The average genus of a 2-bridge knot is asymptotically linear

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ABSTRACT. Experimental work suggests that the Seifert genus of a knot grows linearly with respect to the crossing number of the knot. In this article, we use a billiard table model for 2-bridge or rational knots to show that the average genus of a 2-bridge knot with crossing number c asymptotically approaches $c/4 + 1/12$.

CONTENTS

1. Introduction

The Seifert genus $g(K)$ of a knot K in S^3 is the minimum genus of any oriented surface embedded in S^3 whose boundary is the knot K. Dunfield et al. [Dun14] presented experimental data that suggests the Seifert genus of a knot grows linearly with respect to crossing number. Using a billiard table model for 2-bridge knots developed by Koseleff and Pecker [KP11b, KP11a], Cohen [Coh23] gave a lower bound on the average genus of a 2-bridge knot.

In this paper, we compute the average genus $\overline{g}_c$ of 2-bridge knots with crossing number c and show that $\overline{g}_c$ is asymptotically linear with respect to c. Define $\mathcal{K}_c$ be the set of unoriented 2-bridge knots with c crossings, where if a 2-bridge knot K with crossing number c is chiral, only one of K and its mirror image $\overline{K}$ is in $\mathcal{K}_c$. For example $|\mathcal{K}_3| = 1$ and contains one of the right- or left-handed trefoil. Define the average genus $\overline{g}_c$ by

$$\overline{g}_c = \frac{\sum_{K \in \mathcal{K}_c} g(K)}{|\mathcal{K}_c|}. \tag{1.1}$$

Since the genus of a knot and the genus of its mirror image are the same, $\overline{g}_c$ is independent of the choice of each knot or its mirror image as elements in $\mathcal{K}_c$.

Theorem 1.1. *Let $c \geq 3$. The average genus $\overline{g}_c$ of a 2-bridge knot with crossing number c is*

$$\overline{g}_c = \frac{c}{4} + \frac{1}{12} + \varepsilon(c),$$

where

$$\varepsilon(c) = \begin{cases} \dfrac{2^{\frac{c-4}{2}} - 4}{12\left(2^{c-3} + 2^{\frac{c-4}{2}}\right)} & \textit{if } c \equiv 0 \bmod 4, \\[2em] \dfrac{1}{3 \cdot 2^{\frac{c-3}{2}}} & \textit{if } c \equiv 1 \bmod 4, \\[2em] \dfrac{2^{\frac{c-4}{2}} + 3c - 11}{12\left(2^{c-3} + 2^{\frac{c-4}{2}} - 1\right)} & \textit{if } c \equiv 2 \bmod 4, \textit{ and} \\[2em] \dfrac{2^{\frac{c+1}{2}} + 11 - 3c}{12\left(2^{c-3} + 2^{\frac{c-3}{2}} + 1\right)} & \textit{if } c \equiv 3 \bmod 4. \end{cases}$$

Since $\varepsilon(c) \to 0$ as $c \to \infty$, the average genus $\overline{g}_c$ approaches $\frac{c}{4} + \frac{1}{12}$ as $c \to \infty$.

Suzuki and Tran [ST24] independently proved this formula for $\overline{g}_c$. Ray and Diao [RD23] expressed $\overline{g}_c$ using sums of products of certain binomial coefficients. Baader, Kjuchukova, Lewark, Misev, and Ray [BKLMR19] previously showed that if c is sufficiently large, then $\frac{c}{4} \leq \overline{g}_c$.

The proof of Theorem 1.1 uses the Chebyshev billiard table model for knot diagrams of Koseleff and Pecker [KP11b, KP11a] as presented by Cohen and Krishnan [CK15] and with Even-Zohar [CEZK18]. This model yields an explicit enumeration of the elements of $\mathcal{K}_c$ as well as an alternating diagram in the format of Figure 2 for each element of $\mathcal{K}_c$. Murasugi [Mur58] and Crowell [Cro59] proved that the genus of an alternating knot is the genus of the surface obtained by applying Seifert's algorithm [Sei35] to an alternating diagram of the knot. The proof of Theorem 1.1 proceeds by applying Seifert's algorithm to the alternating diagrams obtained from our explicit enumeration of $\mathcal{K}_c$ and averaging the genera of those surfaces.

This paper is organized as follows. In Section 2, we recall how the Chebyshev billiard table model for 2-bridge knots diagrams can be used to describe the set $\mathcal{K}_c$ of 2-bridge knots. In Section 3, we find recursive formulas that allow us to count the total number of Seifert circles among all 2-bridge knots with crossing number c. Finally in Section 4, we find a closed formula for the number of Seifert circles among all 2-bridge knots and use that to prove Theorem 1.1.

2. Background

The average genus of 2-bridge knots with crossing number c is the quotient of the sum of the genera of all 2-bridge knots with crossing number c and the number of 2-bridge knots with crossing number c. Ernst and Sumners [ES87] proved formulas for the number $|\mathcal{K}_c|$ of 2-bridge knots.

Theorem 2.1 (Ernst-Sumners [ES87], Theorem 5). *The number $|\mathcal{K}_c|$ of 2-bridge knots with c crossings where chiral pairs are* not *counted separately is given by*

$$
|\mathcal{K}_c| = \begin{cases}
\frac{1}{3}\left(2^{c-3} + 2^{\frac{c-4}{2}}\right) & \text{for } 4 \leq c \equiv 0 \bmod 4, \\[2mm]
\frac{1}{3}\left(2^{c-3} + 2^{\frac{c-3}{2}}\right) & \text{for } 5 \leq c \equiv 1 \bmod 4, \\[2mm]
\frac{1}{3}\left(2^{c-3} + 2^{\frac{c-4}{2}} - 1\right) & \text{for } 6 \leq c \equiv 2 \bmod 4, \text{ and} \\[2mm]
\frac{1}{3}\left(2^{c-3} + 2^{\frac{c-3}{2}} + 1\right) & \text{for } 3 \leq c \equiv 3 \bmod 4.
\end{cases}
$$

A billiard table diagram of a knot is constructed as follows. Let a and b be relatively prime positive integers with $a < b$, and consider an $a \times b$ grid. Draw a sequence of line segments along diagonals of the grid as follows. Start at the bottom left corner of the grid with a line segment that bisects the right angle of the grid. Extend that line segment until it reaches an outer edge of the grid, and then start a new segment that is reflected 90°. Continue in this fashion until a line segment ends in a corner of the grid. Connecting the beginning of the first line segment with the end of the last line segment results in a piecewise linear closed curve in the plane with only double-point self-intersections. If each such double-point self-intersection is replaced by a crossing, then one obtains a *billiard table diagram* of a knot. See Figure 1.

FIGURE 1. A billiard table projection and a billiard table diagram of a knot on a 3×8 grid. The diagram corresponds to the word $+ - + + - - +$. We do not draw the arc connecting the ends but understand it to be present.

Billiard table diagrams on a $3 \times b$ grid have bridge number either one or two, that is, such a knot is either the unknot or a 2-bridge knot. In a $3 \times b$ billiard table diagram, there is one crossing on each vertical grid line except the first and the last. A string of length $b - 1$ in the symbols $\{+, -\}$ determines a 2-bridge knot or the unknot, as follows. A crossing corresponding to a $+$ looks like $\diagdown\!\diagup$, and a crossing corresponding to a $-$ looks like $\diagup\!\diagdown$. Figure 1 shows an example.

A given 2-bridge knot has infinitely many descriptions as strings of various lengths in the symbols $\{+, -\}$. Cohen, Krishnan, and Even-Zohar's work [CK15, CEZK18] lets us describe 2-bridge knots in this manner but with more control on the number of strings representing a given 2-bridge knot.

Definition 2.2. Define the *partially double-counted set $T(c)$ of 2-bridge words with crossing number c* as follows. Each word in $T(c)$ is a word in the symbols $\{+, -\}$. If c is odd, then a word w is in $T(c)$ if and only if it is of the form

$$(+)^{\varepsilon_1}(-)^{\varepsilon_2}(+)^{\varepsilon_3}(-)^{\varepsilon_4} \ldots (-)^{\varepsilon_{c-1}}(+)^{\varepsilon_c},$$

where $\varepsilon_i \in \{1, 2\}$ for $i \in \{1, \ldots, c\}$, $\varepsilon_1 = \varepsilon_c = 1$, and the length of the word $\ell = \sum_{i=1}^{c} \varepsilon_i \equiv 1 \bmod 3$. Similarly, if c is even, then a word w is in $T(c)$ if and only if it is of the form

$$(+)^{\varepsilon_1}(-)^{\varepsilon_2}(+)^{\varepsilon_3}(-)^{\varepsilon_4} \ldots (+)^{\varepsilon_{c-1}}(-)^{\varepsilon_c},$$

where $\varepsilon_i \in \{1, 2\}$ for $i \in \{1, \ldots, c\}$, $\varepsilon_1 = \varepsilon_c = 1$, and the length of the word $\ell = \sum_{i=1}^{c} \varepsilon_i \equiv 1 \bmod 3$.

The set $T(c)$ is described as partially double-counted because every 2-bridge knot is represented by exactly one or two words in $T(c)$, as described in Theorem 2.5 below. Although the billiard table diagram associated with w has ℓ crossings, there is an alternating diagram associated with w that has c crossings, and hence we use the $T(c)$ notation.

The *reverse* rev(w) of a word w of length ℓ is a word whose ith entry is the $(\ell - i + 1)$st entry of w; in other words, rev(w) is just w backwards. The *reverse mirror* $\overline{\text{rev}}(w)$ of a word w of length ℓ is the word of length ℓ where each entry disagrees with the corresponding entry of rev(w); in other words, $\overline{\text{rev}}(w)$ is obtained from w by reversing the order and then changing every $+$ to a $-$ and vice versa.

Definition 2.3. The subset $T_p(c) \subset T(c)$ of *words of palindromic type* consists of words $w \in T(c)$ such that $w = \text{rev}(w)$ when c is odd and $w = \overline{\text{rev}}(w)$ when c is even.

Example 2.4. It can easily be seen that $T(3) = \{+ -\ -\ +\} = T_p(3)$ and $T(4) = \{+ -\ +-\} = T_p(4)$.

The following theorem says exactly which 2-bridge knots are represented by two words in $T(c)$ and which 2-bridge knots are represented by only one word in $T(c)$. The theorem is based on work by Schubert [Sch56] and Koseleff and Pecker [KP11a]. The version of the theorem we state below comes from Lemma 2.1 and Assumption 2.2 in [Coh23].

Theorem 2.5. *Let $c \geq 3$. Every 2-bridge knot is represented by a word in $T(c)$. If a 2-bridge knot K is represented by a word w of palindromic type, that is, a word in $T_p(c)$, then w is the only word in $T(c)$ that represents K. If a 2-bridge knot K is represented by a word w that is not in $T_p(c)$, then there are exactly two words in $T(c)$ that represent K, namely w and rev(w) when c is odd or w and $\overline{\text{rev}}(w)$ when c is even.*

A billiard table diagram associated with a word w in $T(c)$ is not necessarily alternating; however the billiard table diagram associated with w can be transformed into an alternating diagram D of the same knot as follows. A *run* in w is a subword of w consisting of all the same symbols (either all $+$ or all $-$) that is not properly contained in a single-symbol subword of longer length. By construction, if $w \in T(c)$, then it is made up of c runs all of length one or two. The run $+$ is replaced by σ_1, the run $++$ is replaced by σ_2^{-1}, the run $-$ is replaced by σ_2^{-1} and the run $--$ is replaced by σ_1, as summarized by pictures in Table 1.

The left side of the diagram has a strand entering from the bottom left and a cap on the top left. If the last term is σ_1, then the right side of the diagram has a strand exiting to the bottom right and a cap to the top right, and if the last term is σ_2^{-1}, then the right side of the diagram has a strand exiting to the top right and a cap on the bottom right. See Figure 2 for an example. Theorem 2.4 and its proof in [Coh23] explain this correspondence.

Run in billiard table diagram word w	$(+)^1$	$(+)^2$	$(-)^1$	$(-)^2$
Crossing in alternating diagram D	σ_1	σ_2^{-1} $\times$	σ_2^{-1} $\times$	σ_1
	$\times$			$\times$

TABLE 1. Transforming a billiard table diagram into an alternating diagram, as seen in [Coh23, Table 1].
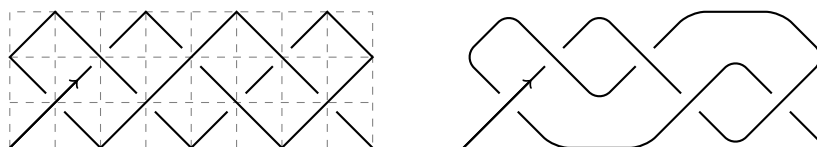
FIGURE 2. The billiard table diagram knot corresponding to the word $+-++-+$ has alternating diagram $\sigma_1 \sigma_2^{-2} \sigma_1^2$.

Murasugi [Mur58] and Crowell [Cro59] proved that the genus of an alternating knot K is the genus of the Seifert surface obtained from Seifert's algorithm on an alternating diagram of K. Therefore, the average genus $\overline{g}_c$ is

$$\overline{g}_c = \frac{1}{2}\left(1 + c - \overline{s}_c\right), \tag{2.1}$$

where $\bar{s}_c$ is the average number of Seifert circles in the alternating diagrams of all 2-bridge knots with crossing number c. In Section 3, we find recursive formulas for the total number of Seifert circles in the alternating diagrams associated with words in $T(c)$ and $T_p(c)$, named $s(c)$ and $s_p(c)$, respectively. Theorem 2.5 implies that

$$\bar{s}_c = \frac{s(c) + s_p(c)}{2|\mathcal{K}_c|}. \tag{2.2}$$

Seifert's algorithm uses the orientation of a knot diagram to construct a Seifert surface. Lemma 3.3 in [Coh23] keeps track of the orientations of the crossings in the alternating diagram D associated with a word w in $T(c)$. See also Property 7.1 in [Coh21a].

Lemma 2.6. [Coh23, Lemma 3.3] *The following conventions determine the orientation of every crossing in the alternating diagram D associated with a word w in $T(c)$.*

(1) *Any vertical line that intersects D away from the crossings intersects the diagram precisely three times. Two of the three strands intersecting any such vertical line are locally oriented to the right.*

(2) *If either a single $+$ or a single $-$ appears in a position congruent to 1 modulo 3 in w, then it corresponds to a single crossing in the alternating diagram D that is horizontally-oriented.*

(3) *If either a double $++$ or a double $--$ appears in two positions congruent to 2 and 3 modulo 3 in w, then they correspond to a single crossing in the alternating diagram D that is horizontally-oriented.*

(4) *The remaining crossings in D are vertically-oriented.*

These conventions are summarized in Table 2.

3. Recursive formulas for Seifert circles

In this section, we recursively compute the number of elements in the general case $T(c)$ and the palindromic case $T_p(c)$ and the number of Seifert circles in the alternating diagrams coming from $T(c)$ and $T_p(c)$. We split the section into two subsections that are organized similarly. In the first, we handle $T(c)$, and in the second, we handle $T_p(c)$.

3.1. General case. The goal of this subsection is to find a recursive formula for the total number $s(c)$ of Seifert circles obtained when Seifert's algorithm is applied to the alternating diagrams associated to words in $T(c)$. Our method is to modify the last three crossings of the alternating diagrams associated with a word w in $T(c)$ to obtain an alternating diagram for a new word $\text{rep}(w)$, called the replacement word of w.

In order to relate the number of Seifert circles before and after the replacement, we want the modifications of the last three crossings to preserve the orientations of the portion of the diagram that remains. Furthermore, we want the

Run in $T(c)$	+	+	−	−
Position mod 3	1	2 or 3	1	2 or 3
Possible orientation(s)	*(crossing diagram)*	*(crossing diagram)* or *(crossing diagram)*	*(crossing diagram)*	*(crossing diagram)* or *(crossing diagram)*

Run in $T(c)$	++	++	−−	−−
Positions mod 3	2-3	3-1 or 1-2	2-3	3-1 or 1-2
Possible orientation(s)	*(crossing diagram)*	*(crossing diagram)* or *(crossing diagram)*	*(crossing diagram)*	*(crossing diagram)* or *(crossing diagram)*

TABLE 2. Orientations of crossings in the alternating diagram associated with a word in $T(c)$, as determined by Lemma 2.6.

modification to result in a diagram with fewer crossings so that we can recursively count the number of knots and the number of Seifert circles. Additionally we need to preserve the length condition modulo 3 so that we obtain words in $T(c')$ for $c' < c$.

We motivate our choices of replacements by showing examples with up to six crossings. Our convention is to enclose the final three runs that are being replaced and the subword replacing those final three runs with parentheses. The subwords before the parentheses are unchanged; these further instruct our decisions for replacements.

Define $t(c)$ to be the number of elements in the set $T(c)$.

Example 3.1. We replace the final three runs of each word in $T(5)$ to obtain a word in $T(3)$ or $T(4)$, as depicted in Figure 3. The words $+ - -(+ - -+)$ and $+ - -(+ + -+)$ in $T(5)$ are both replaced with $+ - -(+)$ in $T(3)$, nominating the (perhaps obvious) replacement that deletes a σ_1^2 or a σ_2^{-2}. The word $+ - (+ + - - +)$ in $T(5)$ is replaced with $+ - (+-)$ in $T(4)$, nominating the less-obvious replacement of $\sigma_2^{-1}\sigma_1^2$ by $\sigma_1\sigma_2^{-1}$.

In Figure 3, the portion of the diagram to the right of the blue segment is replaced. The portion being replaced consists of two strands: one that starts and ends on the blue segment and one that starts on the blue segment and ends to the right, either at the bottom or top. The replacement preserves the starting points and orientations of the two different strands being replaced.

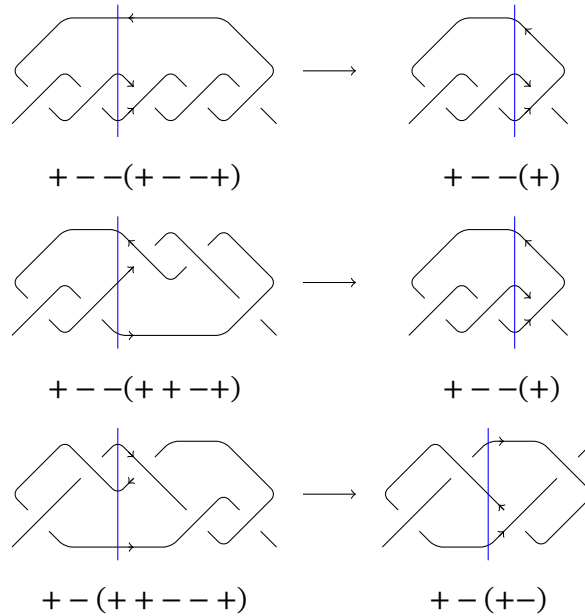This example implies that $t(5) = t(4) + 2t(3)$.

$$+ - -(+ - -+) \qquad\qquad + - -(+)$$

$$+ - -(+ + -+) \qquad\qquad + - -(+)$$

$$+ - (+ + - - +) \qquad\qquad + - (+-)$$

FIGURE 3. Replacements of the three final runs of words in $T(5)$ result in words in $T(3)$ or $T(4)$, as in Example 3.1. Crossings to the left of the blue segment are unchanged while crossings to the right of the vertical segment are modified.

In the next example, since $c = 6$ is even, the final crossing is determined by a single $-$ instead of a single $+$, switching all of the pluses and minuses from the previous example.

Example 3.2. We replace the final three runs of each word in $T(6)$ to obtain a word in $T(4)$ or $T(5)$, as depicted in Figure 4.

In the first three words we still have the obvious replacements $(- + +-)$ by $(-)$ and $(- - +-)$ by $(-)$ like before, as well as the less-obvious replacement $(- - + + -)$ by $(-+)$. Finally the word $+ - + + (- + -)$ in $T(6)$ is replaced by the word $+ - + + (- - +)$ in $T(5)$, and the word $+ - - + (- + -)$ in $T(6)$ is replaced by the word $+ - - + (- - +)$ in $T(5)$, nominating the less-obvious replacement of $\sigma_2^{-1} \sigma_1 \sigma_2^{-1}$ by σ_1^2.

This example implies that $t(6) = t(5) + 2t(4)$.

We now provide proof of concept of these replacements by showing they continue to work in the next example.

Example 3.3. We replace the final three runs of each word in $T(7)$ to obtain a word in $T(5)$ or $T(6)$, as depicted in Table 3.

In what is becoming a pattern, we see that $t(7) = t(6) + 2t(5)$.

$$+ - +(- + + -) \qquad\qquad + - (+-)$$

$$+ - +(- - + -) \qquad\qquad + - (+-)$$

$$+ - - + +(- - + + -) \qquad\qquad + - - + +(-+)$$

$$+ - + + (- + -) \qquad\qquad + - + + (- - +)$$

$$+ - - + (- + -) \qquad\qquad + - - + (- - +)$$
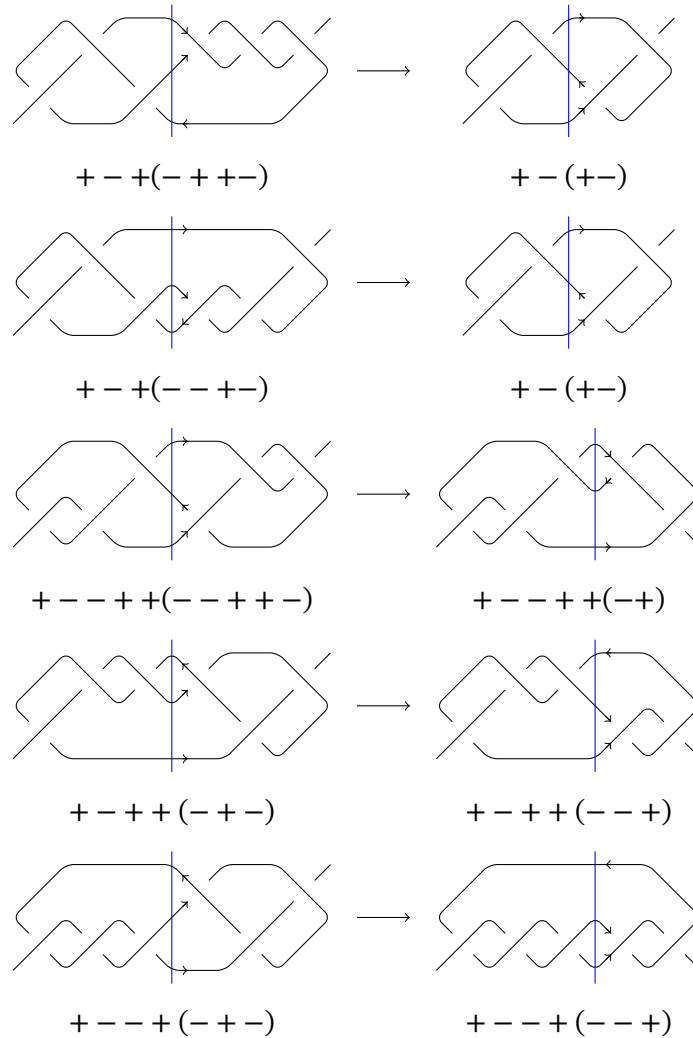
FIGURE 4. Replacements of the three final runs of words in $T(6)$ result in words in $T(4)$ or $T(5)$, as in Example 3.2. Crossings to the left of the blue segment are unchanged while crossings to the right of the vertical segment are modified.

Examples 3.1, 3.2, and 3.3 suggest a general definition for how we replace the final strings of a word in $T(c)$ to obtain words for fewer crossings. Definition 3.4 below formalizes this construction, while Table 4 shows how the replacement function changes the final few crossings of a diagram.

Word w in $T(7)$	Replacement word rep(w)	
$+--++-(+--+)$	$+--++-(+)$	
$+-++--(+--+)$	$+-++--(+)$	$T(5)$
$+--+--(+--+)$	$+--+--(+)$	
$+--++-(++-+)$	$+--++-(+)$	
$+-++--(++-+)$	$+-++--(+)$	$T(5)$
$+--+--(++-+)$	$+--+--(+)$	
$+-+--(++--+)$	$+-+--(+-)$	
$+-++-(++--+)$	$+-++-(+-)$	
$+--+-(++--+)$	$+--+-(+-)$	$T(6)$
$+-+-(+-+)$	$+-+-(++-)$	
$+--++--(+-+)$	$+--++--(++-)$	

TABLE 3. Replacements of the three final runs of words in $T(7)$ result in words in $T(5)$ or $T(6)$, as in Example 3.3.

Partition $T(c)$ into four subsets, as follows. The final run of each of word w in $T(c)$ is fixed by construction; if c is odd, then w ends in a single $+$, and if c is even, then w ends in a single $-$.

If c is odd, then the final three runs in a word w in $T(c)$ must be exactly one of the following cases.

(1) The final three runs of w are $+-+$.
(2) The final three runs of w are $++--+$.
(3) The final three runs of w are $+--+$.
(4) The final three runs of w are $++-+$.

If c is even, then interchange $+$ and $-$ to define analogous cases (1) - (4). For each c, the four cases partition $T(c)$. Recall that $t(c)$ is the number of words in $T(c)$, and for each $1 \leq i \leq 4$, define $t_i(c)$ to be the number of words in case i of $T(c)$.

Definition 3.4. Define the *replacement function* rep : $T(c) \rightarrow T(c-1) \cup T(c-2)$ as follows. Suppose c is odd, and let $w \in T(c)$.

(1) If the final three runs of w are $(+-+)$, then rep(w) is the word obtained by replacing the final three runs of w with $(++-)$. Thus rep(w) $\in T(c-1)$.
(2) If the final three runs of w are $(++-+)$, then rep(w) is the word obtained by replacing the final three runs of w with $(+-)$. Thus rep(w) $\in T(c-1)$.
(3) If the final three runs of w are $(+--+)$, then rep(w) is the word obtained by replacing the final three runs of w with $(+)$. Thus rep(w) $\in T(c-2)$.
(4) If the final three runs of w are $(++-+)$, then rep(w) is the word obtained by replacing the final three runs of w with $(+)$. Thus rep(w) $\in T(c-2)$.

If c is even, then we define $\text{rep}(w)$ in the same way as above where $+$ and $-$ are interchanged.

In each of the cases above, the length of w and $\text{rep}(w)$ are congruent modulo 3.
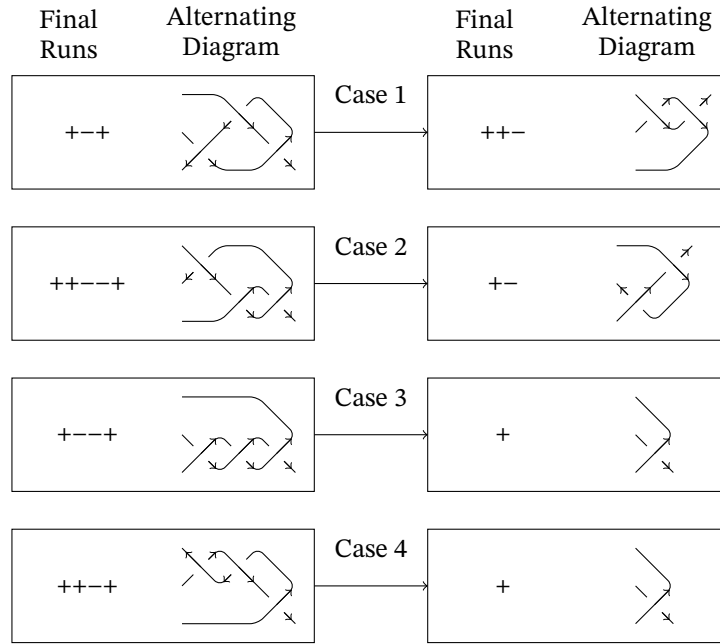

TABLE 4. Changes in alternating diagrams induced by the replacement function for each case (when c is odd).

Recall the final conclusions in Examples 3.1, 3.2, and 3.3 above regarding $t(5)$, $t(6)$, and $t(7)$. In the next lemma, we use the replacement function to establish bijections between subsets of $T(c)$ and subsets of $T(c-1)$ and $T(c-2)$.

Lemma 3.5. *The restriction of the replacement function to each of the four cases is a bijection onto its image. Therefore*

$$t_1(c) = t_2(c-1) + t_3(c-1), \quad t_2(c) = t_1(c-1) + t_4(c-1),$$
$$t_3(c) = t(c-2), \text{ and} \qquad t_4(c) = t(c-2).$$

Proof. In each of the four cases, the replacement function replaces the final three runs of a word w in $T(c)$ with some other word w' in $\{+, -\}$ resulting in the word $\text{rep}(w)$. This inverse of this operation replaces the word w' with the final three runs in w. Since the restriction of rep to each case is invertible, it is a bijection onto its image.

Suppose c is odd. If $w \in T(c)$ is in case 1, then $\text{rep}(w) \in T(c-1)$ has final two runs $++-$, and hence belongs to cases 2 or 3. Thus $t_1(c) = t_2(c-1) + t_3(c-1)$.

If $w \in T(c)$ is in case 2, then $\text{rep}(w) \in T(c-1)$ has final two runs $+-$, and hence belongs to cases 1 or 4. Thus $t_2(c) = t_1(c-1) + t_4(c-1)$. If $w \in T(c)$ is in cases 3 or 4, then $\text{rep}(w) \in T(c-2)$ has final run $+$, and hence can belong to any case. Thus $t_3(c) = t_4(c) = t(c-2)$.

If c is even, the above argument can be repeated by interchanging $+$ and $-$.
$\square$

Examples 3.1, 3.2, and 3.3 show that when $c = 5$, 6, or 7, the relation $t(c) = t(c-1) + 2t(c-2)$ holds. The Jacobsthal sequence A001045 [SEI22b] is an integer sequence satisfying the same recurrence relation $J(n) = J(n-1) + 2J(n-2)$ with initial values $J(0) = 0$ and $J(1) = 1$. The closed formula for the nth Jacobsthal number is $J(n) = \frac{2^n - (-1)^n}{3}$. We use the Jacobsthal sequence to find a formula for the number of words in $T(c)$.

Proposition 3.6. *The number* $t(c) = \frac{2^{c-2} - (-1)^c}{3}$ *is the Jacobsthal number* $J(c-2)$ *and satisfies the recursive formula* $t(c) = t(c-1) + 2t(c-2)$ *when* $c \geq 5$.

Proof. The base cases of $t(3) = t(4) = 1$ hold by on Example 2.4.

Since the four cases partition $T(c)$, it follows that $t(c) = \sum_{i=1}^{4} t_i(c)$. Lemma 3.5 implies that

$$
\begin{aligned}
t(c) &= \sum_{i=1}^{4} t_i(c) \\
&= t_2(c-1) + t_3(c-1) + t_1(c-1) + t_4(c-1) + 2t(c-2) \\
&= t(c-1) + 2t(c-2).
\end{aligned}
$$

Since $t(c)$ satisfies the Jacobsthal recurrence relation and $t(3) = t(4) = J(1) = J(2) = 1$, it follows that $t(c) = J(c-2) = \frac{2^{c-2} - (-1)^c}{3}$.
$\square$

We find additional expressions for two of the quantities in Lemma 3.5. These expressions will be used in the proof of Theorem 3.8 below where we find a recursive formula for the the total number of Seifert circles.

Lemma 3.7. *For each* $c \geq 6$,

$$
t_1(c) = 2t(c-3) \text{ and } t_2(c) = t(c-2).
$$

Proof. We prove both equalities simultaneously by induction on c. For $c = 6$, the relevant sets are $T_1(6) = \{+--++-+-, +--+-+-\}$ and $T_2(6) = \{+-+--+-\}$. By Example 2.4, we have that $t_1(6) = 2t(3)$ and $t_2(6) = t(4)$.

For the inductive step, we assume that $t_1(c') = 2t(c'-3)$ and $t_2(c') = t(c'-2)$ for all $c' < c$. Lemma 3.5 implies that

$$
t_1(c) = t_2(c-1) + t_3(c-1) = t_2(c-1) + t(c-3).
$$

By the inductive hypothesis, $t_2(c-1) = t(c-3)$, and thus $t_1(c) = 2t(c-3)$. Lemma 3.5 also implies that

$$
t_2(c) = t_1(c-1) + t_4(c-1) = t_1(c-1) + t(c-3).
$$

By the inductive hypothesis, $t_1(c-1) = 2t(c-4)$. Therefore, Proposition 3.6 implies

$$t_2(c) = 2t(c-4) + t(c-3) = t(c-2),$$

completing the proof. □

Recall that $s(c)$ is the total number of Seifert circles coming from $T(c)$. We are nearly ready to prove our recursive formula for $s(c)$. Throughout the proof, we refer to Table 5 below.
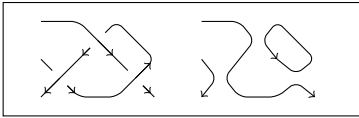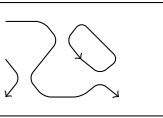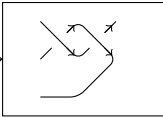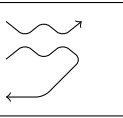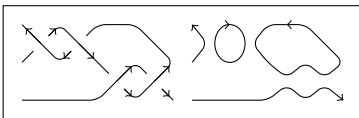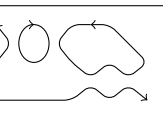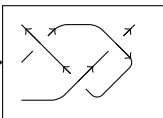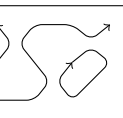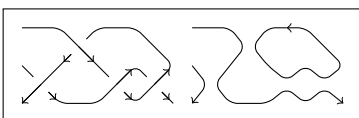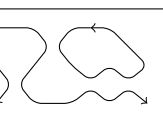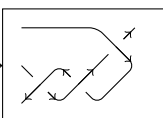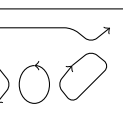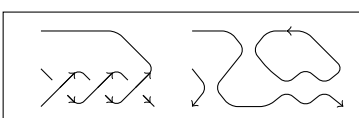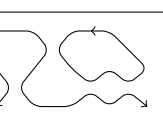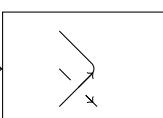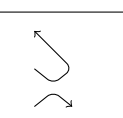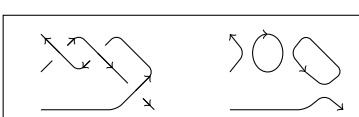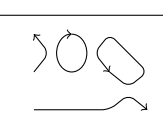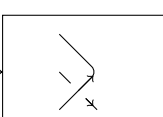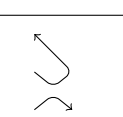
Alternating Diagram	Seifert State		Alternating Diagram	Seifert State
		Case 1 $\Delta s = 1$		
		Case 2A $\Delta s = 1$		
		Case 2B $\Delta s = -1$		
		Case 3 $\Delta s = 1$		
		Case 4 $\Delta s = 2$		

TABLE 5. The pieces of alternating diagrams and Seifert states corresponding to the cases in the proof of Theorem 3.8. The quantity Δs records the difference between the number of Seifert circles of w and $\text{rep}(w)$.

If a word w is in case 2, then the change in the number of Seifert circles between the alternating diagram associated with w and $\text{rep}(w)$ depends on the $(c-3)$rd run of w. If c is odd, we define subcases 2A and 2B depending on the $(c-3)$rd run of w as follows.

(2A) The final four runs of w are $- + + - - +$.
(2B) The final four runs of w are $- - + + - - +$.

If c is even, then interchange $+$ and $-$ to obtain analogous subcases 2A and 2B.

By the same arguments given in the proof of Lemma 3.5, the restriction of the replacement function to each of these two cases is a bijection onto its image. Furthermore, by the definition of these cases, we have

$$t_{2A}(c) = t_1(c-1) \text{ and } t_{2A}(c) = t_4(c-1). \tag{3.1}$$

Let $s_{2A}(c)$ and $s_{2B}(c)$ be the number of Seifert circles coming from cases 2A and 2B, respectively.

Theorem 3.8. *Let $s(c)$ be the total number of Seifert circles obtained when Seifert's algorithm is applied to the alternating 2-bridge diagrams associated with words in $T(c)$. Then $s(c)$ satisfies the recursion*

$$s(c) = s(c-1) + 2s(c-2) + 3t(c-2).$$

Proof. Following the ideas from earlier in this section, we consider the contributions to $s(c)$ from each of the (sub)cases, calling these $s_1(c)$, $s_{2A}(c)$, $s_{2B}(c)$ $s_3(c)$, and $s_4(c)$ so that $s(c) = s_1(c) + s_{2A}(c) + s_{2B}(c) + s_3(c) + s_4(c)$. Refer to Table 5 for pictures of each of the cases, where the orientations of the crossings are determined by Lemma 2.6.

For each word $w \in T(c)$, we compare the number Seifert circles in the alternating diagrams associated with w and the replacement word $\mathrm{rep}(w)$. This comparison can be made by completing the partial Seifert states in Table 5 and is recorded by Δs in Table 5.

In each case, the total number of Seifert circles before replacement is the total number of Seifert circles after replacement plus the product of the change in the number of Seifert cirlces Δs and the number of words in this case. Lemma 3.5 and Equation (3.1) allow us to count the number of Seifert circles before and after replacement, and Table 5 records Δs in each case, implying the following:

$$s_1(c) = s_2(c-1) + s_3(c-1) + t_1(c),$$
$$s_{2A}(c) = s_1(c-1) + t_{2A}(c),$$
$$s_{2B}(c) = s_4(c-1) - t_{2B}(c),$$
$$s_3(c) = s(c-2), \text{ and}$$
$$s_4(c) = s(c-2) + 2t_4(c).$$

Thus Lemma 3.7 implies that

$$
\begin{aligned}
s(c) &= s_1(c) + s_{2A}(c) + s_{2B}(c) + s_3(c) + s_4(c) \\
&= [s_2(c-1) + s_3(c-1) + 2t(c-3)] + [s_1(c-1) + 2t(c-4)] \\
&\quad + [s_4(c-1) - t(c-3)] + [s(c-2)] + [s(c-2) + 2t(c-2)] \\
&= \sum_{i=1}^{4} s_i(c-1) + 2s(c-2) + 2t(c-2) + [t(c-3) + 2t(c-4)] \\
&= s(c-1) + 2s(c-2) + 3t(c-2),
\end{aligned}
$$

completing the proof.   □

3.2. Palindromic case. The goal of this subsection is to find a recursive formula for the total number $s_p(c)$ of Seifert circles obtained when Seifert's algorithm is applied to the alternating diagrams associated to the words in $T_p(c)$. Our process is similar to the general case in the previous subsection. We replace the middle few runs of a word $w \in T_p(c)$ to obtain the palindromic replacement word prep(w) in $T_p(c-2)$ or $T_p(c-4)$. We start with concrete examples for $c = 7, 8, 9$, and 10 motivating our choice of replacements. Then we define our replacements for palindromes and use palindromic replacement to find the recursive formula for $s_p(c)$.

We continue our convention of enclosing the subword being replaced and its replacement subword in parenthesis. Each replacement below preserves the length of the word modulo 3 and the orientations of the crossings not being replaced induced by Lemma 2.6. As these examples will show, the replacement rules for palindromic words look quite different for even and odd crossing numbers.

Example 3.9. We replace the middle runs of each word in $T_p(7)$ to obtain a word in $T_p(3)$ or $T_p(5)$, as shown in Table 6. The words $+(-+-+-)+$ and $+(-++--++-)+$ are both replaced with $+(--)+$ in $T_p(3)$, and the word $+--(+--+)--+$ is replaced by $+--(+)--+$ in $T_p(5)$. This example implies that $t_p(7) = t_p(5) + 2t_p(3)$.

Word w in $T_p(7)$	Replacement word prep(w)	
$+(-+-+-)+$	$+(--)+$	$T_p(3)$
$+(-++--++-)+$	$+(--)+$	$T_p(3)$
$+--(+--+)--+$	$+--(+)--+$	$T_p(5)$

TABLE 6. Replacements of the middle runs of words in $T_p(7)$ result in words in $T_p(3)$ or $T_p(5)$, as in Example 3.9.

Example 3.10. We replace the middle runs of each word in $T_p(8)$ to obtain a word in $T_p(4)$ or $T_p(6)$, as shown in Table 7. The words $+(-+--++-+)-$ and $+(-++-+--+)-$ are both replaced with $+(-+)-$ in $T_p(4)$, and the word $+--(+-+-)++-$ is replaced with $+--(++--)++-$ in $T_p(6)$. This example implies that $t_p(8) = t_p(6) + 2t_p(4)$.

Word w in $T_p(8)$	Replacement word prep(w)	
$+(-+--++-+)-$	$+(-+)-$	$T_p(4)$
$+(-++-+--+)-$	$+(-+)-$	$T_p(4)$
$+--(+-+-)++-$	$+--(++--)++-$	$T_p(6)$

TABLE 7. Replacements of the middle runs of words in $T_p(8)$ result in words in $T_p(4)$ or $T_p(6)$, as in Example 3.10.

Example 3.11. We replace the middle runs of each word in $T_p(9)$ to obtain a word in $T_p(5)$ or $T_p(7)$, as shown in Table 8. This example implies that $t_p(9) = t_p(7) + 2t_p(5)$.

Word w in $T_p(9)$	Replacement word prep(w)	
$+--(++--++--++)--+$	$+--(+)--+$	$T_p(5)$
$+--(++-+-++)--+$	$+--(+)--+$	$T_p(5)$
$+-+(-++-)+-+$	$+-+(-)+-+$	
$+-++(--+--)++-+$	$+-++(--)++-+$	$T_p(7)$
$+--+(--+--)+--+$	$+--+(--)+--+$	

TABLE 8. Replacements of the middle runs of words in $T_p(9)$ result in words in $T_p(5)$ or $T_p(7)$, as in Example 3.11.

Example 3.12. We replace the middle runs of each word in $T_p(10)$ to obtain a word in $T_p(6)$ or $T_p(8)$, as shown in Table 9. This example implies that $t_p(10) = t_p(8) + 2t_p(6)$.

Word w in $T_p(10)$	Replacement word prep(w)	
$+--(++-++--+--)++-$	$+--(++--)++-$	$T_p(6)$
$+--(++--+-++--)++-$	$+--(++--)++-$	$T_p(6)$
$+-+(-+-+)-+-$	$+-+(--++)-+-$	
$+-++(--++--++)--+-$	$+-++(-+)---+-$	$T_p(8)$
$+--+(--++--++)-++-$	$+--+(-+)-++-$	

TABLE 9. Replacements of the middle runs of words in $T_p(10)$ result in words in $T_p(6)$ or $T_p(8)$, as in Example 3.12.

In the previous subsection, we partitioned the set $T(c)$ into four cases in order to find a recursive formula for $t(c)$ in Proposition 3.6. We further partitioned into subcases in order to describe the change in the number of Seifert circles when performing replacements in Theorem 3.8. For the current palindromic case, we define all the cases and subcases at once. We keep the language of subcases in analogy to the general case and because the behavior of the subcases are analogous when c is odd or even.

When $c = 2i$ is even, partition $T_p(c)$ into six subsets, as follows. Since c is even, a word w is in $T_p(c)$ if $w = \overline{\text{rev}}(w)$. Therefore the $(i + 1 - k)$th and $(i + k)$th runs must have the same length but be opposite symbols for each k. If $w \in T_p(c)$ and $c \equiv 0 \bmod 4$, the middle four or six runs of w, that is, the $(i - 1)$st through $(i + 2)$nd runs or the $(i - 2)$nd through $(i + 3)$rd runs, respectively, of w are exactly one of the following six cases.

(1_{pe}) The middle four runs of w are $+-+-$.
(2_{pe}) The middle four runs of w are $++--++--$.
($3A_{pe}$) The middle six runs of w are $-+--++-+$.
($3B_{pe}$) The middle six runs of w are $--+--++-++$.
($4A_{pe}$) The middle six runs of w are $-++-+--+$.
($4B_{pe}$) The middle six runs of w are $--++-+--++$.

If $c \equiv 2 \bmod 4$, then interchange $+$ and $-$ to define analogous cases 1_{pe} through $4B_{pe}$. We say that a word is in case 3_{pe} to mean it is in case $3A_{pe}$ or $3B_{pe}$ and similarly for case 4_{pe}.

When $c = 2i + 1$ is odd, partition $T_p(c)$ into six subsets, as follows. Since c is odd, a word w is in $T_p(c)$ if it is a palindrome, that is, if $w = \mathrm{rev}(w)$. Therefore, for each positive k, the $(i - k)$th run and the $(i + k)$th run of w are the same. We partition $T_p(c)$ according to the middle three or middle five runs of a word w, that is, according to the $(i - 1)$st through the $(i + 1)$st runs or the $(i - 2)$nd through $(i + 2)$nd runs respectively. If $w \in T_p(c)$ and $c \equiv 1 \bmod 4$, then middle three or five runs of w are exactly one of the following six cases.

(1_{po}) The middle three runs of w are $-+ +-$.
(2_{po}) The middle three runs of w are $--+--$.
($3A_{po}$) The middle five runs of w are $+-+-+$.
($3B_{po}$) The middle five runs of w are $++-+-++$.
($4A_{po}$) The middle five runs of w are $+--++--+$.
($4B_{po}$) The middle five runs of w are $++--++--++$.

If $c \equiv 3 \bmod 4$, then interchange $+$ and $-$ to define analogous cases 1_{po} through $4B_{po}$. We say that a word is in case 3_{po} to mean it is in case $3A_{po}$ or $3B_{po}$ and similarly for case 4_{po}.

Definition 3.13. As in Definition 3.4, we define the *palindromic replacement function* prep : $T_p(c) \to T_p(c - 2) \cup T_p(c - 4)$ as follows. Suppose $c \equiv 0 \bmod 4$ and $w \in T_p(c)$.

(1_{pe}) If the middle four runs of w are $(+ - +-)$, then prep(w) is the word obtained by replacing the middle four runs of w with $(+ + --)$. Thus prep(w) $\in T_p(c - 2)$.
(2_{pe}) If the middle four runs of w are $(+ + - - + + --)$, then prep(w) is the word obtained by replacing the middle four runs of w with $(+-)$. Thus prep(w) $\in T_p(c - 2)$.
($3A_{pe}$) If the middle six runs of w are $(- + - - + + -+)$, then prep(w) is the word obtained by replacing the middle six runs of w with $(-+)$. Thus prep(w) $\in T_p(c - 4)$.
($3B_{pe}$) If the middle six runs of w are $(- - + - - + + - ++)$, then prep(w) is the word obtained by replacing the middle six runs of w with $(- - ++)$. Thus prep(w) $\in T_p(c - 4)$.
($4A_{pe}$) If the middle six runs of w are $(- + + - + - -+)$, then prep(w) is the word obtained by replacing the middle six runs of w with $(-+)$. Thus prep(w) $\in T_p(c - 4)$.

($4B_{pe}$) If the middle six runs of w are $(- - + + - + - - + +)$, then prep(w) is
the word obtained by replacing the middle six runs of w with $(- - + +)$.
Thus prep(w) $\in T_p(c - 4)$.

If $c \equiv 2 \bmod 4$, then we define prep(w) in the same way as above, but where $+$
and $-$ are interchanged.

Suppose $c \equiv 1 \bmod 4$ and $w \in T_p(c)$.

(1_{po}) If the middle three runs of w are $(- + + -)$, then prep(w) is the word ob-
tained by replacing the middle three runs of w with $(-)$. Thus prep(w) $\in$
$T_p(c - 2)$.

(2_{po}) If the middle three runs of w are $(- - + - -)$, then prep(w) is the
word obtained by replacing the middle three runs of w with $(- -)$. Thus
prep(w) $\in T_p(c - 2)$.

($3A_{po}$) If the middle five runs of w are $(+ - + - +)$, then prep(w) is the word ob-
tained by replacing the middle five runs of w with $(+ +)$. Thus prep(w) $\in$
$T_p(c - 4)$.

($3B_{po}$) If the middle five runs of w are $(+ + - + - + +)$, then prep(w) is the
word obtained by replacing the middle five runs of w with $(+)$. Thus
prep(w) $\in T_p(c - 4)$.

($4A_{po}$) If the middle five runs of w are $(+ - - + + - - +)$, then prep(w) is the
word obtained by replacing the middle five runs of w by $(+ +)$. Thus
prep(w) $\in T_p(c - 4)$.

($4B_{po}$) If the middle five runs of w are $(+ + - - + + - - + +)$, then prep(w) is
the word obtained by replacing the middle five runs of w by $(+)$. Thus
prep(w) $\in T_p(c - 4)$.

If $c \equiv 3 \bmod 4$, then we define prep(w) in the same way as above, but where $+$
and $-$ are interchanged.

See Tables 10 and 11 for figures depicting how palindromic replacement
changes the alternating diagram and the Seifert states associated to a palin-
dromic word.

Recall that $t_p(c)$ is the number of elements in $T_p(c)$, and define $t_{pj}(c)$ to be
the number of words in case j_{pe} if c is even and the number of words in case
j_{po} if c is odd where $j \in \{1, 2, 3A, 3B, 4A, 4B\}$.

Lemma 3.14. *The restriction of the palindromic replacement function to each
case is a bijection onto its image. Moreover, if $c \geq 7$, then*

$$t_{p1}(c) = t_{p2}(c - 2) + t_{p3}(c - 2), \qquad t_{p2}(c) = t_{p1}(c - 2) + t_{p4}(c - 2),$$
$$t_{p3A}(c) = t_{p1}(c - 4) + t_{p4}(c - 4), \qquad t_{p3B}(c) = t_{p2}(c - 4) + t_{p3}(c - 4),$$
$$t_{p4A}(c) = t_{p1}(c - 4) + t_{p4}(c - 4), \text{ and } \quad t_{p4B}(c) = t_{p2}(c - 4) + t_{p3}(c - 4).$$

Proof. The palindromic replacement function replaces the middle few runs of
a word w with another word w' in $\{+, -\}$, resulting in the word prep(w). The

TABLE 10. Alternating diagrams and Seifert states correspond-
ing to the $c \equiv 0 \bmod 4$ case in the proof of Theorem 3.16.
The quantity Δs records the difference between the number of
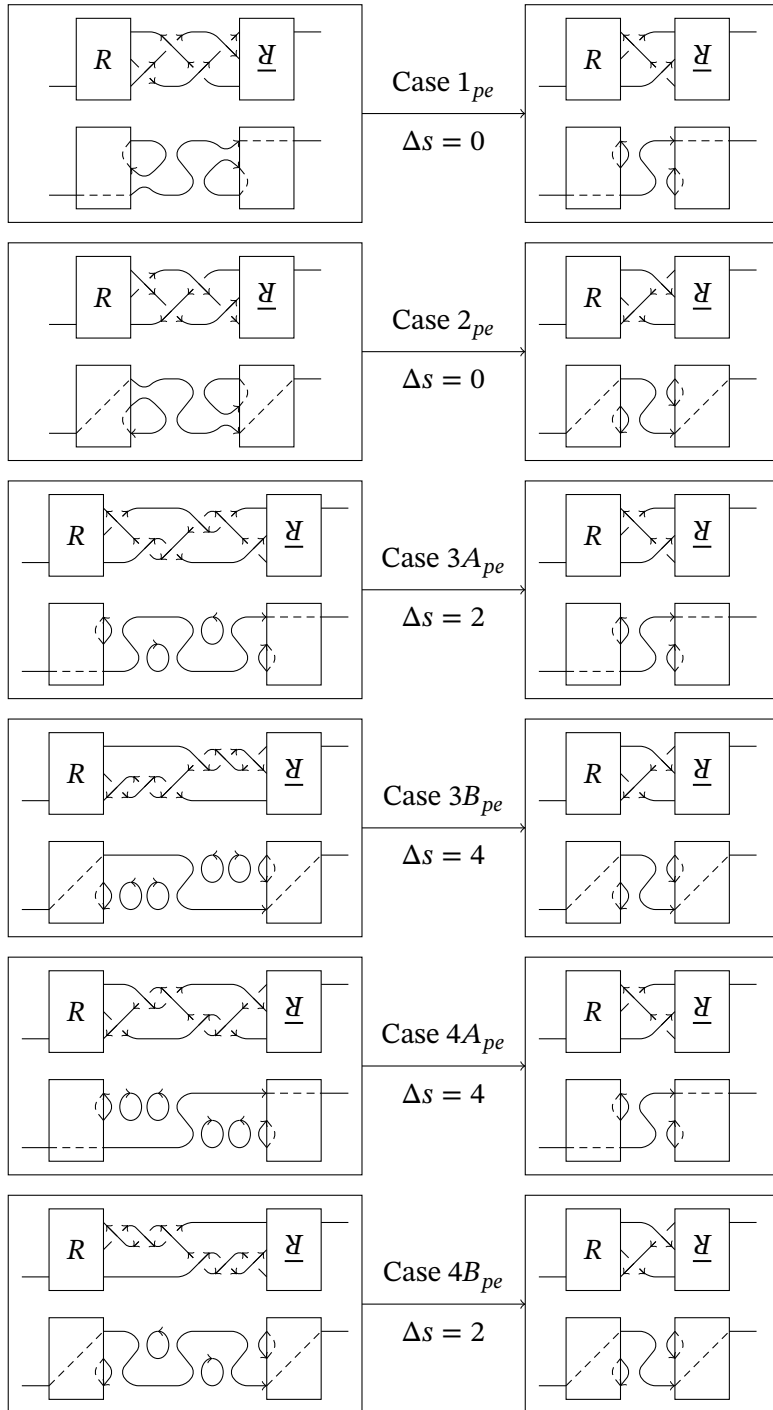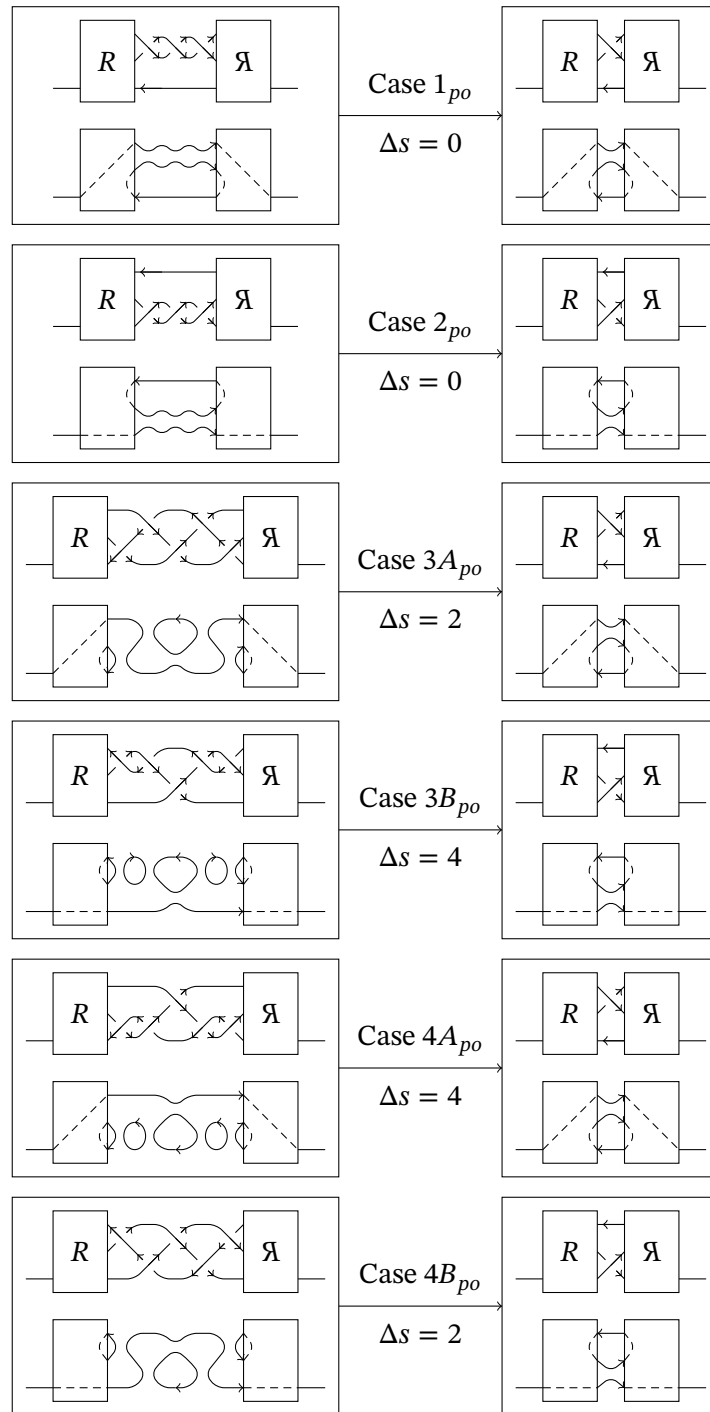Seifert circles of w and $\mathrm{prep}(w)$.

TABLE 11. Alternating diagrams and Seifert states corresponding to the $c \equiv 1 \mod 4$ case in the proof of Theorem 3.16. The quantity Δs records the difference between the number of Seifert circles of w and $\mathrm{prep}(w)$.

inverse operation replaces w' with the appropriate middle few runs of w. Since the restriction of prep to each case is invertible, it is a bijection onto its image.

Suppose $c \equiv 0 \mod 4$. If $w \in T_p(c)$ is in case 1_{pe}, then $\text{prep}(w) \in T_p(c - 2)$ has middle two runs $+ + --$, and hence belongs to case 2_{pe} or 3_{pe}. Thus $t_{p1}(c) = t_{p2}(c - 2) + t_{p3}(c - 2)$. If $w \in T_p(c)$ is in case 2_{pe}, then $\text{prep}(w) \in T_p(c - 2)$ has middle two runs $+-$, and hence belongs to case 1_{pe} or 4_{pe}. Thus $t_{p2}(c) = t_{p1}(c - 2) + t_{p4}(c - 2)$. If $w \in T_p(c)$ is in case $3A_{pe}$ or $4A_{pe}$, then $\text{prep}(w) \in T_p(c - 4)$ has middle two runs $-+$, and hence belongs to case 1_{pe} or 4_{pe}. Thus $t_{p3A}(c) = t_{p4A}(c) = t_{p1}(c - 4) + t_{p4}(c - 4)$. If $w \in T_p(c)$ is in case $3B_{pe}$ or $4B_{pe}$, then $\text{prep}(w) \in T_p(c - 4)$ has middle two runs $-- + +$, and hence belongs to case 2_{pe} or 3_{pe}. Thus $t_{p3B}(c) = t_{p4B}(c) = t_{p2}(c - 4) + t_{p4}(c - 4)$. If $c \equiv 2 \mod 4$, the argument in this paragraph can be repeated with $+$ and $-$ interchanged.

Suppose $c \equiv 1 \mod 4$. If $w \in T_p(c)$ is in case 1_{po}, then $\text{prep}(w)$ in $T_p(c - 2)$ has middle run $-$, and hence belongs to case 2_{po}, or 3_{po}. Thus $t_{p1}(c) = t_{p2}(c - 2) + t_{p3}(c - 2)$. If $w \in T_p(c)$ is in case 2_{po}, then $\text{prep}(w) \in T_p(c - 2)$ has middle run $--$, and hence belongs to case 1_{po} or 4_{po}. Thus $t_{p2}(c) = t_{p1}(c - 2) + t_{p4}(c - 2)$. If $w \in T_p(c)$ is in case $3A_{po}$ or $4A_{po}$, then $\text{prep}(w) \in T_p(c - 4)$ has middle run $++$, and hence belongs to case 1_{po} or 4_{po}. Thus $t_{p3A}(c) = t_{p4A}(c) = t_{p1}(c - 4) + t_{p4}(c - 4)$. If $w \in T_p(c)$ is in case $3B_{po}$ or $4B_{po}$, then $\text{prep}(2) \in T_p(c - 4)$ has middle run $+$, and hence belongs to case 2_{po} or 3_{po}. Thus $t_{p3B}(c) = t_{p4B}(c) = t_{p2}(c - 4) + t_{p3}(c - 4)$. If $c \equiv 3 \mod 4$, the argument in this paragraph can be repeated with $+$ and $-$ interchanged. □

Lemma 3.14 implies the following proposition giving recursive and closed formulas for the number $t_p(c)$ of words in $T_p(c)$. When restricting parity, the recursive formula for $t_p(c)$ follows a similar pattern as the recursion $t(c) = t(c - 1) + 2t(c - 2)$.

Proposition 3.15. *If $c \geq 7$, the number $t_p(c)$ of words of palindromic type in $T_p(c)$ satisfies the recursion $t_p(c) = t_p(c - 2) + 2t_p(c - 4)$. Moreover,*

$$t_p(c) = \begin{cases} J\left(\frac{c-2}{2}\right) = \frac{2^{(c-2)/2} - (-1)^{(c-2)/2}}{3} & \textit{if } c \textit{ is even and} \\ J\left(\frac{c-1}{2}\right) = \frac{2^{(c-1)/2} - (-1)^{(c-1)/2}}{3} & \textit{if } c \textit{ is odd,} \end{cases}$$

where $J(n)$ is the nth Jacobsthal number.

Proof. Examples 3.9 and 3.10 show that $t_p(3) = t_p(4) = t_p(5) = t_p(6) = 1$. Lemma 3.14 implies that

$$t_{p3}(c) = t_{p3A}(c) + t_{p3B}(c) = \sum_{j=1}^{4} t_{pj}(c - 4) = t_p(c - 4)$$

and similarly that

$$t_{p4}(c) = t_{p4A}(c) + t_{p4B}(c) = \sum_{j=1}^{4} t_{pj}(c-4) = t_p(c-4).$$

Therefore

$$\begin{aligned}
t_p(c) &= \sum_{j=1}^{4} t_{pj}(c) \\
&= (t_{p2}(c-2) + t_{p3}(c-2)) + (t_{p1}(c-2) + t_{p4}(c-2)) + 2t_p(c-4) \\
&= t_p(c-2) + 2t_p(c-4).
\end{aligned}$$

When c is even, the terms $t_p(2i+2)$ form the Jacobsthal sequence, and when c is odd, the terms $t_p(2i+1)$ form the Jacobsthal sequence. Hence, the result follows.   □

We end this section by finding a recursive formula for $s_p(c)$, the total number of Seifert circles from $T_p(c)$.

Theorem 3.16. *Let $s_p(c)$ be the total number of Seifert circles over all 2-bridge knots of palindromic type with crossing number c for all knots appearing in $T_p(c)$. Then $s_p(c)$ satisfies the recursion*

$$s_p(c) = s_p(c-2) + 2s_p(c-4) + 6t_p(c-4).$$

Proof. For each word $w \in T_p(c)$, we compare the number of Seifert circles in the alternating diagrams associated with w and prep(w). This comparison can be made by examining the Seifert states in Tables 10 and 11 depending on whether c is odd or even.

In each case, the total number of Seifert circles before replacement is the total number of Seifert circles after replacement plus the product of the change in the number of Seifert cirlces Δs and the number of words in this case. Lemma 3.14 allows us to count the number of Seifert circles before and after replacement. Table 10 records Δs when $c \equiv 0 \bmod 4$, and the case where $c \equiv 2 \bmod 4$ is analogous. Table 11 records Δs when $c \equiv 1 \bmod 4$, and the case where $c \equiv 3 \bmod 4$ is analogous. Since Δs is the same for each corresponding even and odd case, we conclude the following:

$$\begin{aligned}
s_{p1}(c) &= s_{p2}(c-2) + s_{p3}(c-2), \\
s_{p2}(c) &= s_{p1}(c-2) + s_{p4}(c-2), \\
s_{p3A}(c) &= s_{p1}(c-4) + s_{p4}(c-4) + 2t_{p3A}(c), \\
s_{p3B}(c) &= s_{p2}(c-4) + s_{p3}(c-4) + 4t_{p3B}(c), \\
s_{p4A}(c) &= s_{p1}(c-4) + s_{p4}(c-4) + 4t_{p4A}(c), \text{ and} \\
s_{p4B}(c) &= s_{p2}(c-4) + s_{p3}(c-4) + 2t_{p4B}(c).
\end{aligned}$$

Lemma 3.14 implies that

$$s_{p3}(c) + s_{p4}(c) = s_{p3A}(c) + s_{p3B}(c) + s_{p4A}(c) + s_{p4B}(c)$$

$$= 2 \sum_{j=1}^{4} s_{pj}(c-4) + 6 \sum_{j=1}^{4} t_{pj}(c-4)$$

$$= 2s_p(c-4) + 6t_p(c-4).$$

Therefore

$$s_p(c) = \sum_{j=1}^{4} s_{pj}(c) = s_p(c-2) + 2s_p(c-4) + 6t_p(c-4),$$

as desired. □

4. Seifert circles and average genus

In Section 3, we find recursive formulas for the total number of Seifert circles $s(c)$ and $s_p(c)$ coming from the alternating diagrams associated to words in $T(c)$ and $T_p(c)$, respectively. In this section, we find closed formulas for $s(c)$ and $s_p(c)$, and then use those formulas to prove Theorem 1.1.

The total number $s(c)$ of Seifert circles in the alternating diagrams coming from words in $T(c)$ is given by the following theorem.

Theorem 4.1. *Let $c \geq 3$. The number $s(c)$ of Seifert circles in the alternating diagrams with crossing number c coming from words in $T(c)$ can be expressed as*

$$s(c) = \frac{(3c+5)2^{c-3} + (-1)^c(5-3c)}{9}.$$

Proof. Recall that $s(c)$ satisfies the recurrence relation $s(c) = s(c-1) + 2s(c-2) + 3t(c-2)$ with initial conditions $s(3) = 2$ and $s(4) = 3$ and that $3t(c-2) = 2^{c-4} - (-1)^{c-4}$.

Proceed by induction. The base cases of $s(3) = 2$ and $s(4) = 3$ can be shown by direct computation. The recurrence relation is satisfied because

$$s(c-1) + 2s(c-2) + 3t(c-2)$$

$$= \frac{[3(c-1)+5]2^{(c-1)-3} + (-1)^{c-1}[5-3(c-1)]}{9}$$

$$+ 2\left(\frac{[3(c-2)+5]2^{(c-2)-3} + (-1)^{c-2}[5-3(c-2)]}{9} \right) + 2^{c-4} - (-1)^{c-4}$$

$$= \frac{(3c+2)2^{c-4} + (-1)^c(3c-8)}{9} + \frac{(3c-1)2^{c-4} + (-1)^c(22-6c)}{9}$$

$$+ \frac{9 \cdot 2^{c-4} - 9(-1)^c}{9}$$

$$= \frac{(6c+10)2^{c-4} + (-1)^c[(3c-8) + (22-6c) - 9]}{9}$$

$$= \frac{(3c+5)2^{c-3} + (-1)^c(5-3c)}{9}.$$

□

The total number $s_p(c)$ of Seifert circles in the alternating diagrams coming from words of palindromic type in $T_p(c)$ is given by the following theorem.

Theorem 4.2. *Let $c \geq 3$. The number $s_p(c)$ of Seifert circles in the alternating diagrams coming from words of palindromic type in $T_p(c)$ can be expressed as*

$$s_p(c) = \begin{cases} \dfrac{(3c+1)2^{(c-3)/2} + (-1)^{(c-1)/2}(1-3c)}{9} & \text{if } c \text{ is odd,} \\[3mm] \dfrac{(3c+4)2^{(c-4)/2} + (-1)^{(c-2)/2}(1-3c)}{9} & \text{if } c \text{ is even.} \end{cases}$$

Proof. Recall that $s_p(c)$ satisfies the recurrence relation $s_p(c) = s_p(c-2) + 2s_p(c-4) + 6t_p(c-4)$ with initial conditions $s_p(3) = 2$, $s_p(4) = 3$, $s_p(5) = 2$, and $s_p(6) = 3$.

Proceed by induction. One may verify the initial conditions by direct computation. Since the recursion relation for $s_p(c)$ either involves only odd indexed terms or only even indexed terms, we handle each case separately. Suppose c is odd. Then Proposition 3.15 implies that $t_p(c-4) = J(\frac{c-5}{2}) = \frac{2^{(c-5)/2} - (-1)^{(c-5)/2})}{3}$. Thus

$$\begin{aligned} & s_p(c-2) + 2s_p(c-4) + 6t_p(c-4) \\ = & \frac{(3(c-2)+1)2^{((c-2)-3)/2} + (-1)^{((c-2)-1)/2}(1-3(c-2))}{9} \\ & + 2\left(\frac{(3(c-4)+1)2^{((c-4)-3)/2} + (-1)^{((c-4)-1)/2}(1-3(c-4))}{9} \right) \\ & + 6\left(\frac{2^{(c-5)/2} - (-1)^{(c-5)/2}}{3} \right) \\ = & \frac{(3c-5)2^{(c-5)/2} + (-1)^{(c-3)/2}(7-3c)}{9} \\ & + \frac{(3c-11)2^{(c-5)/2} + (-1)^{(c-5)/2}(26-6c)}{9} \\ & + \frac{18 \cdot 2^{(c-5)/2} - (-1)^{(c-5)/2} \cdot 18}{9} \\ = & \frac{(6c+2)2^{(c-5)/2} + (-1)^{(c-1)/2}((3c-7) + (26-6c) - 18)}{9} \\ = & \frac{(3c+1)2^{(c-3)/2} + (-1)^{(c-1)/2}(1-3c)}{9}. \end{aligned}$$

Suppose c is even. Then Proposition 3.15 implies $t_p(c-4) = J(\frac{c-6}{2}) = \frac{2^{(c-6)/2} - (-1)^{(c-6)/2}}{3}$. Thus

$$\begin{aligned} & s_p(c-2) + 2s_p(c-4) + 6t_p(c-4) \\ = & \frac{(3(c-2)+4)2^{((c-2)-4)/2} + (-1)^{((c-2)-2)/2}(1-3(c-2))}{9} \end{aligned}$$

$$+ 2\left(\frac{(3(c-4)+4)2^{((c-4)-4)/2} + (-1)^{((c-4)-2)/2}(1 - 3(c-4))}{9}\right)$$

$$+ 6\left(\frac{2^{(c-6)/2} - (-1)^{(c-6)/2}}{3}\right)$$

$$= \frac{(3c - 2)2^{(c-6)/2} + (-1)^{(c-4)/2}(7 - 3c)}{9}$$

$$+ \frac{(3c - 8)2^{(c-6)/2} + (-1)^{(c-6)/2}(26 - 6c)}{9}$$

$$+ \frac{18 \cdot 2^{(c-6)/2} - (-1)^{(c-6)/2} \cdot 18}{9}$$

$$= \frac{(6c + 8)2^{(c-6)/2} + (-1)^{(c-2)/2}((3c - 7) + (26 - 6c) - 18)}{9}$$

$$= \frac{(3c + 4)2^{(c-4)/2} + (-1)^{(c-2)/2}(1 - 3c)}{9}.$$

$\square$

Although the proofs of Theorems 4.1 and 4.2 are straightforward, finding the formulas for $s(c)$ and $s_p(c)$ involved combining several closed formulas found in the Online Encyclopedia of Integer Sequences [SEI22a]. We use the formulas for $|\mathcal{K}_c|$, $s(c)$, and $s_p(c)$ in Theorems 2.1, 4.1, and 4.2, respectively to prove Theorem 1.1.

Proof of Theorem 1.1. If K is an alternating knot, then Crowell [Cro59] and Murasugi [Mur58] showed that its genus is $g(K) = \frac{1}{2}(1 + c(K) - s(K))$ where $c(K)$ and $s(K)$ are the crossing number and number of components in the Seifert state of a reduced alternating diagram of K. Theorem 2.5 implies that

$$\sum_{K \in \mathcal{K}_c} s(K) = \frac{1}{2}(s(c) + s_p(c)).$$

As in Equation (1.1), the average genus $\overline{g}_c$ satisfies

$$\overline{g}_c = \frac{\sum_{K \in \mathcal{K}_c} g(K)}{|\mathcal{K}_c|} = \frac{\sum_{K \in \mathcal{K}_c}(1 + c - s(K))}{2|\mathcal{K}_c|} = \frac{1}{2} + \frac{c}{2} - \frac{s(c) + s_p(c)}{4|\mathcal{K}_c|}.$$

Theorems 2.1, 4.1 and 4.2 contain expressions for $|\mathcal{K}_c|$, $s(c)$, and $s_p(c)$ that depend on $c \bmod 4$. If $c \equiv 0 \bmod 4$, then

$$\frac{s(c) + s_p(c)}{4|\mathcal{K}_c|} = \frac{(3c + 5)2^{c-3} + (5 - 3c) + (3c + 4)2^{(c-4)/2} + (3c - 1)}{12(2^{c-3} + 2^{(c-4)/2})}$$

$$= \frac{(3c + 5)2^{c-3} + (3c + 5)2^{(c-4)/2} - 2^{(c-4)/2} + 4}{12(2^{c-3} + 2^{(c-4)/2})}$$

$$= \frac{(3c + 5)(2^{c-3} + 2^{(c-4)/2})}{12(2^{c-3} + 2^{(c-4)/2})} + \frac{4 - 2^{(c-4)/2}}{12(2^{c-3} + 2^{(c-4)/2})}$$

$$= \frac{c}{4} + \frac{5}{12} + \frac{4 - 2^{(c-4)/2}}{12(2^{c-3} + 2^{(c-4)/2})}.$$

When $c \equiv 0 \bmod 4$, the average genus is

$$\overline{g}_c = \frac{c}{4} + \frac{1}{12} + \frac{2^{(c-4)/2} - 4}{12(2^{c-3} + 2^{(c-4)/2})}.$$

The cases where $c \equiv 1, 2,$ or $3 \bmod 4$ are similar.

$\square$

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