be equal to 1 modulo k. A computation reveals $\nabla_{K_n} = 1 - nz^2$, thus k has to be a divisor of n. For the last part of the first item, we compute the HOMFLY polynomial of K_n, via the skein relation:

$$P_{K_{n+1}}(a,z) = a^2 P_{K_n}(a,z) + azP_{H^-}(a,z),$$

for all $n \in \mathbb{N}$, where H^- denotes the Hopf link with two negative crossings. Using $P_{K_0}(a,z) = P_O(a,z) = 1$ and $azP_{H^-}(a,z) = a^{-2} - 1 - z^2$, we find that the terms of lowest and highest a-degree of $P_{K_n}(a,z)$ are a^{-2} and a^{2n}, respectively. As a consequence, for all $k \geq 3$,

$$a\text{-span}(P_{K_n}(a, \zeta_{2k} - \zeta_{2k}^{-1})) = a\text{-span}(P_{K_n}(a,z)),$$

and the reduction $P_{K_n}(a,0) \in \mathbb{F}_2[a^{\pm 1}]$ also shares the same a-span. Now Proposition 2.1 and Proposition 4.1 imply that for all $k \geq 2$, all knots K' related to K_n by a finite sequence of t_{2k}-moves satisfy

$$2b(K') \geq a\text{-span}(P_{K'}(a,z)) + 2 \geq a\text{-span}(P_{K_n}(a,z)) + 2 = 2b(K_n).$$

The last equality is again a consequence of Murasugi's result on two-bridge knots.

For the second statement, suppose that k is a divisor of n or $n+1$. Then a sequence of $(t_{2k})^{-1}$-moves transforms $T(2,2n+1)$ into the trivial knot, in the guise of $T(2,1)$ or $T(2,-1)$, respectively. This always works for $k = 1, 2$. Next, suppose that $T(2,2n+1)$ can be unknotted by a finite sequence of $(t_{2k})^{\pm 1}$-moves, for $k \geq 3$. Then, by Proposition 2.1, the polynomial $P_{T(2,2n+1)}(a, \zeta_{2k} - \zeta_{2k}^{-1})$ must be a power of a^{2k}. An induction shows that $P_{T(2,2n+1)}(a,z)$ takes the form $a^{2n}f(z) + a^{2n+2}g(z)$. We conclude that k divides n or $n+1$. The very last statement is an immediate consequence of Theorem 1.1. □

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