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The random Markov-Kakutani fixed point theorem in a random locally convex module

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ABSTRACT. Based on the recently developed theory of σ-stable sets and stable compactness, we first establish the random Markov-Kakutani fixed point theorem in a random locally convex module: let $(E, \mathcal{P})$ be a random locally convex module and G be a nonempty stably compact L^0-convex subset of E, then every commutative family of $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous L^0-affine mappings from G to G has a common fixed point, where $\mathcal{P}_{cc}$ is the σ-stable hull of $\mathcal{P}$ and $\mathcal{T}_c(\mathcal{P}_{cc})$ is the locally L^0-convex topology induced by $\mathcal{P}_{cc}$. Second, we prove that the random Markov-Kakutani fixed point theorem implies the algebraic form of the known random Hahn-Banach theorem. Finally, we establish a more general strict separation theorem in a random locally convex module, which provides not only a more general geometric form of the random Hahn-Banach theorem but also another proof for the random Markov-Kakutani fixed point theorem. Therefore, as a byproduct, the work of this paper also shows that the algebraic and geometric forms of the random Hahn-Banach theorem are equivalent. It should be pointed out that the main challenge in this paper lies in overcoming noncompactness since a stably compact set is generally noncompact.

CONTENTS

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1. Introduction

The celebrated Markov-Kakutani fixed point theorem [27, 31] states that let E be a Hausdorff locally convex space and G be a nonempty compact convex subset of E, then every commutative family of continuous affine mappings from G to itself has a common fixed point. It was first proved by Markov [31] by using the Tychonoff fixed point theorem [35]. In 1938, Kakutani [27] presented another constructive direct proof and showed that the Markov-Kakutani fixed point theorem implies the Hahn-Banach theorem. Since then, the Markov-Kakutani fixed point theorem had stimulated a series of subsequent studies of common fixed point theorems for commutative family of mappings in Banach spaces or locally convex spaces [1, 2, 3, 6, 28, 32, 33]. In 1992, Werner [36] proved that the Hahn-Banach theorem implies the Markov-Kakutani fixed point theorem. The equivalence between the Markov-Kakutani fixed point theorem and the Hahn-Banach theorem further shows that the Markov-Kakutani fixed point theorem is of fundamental importance in analysis. The purpose of this paper is to prove the random Markov-Kakutani theorem in a random locally convex module and establish its corresponding connection with the known random Hahn-Banach theorem [9, 10, 14].

Random functional analysis is based on the idea of randomizing the traditional space theory in functional analysis. Over the last three decades, random functional analysis has formed its basic frameworks such as random normed modules and random locally convex modules together with the theory of random conjugate spaces — a basic tool for the development of these frameworks [14]. Random normed modules and random locally convex modules are a random generalization of ordinary normed spaces and locally convex spaces, respectively, which were introduced by Guo [9, 10, 13]. Before 2009, random normed modules and random locally convex modules are often endowed with the (ε, λ)-linear topology, and are generally locally nonconvex topological modules. It is in order to overcome the nonconvexity of this linear topology that the theory of random conjugate spaces was developed by establishing the Hahn-Banach theorem for random linear functionals in [9, 10]. From 1991 to 2009, the theory of random conjugate spaces already obtained a rapid and deep development, for example, the work on the representation theorems of random conjugate spaces [11, 19] as well as the characterization of random reflexivity [15] was carried out during this time and has played an essential role in the recent development of nonsmooth differential geometry [8, 29, 30], see [16] for details and in particular for Gigli's independent contribution. In 2009, motivated by financial applications, Filipović, et al introduced the notion of locally L^0-convex modules in order to provide a generalized convex analysis (called random convex analysis) suited to the development of conditional or dynamic risk measures [7], where another kind of topology for random locally convex modules, called the locally L^0-convex topology, was introduced. The locally L^0-convex topology is much stronger than the (ε, λ)-topology so that the locally L^0-convex topology can assure the L^0-convex subsets in question to have nonempty topological interiors, which makes it possible to establish some basic results such as continuity and subdifferentiability theorems for L^0-convex functions. However, the locally L^0-convex topology is too strong to be a linear topology, which makes the earlier work [7] on random convex analysis not deeply developed. Following [7], Guo [14] introduced the notion of σ-stable sets and established the connection between some basic results derived from the two kinds of topologies. Thus, the advantages of the two kinds of topologies can be combined together to lead to a subsequent deep development of random convex analysis with applications [21, 22, 23, 24].

Just as pointed out in [17, 20], with the deep development of financial applications of random functional analysis, the current central task of random functional analysis is extending fixed point theory from Banach spaces or locally convex spaces to complete random normed modules and random locally convex modules, whereas the biggest challenge lies in overcoming noncompactness: since the classical topological fixed point theorems such as Brouwer, Schauder and Tychonoff fixed point theorems are established on compact convex sets, however, the closed L^0-convex sets frequently encountered in random functional analysis and its applications are generally noncompact [12, 26]. Fortunately, the theory of stable compactness, as a proper generalization of ordinary compactness, has been successfully developed in [4, 16, 17, 18]. On the basis of these developments on stable compactness, this paper returns to the establishment of random Markov-Kakutani fixed point theorem together with its connection with the random Hahn-Banach theorem.

Although our proofs of the main results in this paper are considerably motivated by the works in [27] and [36], it should be pointed out that our methods used in this paper are not a simple translation of the classical methods since our proofs require overcoming lots of challenges. First, since stably compact sets in a random locally convex module are generally not compact and thus the related arguments are quite complicated, for example, the open covering characterization of stably compact sets states that any σ-stable covering composed of σ-stable open sets must have a stable finite subcovering (see Proposition 2.14), which makes many methods used in [27] not applicable to our proofs, requiring us to devise new approaches. Second, our investigations focus on stably compact L^0-convex sets in random locally convex modules, requiring that the sets and mappings constructed in our proofs, for example S and $T_{\tilde{n}}$ in the proof of Proposition 3.3, be σ-stable, which forces us to overcome lots of new difficulties not present in the classical cases. In particular, in the proof of Proposition 3.3, in order to obtain $\{T_{\tilde{n}}(G) : \tilde{n} \in L^0(\mathcal{F}, \mathbb{N})\}$ has a nonempty intersection, we must show that $\{T_{\tilde{n}}(G) : \tilde{n} \in L^0(\mathcal{F}, \mathbb{N})\}$ is a σ-stable family. Finally, our proofs require the consideration of complicated stratification structures, as detailed in Lemma 4.1 and Theorem 4.2.

The remainder of this paper is organized as follows: Section 2 first recapitulates some basic notions and known facts, then studies some basic properties of σ-stable sets and stably compact sets in random locally convex modules, presenting several useful lemmas. In section 3, we establish the random Markov-Kakutani fixed point theorem in a random locally convex module and employ it to give a new proof of the algebraic form of the know random Hahn-Banach theorem. In Section 4, we first establish a more general strict separation theorem in a random locally convex module, which provides not only a more general geometric form of the random Hahn-Banach theorem but also another proof for the random Markov-Kakutani fixed point theorem. Therefore, as a byproduct, the work of this paper also shows that the algebraic and geometric forms of the random Hahn-Banach theorem are equivalent.

Throughout this paper, for a family $\mathcal{E}$ of sets, we always use $\bigcup \mathcal{E}$ and $\bigcap \mathcal{E}$ for the union and intersection of the sets in $\mathcal{E}$, respectively, here we would like to remind the reader of the conventional use in Proposition 2.14 and Lemmas 2.10, 2.17, 2.18 and 4.1.

2. On σ-stable sets and stable compactness in random locally convex modules

Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space, $\mathbb{N}$ the set of positive integers, $\mathbb{K}$ either the scalar field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, $\bar{\mathbb{R}}$ the set of extended real numbers, $L^0(\mathcal{F}, \mathbb{K})$ the algebra of equivalence classes of $\mathbb{K}$-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$, $L^0(\mathcal{F}, \mathbb{N})$ the set of equivalence classes of $\mathbb{N}$-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$. Specially, $L^0(\mathcal{F}) := L^0(\mathcal{F}, \mathbb{R})$ and $\bar{L}^0(\mathcal{F})$ denotes the set of equivalence classes of extended real valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

Proposition 2.1 below can be regarded as the randomized version of the order completeness of $\mathbb{R}$.

Proposition 2.1 ([5]). *Define a partial order $\leq$ on $\bar{L}^0(\mathcal{F})$ as follows: $\xi \leq \eta$ if $\xi^0(\omega) \leq \eta^0(\omega)$ for almost all ω in Ω (briefly, $\xi^0 \leq \eta^0$ a.s.), where ξ^0 and η^0 are respectively arbitrarily chosen representatives of ξ and η in $\bar{L}^0(\mathcal{F})$. Then $(\bar{L}^0(\mathcal{F}), \leq)$ is a complete lattice. For a nonempty subset H of $\bar{L}^0(\mathcal{F})$, $\bigvee H$ and $\bigwedge H$ respectively stand for the supremum and infimum of a subset H of $\bar{L}^0(\mathcal{F})$. Furthermore, the following statements hold:*

(1) There exist two sequences $\{a_n, n \in N\}$ and $\{b_n, n \in N\}$ in H such that $\bigvee H = \bigvee_n a_n$ and $\bigwedge H = \bigwedge_n b_n$.
(2) If H is directed upwards (resp., downwards), namely for any two elements h_1 and h_2 in H there exists some h_3 in H such that $h_1 \vee h_2 \leq h_3$ (resp., $h_3 \leq h_1 \wedge h_2$), then $\{a_n, n \in N\}$ (resp., $\{b_n, n \in N\}$) stated above can be chosen as nondecreasing (resp., nonincreasing).
(3) $(L^0(\mathcal{F}), \leq)$ is a Dedekind complete lattice.

As usual, for any $\xi, \eta \in \bar{L}^0(\mathcal{F})$, $\xi > \eta$ means $\xi \geq \eta$ and $\xi \neq \eta$, whereas, for any $A \in \mathcal{F}$, $\xi < \eta$ on A means $\xi^0(\omega) < \eta^0(\omega)$ for almost all ω in A, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively.

The following notations are employed throughout this paper.
$L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) : \xi \geq 0\}$;
$L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) : \xi > 0 \text{ on } \Omega\}$;
$\mathcal{F}_+ = \{A \in \mathcal{F} : P(A) > 0\}$.
$\tilde{I}_A$ always stands for the equivalence class of I_A for any $A \in \mathcal{F}$, where I_A is the characteristic function of A, namely $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ otherwise.

Definition 2.2 ([13]). *An ordered pair $(E, \mathcal{P})$ is called a random locally convex module (briefly, an RLC module) over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ if E is a left module over the algebra $L^0(\mathcal{F}, \mathbb{K})$ (briefly, an $L^0(\mathcal{F}, \mathbb{K})$-module) and $\mathcal{P}$ is a family of mappings from E to $L^0_+(\mathcal{F})$ such that the following conditions are satisfied:*
(1) $\bigvee\{\|x\| : \|\cdot\| \in \mathcal{P}\} = 0$ iff $x = \theta$ (the null element of E);
(2) $\|\xi x\| = |\xi|\|x\|$, $\forall \xi \in L^0(\mathcal{F}, \mathbb{K})$, $x \in E$ and $\|\cdot\| \in \mathcal{P}$;
(3) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in E$ and $\|\cdot\| \in \mathcal{P}$.
Furthermore, a mapping $\|\cdot\| : E \to L^0_+(\mathcal{F})$ satisfying (2) and (3) is called an L^0-seminorm on E; in addition, if $\|x\| = 0$ also implies $x = \theta$, then it is called an L^0-norm, in which case $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. Thus, a random normed module is a special case of a random locally convex module when $\mathcal{P}$ consists of a single L^0-norm $\|\cdot\|$.

When $(\Omega, \mathcal{F}, P)$ is trivial, namely $\mathcal{F} = \{\Omega, \emptyset\}$, it is clear that an *RLC* module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ just reduces to an ordinary locally convex space over $\mathbb{K}$. The simplest nontrivial *RN* module is $(L^0(\mathcal{F}, \mathbb{K}), |\cdot|)$, where $|\cdot|$ is the absolute value mapping.

For an *RLC* module $(E, \mathcal{P})$ over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, $\mathcal{P}_f$ denotes the set of finite subfamilies of $\mathcal{P}$. For any $Q \in \mathcal{P}_f$, the L^0-seminorm $\|\cdot\|_Q$ is defined by $\|x\|_Q = \bigvee\{\|x\| : \|\cdot\| \in Q\}$, $\forall x \in E$. For each sequence $\{Q_n, n \in \mathbb{N}\}$ in $\mathcal{P}_f$ and each countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ (namely, each $A_n \in \mathcal{F}$, $\cup_{n=1}^{\infty} A_n = \Omega$ and $A_i \cap A_j = \emptyset$ when $i \neq j$), $\sum_{n=1}^{\infty} \tilde{I}_{A_n}\|\cdot\|_{Q_n} : E \to L^0_+(\mathcal{F})$ is defined by $(\sum_{n=1}^{\infty} \tilde{I}_{A_n}\|\cdot\|_{Q_n})(x) = \sum_{n=1}^{\infty} \tilde{I}_{A_n}\|x\|_{Q_n}$, $\forall x \in E$. Then $\sum_{n=1}^{\infty} \tilde{I}_{A_n}\|\cdot\|_{Q_n}$ is well defined and still an L^0-seminorm, the set of all such L^0-seminorms is called the σ-stable hull of $\mathcal{P}$, denoted by $\mathcal{P}_{cc}$.

RLC modules were often endowed with the (ε, λ)-topology before 2009, which inherits from the theory of probability metric spaces (see [34] for details).

Proposition 2.3 ([13, 14]). *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. For any real numbers ε and λ such that $\varepsilon > 0$ and $0 < \lambda < 1$, and any $Q \in \mathcal{P}_f$, let $U_\theta(Q, \varepsilon, \lambda) = \{x \in E : P\{\omega \in \Omega : \|x\|_Q(\omega) < \varepsilon\} > 1 - \lambda\}$ and $\mathcal{U}_\theta(\mathcal{P}) = \{U_\theta(Q, \varepsilon, \lambda) : Q \in \mathcal{P}_f,\ \varepsilon > 0,\ 0 < \lambda < 1\}$, then $\mathcal{U}_\theta(\mathcal{P})$ is a local base of some Hausdorff linear topology, called the (ε, λ)-topology induced by $\mathcal{P}$. Further, we have the following statements:*
(1) $L^0(\mathcal{F}, \mathbb{K})$ endowed with its (ε, λ)-topology is a topological algebra over $\mathbb{K}$, it is easy to see that the (ε, λ)-topology exactly the topology of convergence in probability P;

(2) E is a Hausdorff topological module over the topological algebra $L^0(\mathcal{F},\mathbb{K})$ when E and $L^0(\mathcal{F},\mathbb{K})$ are endowed with their respective (ε,λ)-topologys;
(3) A net $\{x_\alpha,\alpha\in\Lambda\}$ in E converges in the (ε,λ)-topology to $x\in E$ iff $\{\|x_\alpha-x\|,\alpha\in\Lambda\}$ converges in probability P to θ for each $\|\cdot\|\in\mathcal{P}$.

In 2009, Filipović et al. [7] introduced the following locally L^0-convex topology.

Proposition 2.4 ([7]). *Let $(E,\mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$. For any $\varepsilon\in L^0_{++}(\mathcal{F})$ and any $Q\in\mathcal{P}_f$, let $V(\theta,Q,\varepsilon)=\{x\in E:\|x\|_Q<\varepsilon$ on $\Omega\}$, then $\mathcal{V}_\theta(\mathcal{P})=\{V(\theta,Q,\varepsilon):Q\in\mathcal{P}_f,\ \varepsilon\in L^0_{++}(\mathcal{F})\}$ is a local base of some Hausdorff locally L^0-convex topology, called the locally L^0-convex topology induced by $\mathcal{P}$. Further, the following statements hold:*

(1) $L^0(\mathcal{F},\mathbb{K})$ endowed with its locally L^0-convex topology is a topological ring ;
(2) E is a Hausdorff topological module over the topological ring $L^0(\mathcal{F},\mathbb{K})$ when E and $L^0(\mathcal{F},\mathbb{K})$ are endowed with their respective locally L^0-convex topology;
(3) A net $\{x_\alpha,\alpha\in\Lambda\}$ in E converges in the locally L^0-convex topology to $x\in E$ iff $\{\|x_\alpha-x\|,\alpha\in\Lambda\}$ converges in the locally L^0-convex topology of $L^0(\mathcal{F},\mathbb{K})$ to θ for each $\|\cdot\|\in\mathcal{P}$.

From now on, for an *RLC* module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$, we always use $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$ (resp., $\mathcal{T}_c(\mathcal{P})$) for the (ε,λ)-topology (resp., locally L^0-convex topology) induced by $\mathcal{P}$, and $\mathcal{U}_\theta(\mathcal{P})$ (resp., $\mathcal{V}_\theta(\mathcal{P})$) for the local base of $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$ (resp., $\mathcal{T}_c(\mathcal{P})$). Similarly, one can understand $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P}_{cc})$ (resp., $\mathcal{T}_c(\mathcal{P}_{cc})$) and $\mathcal{U}_\theta(\mathcal{P}_{cc})$ (resp., $\mathcal{V}_\theta(\mathcal{P}_{cc})$). It is known from [14] that $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})=\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P}_{cc})$, $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$ is weaker than $\mathcal{T}_c(\mathcal{P})$, and $\mathcal{T}_c(\mathcal{P})$ is weaker than $\mathcal{T}_c(\mathcal{P}_{cc})$.

Let E be an $L^0(\mathcal{F},\mathbb{K})$-module and G be a nonempty subset of E. A subset H of E is said to be L^0-absorbed by G if there exists $\xi\in L^0_{++}(\mathcal{F})$ such that $\eta H\subset G$ for any $\eta\in L^0(\mathcal{F},\mathbb{K})$ with $|\eta|\le\xi$. G is said to be L^0-absorbent if G L^0-absorbs every x in E. G is said to be L^0-balanced if $\eta G\subset G$ for any $\eta\in L^0(\mathcal{F},\mathbb{K})$ with $|\eta|\le 1$. G is said to be L^0-convex if $\xi x+(1-\xi)y\in G$ for any $x,y\in G$ and any $\xi\in L^0_+(\mathcal{F})$ with $0\le\xi\le 1$. It is easy to see that every $V(\theta,Q,\varepsilon)$ in $\mathcal{V}_\theta(\mathcal{P})$ is L^0-absorbent and L^0-convex.

Let $(E,\mathcal{P})$ be an *RLC* module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$, a subset G of E is said to be $\mathcal{T}_c(\mathcal{P})$-bounded if G is L^0-absorbed by every V in $\mathcal{V}_\theta(\mathcal{P})$. It is known from Theorem 2.16 of [24] that G is $\mathcal{T}_c(\mathcal{P})$-bounded iff G is *a.s.* bounded (namely, for any $\|\cdot\|\in\mathcal{P}$, $\bigvee\{\|g\|:g\in G\}\in L^0_+(\mathcal{F})$).

To combine the advantages of both (ε,λ)-topology and locally L^0-convex topology for the development of random functional analysis, Guo [14] introduced the concept of σ-stability as follows.

Let E be an $L^0(\mathcal{F},\mathbb{K})$-module, E is said to be regular if E has the following property: for any given two elements x and y in E, if there exists some countable partition $\{A_n,n\in\mathbb{N}\}$ of Ω to $\mathcal{F}$ such that $\tilde{I}_{A_n}x=\tilde{I}_{A_n}y$ for each $n\in\mathbb{N}$, then $x=y$.

In the remainder of this paper, we always denote the set of countable partitions of Ω to $\mathcal{F}$ by $\Pi_{\mathcal{F}}$ and assume that all the $L^0(\mathcal{F},\mathbb{K})$-modules occurring in this paper are regular.

Definition 2.5 ([14, 18]). *Let E be an $L^0(\mathcal{F},\mathbb{K})$-module and G be a nonempty subset of E. G is said to be stable if $\tilde{I}_A x + \tilde{I}_{A^c} y \in G$ for any $x, y \in G$ and any $A \in \mathcal{F}$. G is said to be σ-stable (or to have the countable concatenation property) if for each sequence $\{x_n, n \in \mathbb{N}\}$ in G and each $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$, there exists some $x \in G$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} x_n$ for each $n \in \mathbb{N}$ (x is unique since E is assumed to be regular, usually denoted by $\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n$, called the countable concatenation of $\{x_n, n \in \mathbb{N}\}$ along $\{A_n, n \in \mathbb{N}\}$). By the way, if G is σ-stable and H is a nonempty subset of G, then $\sigma(H) := \{\sum_{n=1}^{\infty} \tilde{I}_{A_n} h_n : \{h_n, n \in \mathbb{N}\}$ is a sequence in H and $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}\}$ is called the σ-stable hull of H.*

Definition 2.6 ([14, 18]). *Let E be an $L^0(\mathcal{F},\mathbb{K})$-module, G_1 and G_2 two nonempty subsets of E and $f : G_1 \to G_2$ a mapping. f is said to be stable if G_1 and G_2 are stable and $f(\tilde{I}_A x + \tilde{I}_{A^c} y) = \tilde{I}_A f(x) + \tilde{I}_{A^c} f(y)$ for any $x, y \in G_1$ and any $A \in \mathcal{F}$. f is said to be σ-stable if G_1 and G_2 are σ-stable and $f(\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} f(x_n)$ for any sequence $\{x_n, n \in \mathbb{N}\}$ in G_1 and any $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$.*

Definition 2.7 ([18]). *Let E be an $L^0(\mathcal{F},\mathbb{K})$-module and G be a σ-stable subset of E. For any sequence of nonempty subsets $\{G_n, n \in \mathbb{N}\}$ of G and any $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$, $\sum_{n=1}^{\infty} \tilde{I}_{A_n} G_n := \{\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n : x_n \in G_n, \forall\, n \in \mathbb{N}\}$ is called the countable concatenation of $\{G_n, n \in \mathbb{N}\}$ along $\{A_n, n \in \mathbb{N}\}$. For a family $\mathcal{E}$ of nonempty subsets of G, $\sigma(\mathcal{E}) := \{\sum_{n=1}^{\infty} \tilde{I}_{A_n} G_n : \{G_n, n \in \mathbb{N}\}$ is a sequence in $\mathcal{E}$ and $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}\}$ is called the σ-stable hull of $\mathcal{E}$; if $\sigma(\mathcal{E}) = \mathcal{E}$, then $\mathcal{E}$ is said to be σ-stable.*

Given an *RLC* module $(E,\mathcal{P})$ over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$, let $E^*_{\varepsilon,\lambda}(\mathcal{P}) = \{f : E \to L^0(\mathcal{F},\mathbb{K}) \mid f$ is a continuous module homomorphism from $(E,\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P}))$ to $(L^0(\mathcal{F},\mathbb{K}),\mathcal{T}_{\varepsilon,\lambda})\}$ and $E^*_c(\mathcal{P}) = \{f : E \to L^0(\mathcal{F},\mathbb{K}) \mid f$ is a continuous module homomorphism from $(E,\mathcal{T}_c(\mathcal{P}))$ to $(L^0(\mathcal{F},\mathbb{K}),\mathcal{T}_c)\}$, which are called the random conjugate spaces of $(E,\mathcal{P})$ under $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$ and $\mathcal{T}_c(\mathcal{P})$, respectively. Similarly, one can understand $E^*_{\varepsilon,\lambda}(\mathcal{P}_{cc})$ and $E^*_c(\mathcal{P}_{cc})$.

Proposition 2.8 below establishes the connection between some basic results derived from the (ε,λ)-topology and the locally L^0-convex topology.

Proposition 2.8 ([14, 18, 23]). *Let $(E,\mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$ and G be a nonempty subset of E. Then we have the following:*

(1) If G is σ-stable, then $G^-_{\varepsilon,\lambda}(\mathcal{P}) = G^-_c(\mathcal{P}) = G^-_c(\mathcal{P}_{cc})$, where $G^-_{\varepsilon,\lambda}(\mathcal{P})$, $G^-_c(\mathcal{P})$ and $G^-_c(\mathcal{P}_{cc})$ denote the closure of G under $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$, $\mathcal{T}_c(\mathcal{P})$ and $\mathcal{T}_c(\mathcal{P}_{cc})$, respectively.

(2) If G is stable, then G is $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$-complete iff G is both σ-stable and $\mathcal{T}_c(\mathcal{P})$-complete. Moreover, if G is σ-stable, then G is $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$-complete iff G is $\mathcal{T}_c(\mathcal{P})$-complete iff G is $\mathcal{T}_c(\mathcal{P}_{cc})$-complete.

(3) $E^*_{\varepsilon,\lambda}(\mathcal{P}) = \sigma(E^*_c(\mathcal{P}))$. Specially, $E^*_{\varepsilon,\lambda}(\mathcal{P}_{cc}) = E^*_c(\mathcal{P}_{cc})$.

To simplify the proofs of the main results in this paper, we present the following Lemmas 2.9 and 2.10.

Lemma 2.9. *Let E be an $L^0(\mathcal{F},\mathbb{K})$-module, G and F two nonempty σ-stable subsets of E, and $T : G \to F$ a σ-stable mapping. Then we have the following statements:*

(1) $T(G)$ is σ-stable and $T^{-1}(F_1) := \{x \in G : T(x) \in F_1\}$ is σ-stable for any nonempty σ-stable subset F_1 of $T(G)$.

(2) For each sequence $\{G_n, n \in \mathbb{N}\}$ of nonempty subsets of G and each $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$, we have

$$T(\sum_{n=1}^{\infty} \tilde{I}_{A_n} G_n) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} T(G_n).$$

(3) For each sequence $\{F_n, n \in \mathbb{N}\}$ of nonempty stable subsets of $T(G)$ and each $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$, we have

$$T^{-1}(\sum_{n=1}^{\infty} \tilde{I}_{A_n} F_n) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} T^{-1}(F_n).$$

Proof. (1) and (2) are straightforward, so are omitted. We focus on proving (3). It is obvious that $\sum_{n=1}^{\infty} \tilde{I}_{A_n} T^{-1}(F_n) \subset T^{-1}(\sum_{n=1}^{\infty} \tilde{I}_{A_n} F_n)$. Conversely, for any $x \in T^{-1}(\sum_{n=1}^{\infty} \tilde{I}_{A_n} F_n)$, there exists $y_n \in F_n$ such that $T(x) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} y_n$. For each y_n, arbitrarily choose $z_n \in T^{-1}(y_n)$ and let $x_n = \tilde{I}_{A_n} x + \tilde{I}_{A_n^c} z_n$, then $T(x_n) = \tilde{I}_{A_n} T(x) + \tilde{I}_{A_n^c} T(z_n) = y_n \in F_n$, which implies that $x_n \in T^{-1}(F_n)$ and

$$x = \sum_{n=1}^{\infty} \tilde{I}_{A_n} x = \sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n \in \sum_{n=1}^{\infty} \tilde{I}_{A_n} T^{-1}(F_n).$$

Thus, (3) holds. □

Lemma 2.10. *Let E be an $L^0(\mathcal{F},\mathbb{K})$-module, G a nonempty σ-stable subset of E and $\mathcal{E} = \{G_i, i \in I\}$ a family of nonempty subsets of G. Then we have the following statements:*

(1) $\bigcap \mathcal{E} = \bigcap \sigma(\mathcal{E})$.

(2) If G_i is σ-stable for any $i \in I$, then $\sigma(\mathcal{E})$ has the finite intersection property iff $\mathcal{E}$ has the finite intersection property.

Proof. (1) It is clear that $\bigcap \sigma(\mathcal{E}) \subset \bigcap \mathcal{E}$ since $\mathcal{E} \subset \sigma(\mathcal{E})$. Conversely, if $\bigcap \mathcal{E} = \emptyset$, of course $\bigcap \mathcal{E} \subset \bigcap \sigma(\mathcal{E})$; if $\bigcap \mathcal{E} \neq \emptyset$, then for any $x \in \bigcap \mathcal{E}$, it always holds that $x \in \sum_{n=1}^{\infty} \tilde{I}_{A_n} G_n$ for any $\{A_n, \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ and any sequence $\{G_n, n \in \mathbb{N}\}$ in $\mathcal{E}$, which implies that $x \in \bigcap \sigma(\mathcal{E})$, and thus $\bigcap \mathcal{E} \subset \bigcap \sigma(\mathcal{E})$.

(2) **Necessity.** It is obvious.

Sufficiency. For any finite subset $\{M_k : k = 1, \cdots, m\}$ of $\sigma(\mathcal{E})$ with $M_k = \sum_{n=1}^{\infty} \tilde{I}_{A_{k,n}} G_{k,n}$ for some $\{A_{k,n}, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ and some sequence $\{G_{k,n}, n \in \mathbb{N}\}$

in $\mathcal{E}$, we have

$$\begin{aligned} M_1 \cap \cdots \cap M_m &= (\sum_{n=1}^{\infty} \tilde{I}_{A_{1,n}} G_{1,n}) \cap \cdots \cap (\sum_{n=1}^{\infty} \tilde{I}_{A_{m,n}} G_{m,n}) \\ &\supset \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} \tilde{I}_{A_{1,n_1} \cap \cdots \cap A_{m,n_m}} (G_{1,n_1} \cap \cdots \cap G_{m,n_m}) \\ &\neq \emptyset. \end{aligned}$$

□

The following concept of stable finite families is crucial for the introduction of stable compactness.

Definition 2.11 ([18, 25])**.** *Let G be a σ-stable subset of an $L^0(\mathcal{F}, \mathbb{K})$-module and $\mathcal{E}$ be a σ-stable family of nonempty σ-stable subset of G. A nonempty subfamily $\tilde{\mathcal{E}}$ of $\mathcal{E}$ is said to be stable finite, if there exist $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ and a sequence $\{\mathcal{E}_n, n \in \mathbb{N}\}$ of nonempty finite subfamilies of $\mathcal{E}$ such that $\tilde{\mathcal{E}} = \sum_{n=1}^{\infty} \tilde{I}_{A_n} \sigma(\mathcal{E}_n)$.*

Definition 2.12 ([18, 25])**.** *Let G be a σ-stable subset of an $L^0(\mathcal{F}, \mathbb{K})$-module. The topology $\mathcal{T}$ defined on G is said to be stable, if it admits a topological base which is a σ-stable family of σ-stable sets. A filter on G is said to be stable if it admits a filter base which is a σ-stable family of σ-stable sets. A stable topological space $(G, \mathcal{T})$ (namely, $\mathcal{T}$ is stable) is said to be stably compact if every stable filter on G has a cluster point in G.*

Example 2.13. *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ and G be a nonempty σ-stable subset of E. For any $x \in G$, $Q \in \mathcal{P}_{cc}$ and $\varepsilon \in L^0_{++}(\mathcal{F})$, let $V_G(x, Q, \varepsilon) := \{y \in G : \|x - y\|_Q < \varepsilon \text{ on } \Omega\}$. Then $\{V_G(x, Q, \varepsilon) : x \in G, Q \in \mathcal{P}_{cc}, \varepsilon \in L^0_{++}(\mathcal{F})\}$ is a base of $\mathcal{T}_c(\mathcal{P}_{cc})|_G$ (namely, the relativization of $\mathcal{T}_c(\mathcal{P}_{cc})$ to G). It is clear that each $V_G(x, Q, \varepsilon)$ is σ-stable. Moreover, for any sequence $\{x_n, n \in \mathbb{N}\}$ in G, any sequence $\{Q_n, n \in \mathbb{N}\}$ in $\mathcal{P}_{cc}$, and any $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$, we have*

$$\sum_{n=1}^{\infty} \tilde{I}_{A_n} V_G(x_n, Q_n, \varepsilon_n) = V_G\left(\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n, \sum_{n=1}^{\infty} \tilde{I}_{A_n} Q_n, \sum_{n=1}^{\infty} \tilde{I}_{A_n} \varepsilon_n\right).$$

Thus,$(G, \mathcal{T}_c(\mathcal{P}_{cc})|_G)$ (briefly, denoted by $(G, \mathcal{T}_c(\mathcal{P}_{cc}))$ is a stable topological space. In general, neither $(G, \mathcal{T}_{\varepsilon,\lambda}(\mathcal{P}))$ nor $(G, \mathcal{T}_c(\mathcal{P}))$ is stable.

Corresponding to classical compact sets, stable compact sets have the following characterization.

Proposition 2.14 ([18, 25])**.** *Let $(X, \mathcal{T})$ be a stable topological space. Then the following are equivalent.*

(1) $(X, \mathcal{T})$ is stably compact.
(2) For every σ-stable family $\mathcal{O}$ of σ-stable open sets with $X = \bigcup \mathcal{O}$, there exists a stable finite subfamily $\tilde{\mathcal{O}}$ of $\mathcal{O}$ such that $X = \bigcup \tilde{\mathcal{O}}$.

(3) For every σ-stable family $\mathcal{E}$ of σ-stable closed sets which has stable finite intersection property (namely, $\bigcap \tilde{\mathcal{E}} \neq \emptyset$ for every stable finite subfamily $\tilde{\mathcal{E}}$ of $\mathcal{E}$), it holds that $\bigcap \mathcal{E} \neq \emptyset$.

Proposition 2.15 below considerably simplifies (3) of Proposition 2.14.

Proposition 2.15 ([18])**.** *Let $(X, \mathcal{T})$ be a stable topological space and $\mathcal{E}$ be a σ-stable family of nonempty subsets of X. Then, $\mathcal{E}$ has stable finite intersection property if and only if $\mathcal{E}$ has finite intersection property. Thus, $(X, \mathcal{T})$ is stably compact if and only if every σ-stable family of σ-stable closed subsets of X has a nonempty intersection whenever the family has finite intersection property.*

Remark 2.16. *(1) In* [18], *Guo introduced the concept of $B_{\mathcal{F}}$-stability, which is a natural generalization of σ-stability. Definitions 2.5, 2.6, 2.7, 2.11 and 2.12 were just introduced under the framework of $B_{\mathcal{F}}$-stable sets in* [18]. *Additionally, Propositions 2.14 and 2.15 were also established in such a general context in* [18]. *Since this paper is focused only on $L^0(\mathcal{F}, \mathbb{K})$-modules, we, in fact, only employ the corresponding special cases of the results of* [18].

(2) Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ and G be a σ-stable subset of E. As pointed out by Guo in [18], *although $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$ and $\mathcal{T}_c(\mathcal{P})$ are generally not stable topologies, Proposition 2.8 shows that $(E, \mathcal{P})$ has the same σ-stable closed sets under the three topologies $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$, $\mathcal{T}_c(\mathcal{P})$ and $\mathcal{T}_c(\mathcal{P}_{cc})$. Hence, according to Proposition 2.15, we can also define stably compact sets under $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$ and $\mathcal{T}_c(\mathcal{P})$: a σ-stable subset G of $(E, \mathcal{P})$ is called $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$-stably compact (resp., $\mathcal{T}_c(\mathcal{P})$-stably compact) if any σ-stable family of σ-stable $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$-closed (resp., $\mathcal{T}_c(\mathcal{P})$-closed) subsets of G with the finite intersection property has a nonempty intersection. Moreover, it is also easy to see that a σ-stable subset G of $(E, \mathcal{P})$ has equivalent stable compactness under the three topologies $\mathcal{T}_{\varepsilon,\lambda}(\mathcal{P})$, $\mathcal{T}_c(\mathcal{P})$ and $\mathcal{T}_c(\mathcal{P}_{cc})$. In the remainder of this paper, for a σ-stable set G, we will simply refer to G as stably compact (or closed) without specifying which topology is employed.*

The following Lemmas 2.17 and 2.18 present the fundamental properties of stably compact sets, which are essential for the proofs of main results in this paper.

Lemma 2.17. *Let $(X, \mathcal{T}_1)$ be a stably compact topological space, $(Y, \mathcal{T}_2)$ a stable topological space and $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ a σ-stable continuous surjective mapping. Then $(Y, \mathcal{T}_2)$ is stably compact.*

Proof. Let $\mathcal{O} = \{O_i : i \in I\}$ be a σ-stable family of σ-stable open sets of Y such that $Y = \cup_{i \in I} O_i$, then $\{f^{-1}(O_i) : i \in I\}$ is a σ-family of σ-stable open sets with $X = \cup_{i \in I} f^{-1}(O_i)$ by (3) of Lemma 2.9.

Since $(X, \mathcal{T}_1)$ is stably compact, there exist $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ and a sequence $\{\mathcal{E}_n, n \in \mathbb{N}\}$ of nonempty finite subfamilies of $\{f^{-1}(O_i) : i \in I\}$ such that $X = \bigcup \sum_{n=1}^{\infty} \tilde{I}_{A_n} \sigma(\mathcal{E}_n)$. Let $\mathcal{E}_n = \{f^{-1}(O_{n,l}) : l = 1 \sim k_n\}$ for each $n \in \mathbb{N}$, then we have

$$
\begin{aligned}
Y &= f(X) \\
&= f(\bigcup \sum_{n=1}^{\infty} \tilde{I}_{A_n}\sigma[f^{-1}(O_{n,1}), \cdots, f^{-1}(O_{n,k_n})]) \\
&= \bigcup \sum_{n=1}^{\infty} \tilde{I}_{A_n}\sigma[f(f^{-1}(O_{n,1})), \cdots, f(f^{-1}(O_{n,k_n}))] \\
&= \bigcup \sum_{n=1}^{\infty} \tilde{I}_{A_n}\sigma(O_{n,1}, \cdots, O_{n,k_n}),
\end{aligned}
$$

which implies that $\sum_{n=1}^{\infty} \tilde{I}_{A_n}\sigma(O_{n,1}, \cdots, O_{n,k_n})$ is a stable finite subfamily of $\mathcal{O}$ with $Y = \bigcup \sum_{n=1}^{\infty} \tilde{I}_{A_n}\sigma\{O_{n,1}, \cdots, O_{n,k_n}\}$. Thus, $(Y, \mathcal{T}_2)$ is stably compact. □

Lemma 2.18. *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ and G be a nonempty stably compact subset of E. Then we have the following statements:*

(1) For any $V \in \mathcal{V}_\theta(\mathcal{P}_{cc})$, there exists $\tilde{n} \in L^0(\mathcal{F}, \mathbb{N})$ such that $G \subset \tilde{n}V$, namely, G is $\mathcal{T}_c(\mathcal{P}_{cc})$-bounded, since V is L^0-balanced.

(2) If $\theta \in G$, then $\cap_{\tilde{n} \in L^0(\mathcal{F},\mathbb{N})} \frac{1}{\tilde{n}} G = \{\theta\}$.

Proof. (1) For any $V \in \mathcal{V}_\theta(\mathcal{P}_{cc})$, V is L^0-absorbent, so $E = \cup_{\xi \in L^0_{++}(\mathcal{F})} \xi V$. Then, $G = \bigcup\{(\xi V) \cap G : \xi \in L^0_{++}(\mathcal{F})$ and $(\xi V) \cap G \neq \emptyset\}$. It is clear that every $(\xi V) \cap G$ is a σ-stable open set in $(G, \mathcal{T}_c(\mathcal{P}_{cc}))$. Furthermore, for each $\{B_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ and each sequence $\{(\xi_n V) \cap G, n \in \mathbb{N}\}$ in $\{(\xi V) \cap G : \xi \in L^0_{++}(\mathcal{F})$ and $(\xi V) \cap G \neq \emptyset\}$, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} \tilde{I}_{B_n}[(\xi_n V) \cap G] &= [(\sum_{n=1}^{\infty} \tilde{I}_{B_n} \xi_n) V] \cap G \\
&\in \{(\xi V) \cap G : \xi \in L^0_{++}(\mathcal{F}) \text{ and } (\xi V) \cap G \neq \emptyset\},
\end{aligned}
$$

which implies that $\{(\xi V) \cap G : \xi \in L^0_{++}(\mathcal{F})$ and $(\xi V) \cap G \neq \emptyset\}$ is a σ-stable family of σ-stable open sets of G with $G = \bigcup\{(\xi V) \cap G : \xi \in L^0_{++}(\mathcal{F})$ and $(\xi V) \cap G \neq \emptyset\}$.

Since G is stably compact, there exist $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ and a sequence $\{\mathcal{E}_n, n \in \mathbb{N}\}$ of nonempty finite subfamilies of $\{(\xi V) \cap G : \xi \in L^0_{++}(\mathcal{F})$ and $(\xi V) \cap G \neq \emptyset\}$ such that $X = \sum_{n=1}^{\infty} \tilde{I}_{A_n}\sigma(\mathcal{E}_n)$. Let $\mathcal{E}_n = \{(\xi_{n,l} V) \cap G : l = 1 \sim k_n\}$, $\xi_n = \bigvee_{l=1}^{k_n} \xi_{n,l}$ and $\xi = \sum_{n=1}^{\infty} \tilde{I}_{A_n} \xi_n$, then $\xi \in L^0_{++}(\mathcal{F})$ and we have

$$
\begin{aligned}
G &= \bigcup \sum_{n=1}^{\infty} \tilde{I}_{A_n}\sigma[(\xi_{n,1} V \cap G), \cdots, (\xi_{n,k_n} V \cap G)] \\
&\subset \sum_{n=1}^{\infty} \tilde{I}_{A_n}(\xi_n V \cap G) \\
&= [(\sum_{n=1}^{\infty} \tilde{I}_{A_n} \xi_n) V] \cap G
\end{aligned}
$$

$$\subset \xi V,$$

since the sets ηV increase with $\eta \in L^0_{++}(\mathcal{F})$.

Let ξ^0 be an arbitrarily chosen representative of ξ and assume, without loss of generality, that $\xi^0(\omega) > 0$ for all $\omega \in \Omega$. Let $A_n = \{\omega \in \Omega : n-1 \leq \xi^0(\omega) < n\}$ and $\tilde{n} = \sum_{n=1}^{\infty} \tilde{I}_{A_n} n$, then $\tilde{n} \in L^0(\mathcal{F}, \mathbb{N})$ and $\xi < \tilde{n}$ on Ω, which implies that $G \subset \xi V \subset \tilde{n} V$.

(2) According to (1), for any $V \in \mathcal{V}_\theta(\mathcal{P}_{cc})$, it holds that $\cap_{\tilde{n} \in L^0(\mathcal{F},\mathbb{N})} \frac{1}{\tilde{n}} G \subset V$. Consequently, we have $\cap_{\tilde{n} \in L^0(\mathcal{F},\mathbb{N})} \frac{1}{\tilde{n}} G \subset \cap_{V \in \mathcal{V}_\theta(\mathcal{P}_{cc})} V$. Moreover, since $\mathcal{T}_c(\mathcal{P}_{cc})$ is a Hausdorff topology on E, it holds that $\cap_{V \in \mathcal{V}_\theta(\mathcal{P}_{cc})} V = \{\theta\}$. Therefore, we conclude that $\cap_{\tilde{n} \in L^0(\mathcal{F},\mathbb{N})} \frac{1}{\tilde{n}} G = \{\theta\}$. □

3. The random Markov-Kakutani fixed point theorem in a random locally convex module

The aim of this section is to establish the random Markov-Kakutani fixed point theorem in a random locally convex module and employ it to prove the algebraic form of the known random Hahn-Banach theorem. The main results in this section are Theorems 3.4 and 3.9. To arrive at these, we first provide lemmas 3.1 and 3.2.

Let E be an $L^0(\mathcal{F}, \mathbb{K})$-module, G_1 and G_2 two nonempty L^0-convex subsets of E. A mapping $f : G_1 \to G_2$ is said to be L^0-affine if $f(\xi x + (1-\xi)y) = \xi f(x) + (1-\xi) f(y)$ for any $x, y \in G_1$ and any $\xi \in L^0_+(\mathcal{F})$ with $0 \leq \xi \leq 1$.

Lemma 3.1. *Let E be an $L^0(\mathcal{F}, \mathbb{K})$-module, G a nonempty σ-stable subset of E and $T : G \to G$ an L^0-affine mapping. Then T is σ-stable.*

Proof. For any given $g_0 \in G$, consider $G' = G - g_0 := \{x - g_0 : x \in G\}$ and $T' : G' \to G'$ defined by $T'(x - g_0) = T(x) - g_0$, $\forall x \in G$. Then G' is still σ-stable and T' is σ-stable iff T is σ-stable. So, without loss of generality, we can assume that $\theta \in G$.

For any $x \in G$ and any $A \in \mathcal{F}$, since $T(\tilde{I}_A x) = T(\tilde{I}_A x + \tilde{I}_{A^c}\theta) = \tilde{I}_A T(x) + \tilde{I}_{A^c} T(\theta)$, so $\tilde{I}_A T(\tilde{I}_A x) = \tilde{I}_A T(x)$. Then, for each sequence $\{x_n, n \in \mathbb{N}\}$ in G and each $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$, we have

$$\begin{aligned} T(\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n) &= (\sum_{n=1}^{\infty} \tilde{I}_{A_n}) T(\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n) \\ &= \sum_{n=1}^{\infty} \tilde{I}_{A_n} T(\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n) \\ &= \sum_{n=1}^{\infty} \tilde{I}_{A_n} T(\tilde{I}_{A_n} x_n) \\ &= \sum_{n=1}^{\infty} \tilde{I}_{A_n} T(x_n). \end{aligned}$$

Thus, T is σ-stable. □

Lemma 3.2. *Let $(E,\mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$, G be a nonempty σ-stable subset of E, and $\{f_n, n\in\mathbb{N}\}$ be a sequence of σ-stable $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous mappings from G to G (namely, $f_n:(G,\mathcal{T}_c(\mathcal{P}_{cc}))\to(G,\mathcal{T}_c(\mathcal{P}_{cc}))$ is continuous). For any given $\{A_n, n\in\mathbb{N}\}\in\Pi_{\mathcal{F}}$, define $f:G\to G$ by $f(x)=\sum_{n=1}^{\infty}\tilde{I}_{A_n}f_n(x)$, $\forall x\in G$. Then, f is σ-stable and $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous.*

Proof. It is clear that f is well defined and σ-stable, we only need to show that f is $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous. For any given $x\in G$, $Q\in(\mathcal{P}_{cc})_f$ and $\varepsilon\in L^0_{++}(\mathcal{F})$, since f_n is $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous, there exist $Q_n\in(\mathcal{P}_{cc})_f$ and $\varepsilon_n\in L^0_{++}(\mathcal{F})$ such that $f_n[V_G(x,Q_n,\varepsilon_n)]\subset V_G(f_n(x),Q,\varepsilon)$. Since $(G,\mathcal{T}_c(\mathcal{P}_{cc}))$ is a stable topological space, by Example 2.13, $V_G(x,\sum_{n=1}^{\infty}\tilde{I}_{A_n}Q_n,\sum_{n=1}^{\infty}\tilde{I}_{A_n}\varepsilon_n)$ is a $\mathcal{T}_c(\mathcal{P}_{cc})$-neighborhood of x and we have

$$
\begin{aligned}
f(V_G(x,\sum_{n=1}^{\infty}\tilde{I}_{A_n}Q_n,\sum_{n=1}^{\infty}\tilde{I}_{A_n}\varepsilon_n)) &= \sum_{n=1}^{\infty}\tilde{I}_{A_n}[f_n(V_G(x,\sum_{n=1}^{\infty}\tilde{I}_{A_n}Q_n,\sum_{n=1}^{\infty}\tilde{I}_{A_n}\varepsilon_n))]\\
&= \sum_{n=1}^{\infty}\tilde{I}_{A_n}[\sum_{n=1}^{\infty}\tilde{I}_{A_n}f_n(V_G(x,Q_n,\varepsilon_n))]\\
&= \sum_{n=1}^{\infty}\tilde{I}_{A_n}f_n(V_G(x,Q_n,\varepsilon_n))\\
&\subset \sum_{n=1}^{\infty}\tilde{I}_{A_n}V_G(f_n(x),Q,\varepsilon)\\
&= V_G(\sum_{n=1}^{\infty}\tilde{I}_{A_n}f_n(x),Q,\varepsilon)\\
&= V_G(f(x),Q,\varepsilon).
\end{aligned}
$$

Thus, f is $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous. □

Proposition 3.3 below lies in the heart of our proof of the random Markov-Kakutani fixed point theorem, which reduces the discussion of a commutative family of mappings to a single map.

In the proof of Proposition 3.3 below, for any $\tilde{n}\in L^0(\mathcal{F},\mathbb{N})$, $\tilde{n}:=\sum_{k=1}^{\infty}\tilde{I}_{A_k}k$ means that $A_k=\{\omega\in\Omega:\tilde{n}^0(\omega)=k\}$ for each $k\in\mathbb{N}$, where $\tilde{n}^0$ is a chosen representative of $\tilde{n}$.

Proposition 3.3. *Let $(E,\mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$, G a nonempty stably compact L^0-convex subset of E and T a $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous L^0-affine mapping from G to G. Then T has a fixed point. Moreover, the set of all such fixed points is stably compact and L^0-convex.*

Proof. For any $\tilde{n} := \sum_{k=1}^{\infty} \tilde{I}_{A_k} k$ in $L^0(\mathcal{F}, \mathbb{N})$, defining $T_{\tilde{n}} : G \to G$ by

$$T_{\tilde{n}}(x) = \sum_{k=1}^{\infty} \tilde{I}_{A_k}[\frac{1}{k}(\sum_{i=0}^{k-1} T^i(x))],\ \forall x \in G,$$

where T^i denotes the i-iterate of T (hence T^0 is the identity map on G). It is clear that $T_{\tilde{n}}$ is well defined. For any $x, y \in G$ and any $\xi \in L^0_+(\mathcal{F})$ with $0 \leq \xi \leq 1$, we have

$$\begin{aligned}
T_{\tilde{n}}(\xi x + (1-\xi)y) &= \sum_{k=1}^{\infty} \tilde{I}_{A_k}[\frac{1}{k}(\sum_{i=0}^{k-1} T^i(\xi x + (1-\xi)y))] \\
&= \sum_{k=1}^{\infty} \tilde{I}_{A_k}\{\frac{1}{k}\sum_{i=0}^{k-1}[\xi T^i(x) + (1-\xi)T^i(y)]\} \\
&= \sum_{k=1}^{\infty} \tilde{I}_{A_k}[\frac{1}{k}\sum_{i=0}^{k-1}(\xi T^i(x))] + \sum_{k=1}^{\infty} \tilde{I}_{A_k}[\frac{1}{k}\sum_{i=0}^{k-1}((1-\xi)T^i(y))] \\
&= \xi T_{\tilde{n}}(x) + (1-\xi)T_{\tilde{n}}(y),
\end{aligned}$$

which implies that $T_{\tilde{n}}$ is L^0-affine. Therefore, $T_{\tilde{n}}$ is σ-stable by Lemma 3.1. Since T is $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous, it is easy to check that $T_{\tilde{n}}$ is $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous by Lemma 3.2. Furthermore, for any $\tilde{m}, \tilde{n} \in L^0(\mathcal{F}, \mathbb{N})$,

$$\begin{aligned}
T_{\tilde{n}}[T_{\tilde{m}}(x)] &= \sum_{k=1}^{\infty} \tilde{I}_{A_k}[\frac{1}{k}(\sum_{i=0}^{k-1} T^i(T_{\tilde{m}}(x)))] \\
&= \sum_{k=1}^{\infty} \tilde{I}_{A_k}[\frac{1}{k}(\sum_{i=0}^{k-1} T_{\tilde{m}}(T^i(x)))] \\
&= \sum_{k=1}^{\infty} \tilde{I}_{A_k} T_{\tilde{m}}[\frac{1}{k}(\sum_{i=0}^{k-1} T^i(x))] \\
&= T_{\tilde{m}}\{\sum_{k=1}^{\infty} \tilde{I}_{A_k}[\frac{1}{k}(\sum_{i=0}^{k-1} T^i(x))]\} \\
&= T_{\tilde{m}}[T_{\tilde{n}}(x)].
\end{aligned}$$

Thus, $\{T_{\tilde{n}} : \tilde{n} \in L^0(\mathcal{F}, \mathbb{N})\}$ is a commutative family of $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous L^0-affine mappings from G to G.

Let $S = \cap_{\tilde{n} \in L^0(\mathcal{F},\mathbb{N})} T_{\tilde{n}}(G)$. We claim that S is a nonempty stably compact L^0-convex subset of G and that S is exactly the fixed point set of T.

First, we prove that $S \neq \emptyset$. For any $\tilde{n} \in L^0(\mathcal{F}, \mathbb{N})$, since $T_{\tilde{n}}$ is a $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous L^0-affine mapping from G to G, Lemma 2.17 implies that $T_{\tilde{n}}(G)$ is stably compact and L^0-convex, then $T_{\tilde{n}}(G)$ is σ-stable, closed and L^0-convex. Furthermore, for any $x \in G$, any $\{B_l, l \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ and any sequence $\{\tilde{n}_l, l \in \mathbb{N}\}$

in $L^0(\mathcal{F},\mathbb{N})$ with $\tilde{n}_l := \sum_{k=1}^{\infty}\tilde{I}_{A_{l,k}}k$, we have

$$\begin{aligned}\sum_{l=1}^{\infty}\tilde{I}_{B_l}T_{\tilde{n}_l}(x) &= \sum_{l=1}^{\infty}\tilde{I}_{B_l}\{\sum_{k=1}^{\infty}\tilde{I}_{A_{l,k}}[\frac{1}{k}(\sum_{i=0}^{k-1}T^i(x))]\}\\ &= \sum_{k=1}^{\infty}\{\sum_{l=1}^{\infty}\tilde{I}_{B_l\cap A_{l,k}}[\frac{1}{k}(\sum_{i=0}^{k-1}T^i(x))]\}\\ &= \sum_{k=1}^{\infty}\tilde{I}_{\cup_{l=1}^{\infty}(B_l\cap A_{l,k})}[\frac{1}{k}(\sum_{i=0}^{k-1}T^i(x))]\\ &= T_{\sum_{k=1}^{\infty}\tilde{I}_{\cup_{l=1}^{\infty}(B_l\cap A_{l,k})}k}(x)\\ &= T_{\sum_{l=1}^{\infty}\tilde{I}_{B_l}\tilde{n}_l}(x).\end{aligned}$$

Then, it is easy to check that $\sum_{l=1}^{\infty}\tilde{I}_{B_l}T_{\tilde{n}_l}(G) = T_{\sum_{l=1}^{\infty}\tilde{I}_{B_l}\tilde{n}_l}(G)$, which implies that $\{T_{\tilde{n}}(G) : \tilde{n}\in L^0(\mathcal{F},\mathbb{N})\}$ is a σ-stable family of σ-stable closed subsets of the stably compact set G. By Proposition 2.15, we only need to show that $\{T_{\tilde{n}}(G) : \tilde{n}\in L^0(\mathcal{F},\mathbb{N})\}$ has the finite intersection property. For any $T_{\tilde{n}}(G)$ and $T_{\tilde{m}}(G)$ in $\{T_{\tilde{n}}(G) : \tilde{n}\in L^0(\mathcal{F},\mathbb{N})\}$, it holds that $T_{\tilde{n}}(G)\cap T_{\tilde{m}}(G)\supset T_{\tilde{m}}[T_{\tilde{n}}(G)]\cap T_{\tilde{n}}[T_{\tilde{m}}(G)] = T_{\tilde{m}}[T_{\tilde{n}}(G)]\neq\emptyset$. By induction, it is straightforward to verify that $\{T_{\tilde{n}}(G) : \tilde{n}\in L^0(\mathcal{F},\mathbb{N})\}$ indeed has the finite intersection property. Thus, $S\neq\emptyset$.

Second, we prove that S is the fixed point set of T. Clearly every fixed point of T belongs to S. Conversely, for any $y\in S$ and any $\tilde{n} = \sum_{k=1}^{\infty}\tilde{I}_{A_k}k\in L^0(\mathcal{F},\mathbb{N})$, there exists $x_{\tilde{n}}\in G$ such that $T_{\tilde{n}}(x_{\tilde{n}}) = y$, then

$$\begin{aligned}T(y)-y &= T[T_{\tilde{n}}(x_{\tilde{n}})] - T_{\tilde{n}}(x_{\tilde{n}})\\ &= \sum_{k=1}^{\infty}\tilde{I}_{A_k}[\frac{1}{k}(\sum_{i=0}^{k-1}T^i(T(x)))] - \sum_{k=1}^{\infty}\tilde{I}_{A_k}[\frac{1}{k}(\sum_{i=0}^{k-1}T^i(x))]\\ &= \sum_{k=1}^{\infty}\tilde{I}_{A_k}[\frac{1}{k}(T^k(x)-x)]\\ &\in (\sum_{k=1}^{\infty}\tilde{I}_{A_k}\frac{1}{k})(G-G)\\ &= \frac{1}{\tilde{n}}(G-G).\end{aligned}$$

By the arbitrariness of $\tilde{n}$, we have

$$T(y)-y\in\cap_{\tilde{n}\in L^0(\mathcal{F},\mathbb{N})}\frac{1}{\tilde{n}}(G-G).$$

Since $(E,\mathcal{T}_c(\mathcal{P}_{cc}))$ is a topological group with respect to the addition operation, the mapping $f : E\times E\to E$ defined by $f(x,y) = x-y$ is $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous. Furthermore, according to Theorem 5.8 of [25], $G\times G$ is stably compact, and thus, by Lemma 2.17, $G-G$ is stably compact. Since $\theta\in G-G$, by Lemma 2.18, we have

$$\bigcap_{\tilde{n}\in L^0(\mathcal{F},\mathbb{N})} \frac{1}{\tilde{n}}(G-G)=\theta,$$

which implies that $T(y)=y$.

Finally, since S is the intersection of a family of σ-stable closed L^0-convex subsets of G, it is a σ-stable closed L^0-convex subsets of G. Therefore, S is stably compact and L^0-convex. □

Now, we can complete the proof of the random Markov-Kakutani fixed point theorem.

Theorem 3.4. *Let $(E,\mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$, G a nonempty stably compact L^0-convex subset of E and $\mathcal{M}$ a commutative family of $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous L^0-affine mappings from G to G. Then $\mathcal{M}$ has a common fixed point.*

Proof. By Proposition 3.3, for any $T\in\mathcal{M}$, $Fix(T)=\{x\in G : T(x)=x\}$ is nonempty, σ-stable and closed, then it is easy to check that $\sigma(\{Fix(T) : T\in\mathcal{M}\})$ is a σ-stable family of nonempty σ-stable closed subsets of G. We want to show that $\cap_{T\in\mathcal{M}}Fix(T)\neq\emptyset$, since G is stably compact, by Proposition 2.15 and Lemma2.10, it is enough to show that $\{Fix(T) : T\in\mathcal{M}\}$ has the finite intersection property.

For any T_1 and T_2 in $\mathcal{M}$, we have $T_2(T_1(x))=T_1(T_2(x))=T_1(x)$ for all $x\in Fix(T_2)$, which implies that T_1 maps $Fix(T_2)$ into $Fix(T_2)$. According to Proposition 3.3, T_1 has a fixed point in $Fix(T_2)$, hence $Fix(T_2)\cap Fix(T_1)\neq\emptyset$. By induction, $\{Fix(T) : T\in\mathcal{M}\}$ has the finite intersection property. □

Let X be a linear space over $\mathbb{K}$, a linear operator $f : X\to L^0(\mathcal{F},\mathbb{K})$ is called a random linear functional on X. Let E be an $L^0(\mathcal{F},\mathbb{K})$-module, a module homomorphism $f : E\to L^0(\mathcal{F},\mathbb{K})$ is called an L^0-linear function. If $\mathbb{K}=\mathbb{R}$, a mapping $p : E\to L^0(\mathcal{F},\mathbb{R})$ is called an L^0-sublinear linear function if it satisfies the following:

(i) $p(\xi x)=\xi p(x)$, $\forall\xi\in L^0_+(\mathcal{F})$ and $x\in E$;
(ii) $p(x+y)\leq p(x)+p(y)$, $\forall x,y\in E$.

Following is the algebraic form of the random Hahn-Banach theorem, which is known in random functional analysis.

Theorem 3.5 ([14])**.** *Let E be an $L^0(\mathcal{F},\mathbb{R})$-module, $M\subset E$ a submodule, $f : M\to L^0(\mathcal{F},\mathbb{R})$ an L^0-linear function and $p : E\to L^0(\mathcal{F},\mathbb{R})$ an L^0-sublinear function such that $f(x)\leq p(x)$, $\forall x\in M$. Then there exists an L^0-linear function $F : E\to L^0(\mathcal{F},\mathbb{R})$ such that F extends f and $F(x)\leq p(x)$, $\forall x\in E$. When E is an $L^0(\mathcal{F},\mathbb{C})$-module and $M\subset E$ is a submodule, $f : M\to L^0(\mathcal{F},\mathbb{C})$ an L^0-linear function and $p : E\to L^0_+(\mathcal{F})$ an L^0-seminorm such that $|f(x)|\leq p(x)$, $\forall x\in M$. Then there exists an L^0-linear function $F : E\to L^0(\mathcal{F},\mathbb{C})$ such that F extends f and $|F(x)|\leq p(x)$, $\forall x\in E$.*

Following is the analytic form of the random Hahn-Banach theorem, which is also known in [14] and can be derived from Theorem 3.5.

Theorem 3.6 ([14]). *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ and M be a submodule of E. Then every $f \in M^*_{\varepsilon,\lambda}(\mathcal{P})$ (or $M^*_c(\mathcal{P})$) can be extended to an element F in $E^*_{\varepsilon,\lambda}(\mathcal{P})$ (or $E^*_c(\mathcal{P})$).*

Here, we provide another proof of Theorem 3.5 by using Theorem 3.4. For this, we first give Definition 3.7 and Lemma 3.8.

Definition 3.7. *Let $\{(E_i, \mathcal{P}_i), i \in I\}$ be a family of RLC modules over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. Let $E = \prod_{i\in I} E_i$, then in an apparent way E becomes an $L^0(\mathcal{F}, \mathbb{K})$-module. Fixing $i \in I$, $\|\cdot\|\circ\pi_i$ is an L^0-seminorm on E for any $\|\cdot\| \in \mathcal{P}_i$, where π_i is the canonical projection of E onto E_i. Denote $\tilde{\mathcal{P}} = \bigcup_{i\in I}\{\|\cdot\|\circ\pi_i : \|\cdot\| \in \mathcal{P}_i\}$, then $(E, \tilde{\mathcal{P}})$ is obviously an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, called the product of $\{(E_i, \mathcal{P}_i), i \in I\}$.*

On Definition 3.7, it is easy to check that $\mathcal{T}_c(\tilde{\mathcal{P}})$ (resp., $\mathcal{T}_{\varepsilon,\lambda}(\tilde{\mathcal{P}})$) is just the product topology for $\prod_{i\in I}(E_i, \mathcal{T}_c(\mathcal{P}_i))$ (resp., $\prod_{i\in I}(E_i, \mathcal{T}_{\varepsilon,\lambda}(\mathcal{P}_i))$).

Lemma 3.8. *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, let G be a nonempty σ-stable subset of E, and let $f : G \to G$ be a σ-stable $\mathcal{T}_c(\mathcal{P})$-continuous mapping. Then f is $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous.*

Proof. For any $x \in G$, any $Q \in (\mathcal{P}_{cc})_f$ and any $\varepsilon \in L^0_{++}(\mathcal{F})$, there exist a sequence $\{Q_n, n \in \mathbb{N}\}$ in $\mathcal{P}$ and $\{A_n, n \in \mathbb{N}\} \in \Pi_{\mathcal{F}}$ such that $V_G(f(x), Q, \varepsilon) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} V_G(f(x), Q_n, \varepsilon)$. Then, we have

$$
\begin{aligned}
f^{-1}[V_G(f(x), Q, \varepsilon)] &= f^{-1}[\sum_{n=1}^{\infty} \tilde{I}_{A_n} V_G(f(x), Q_n, \varepsilon)] \\
&= \sum_{n=1}^{\infty} \tilde{I}_{A_n} f^{-1}[V_G(f(x), Q_n, \varepsilon)].
\end{aligned}
$$

Since f is $\mathcal{T}_c(\mathcal{P})$-continuous, each set $f^{-1}[V_G(f(x), Q_n, \varepsilon)]$ is a $\mathcal{T}_c(\mathcal{P})$ neighbourhood of x. Therefore,

$$
\sum_{n=1}^{\infty} \tilde{I}_{A_n} f^{-1}[V_G(f(x), Q_n, \varepsilon)]
$$

is a $\mathcal{T}_c(\mathcal{P}_{cc})$-neighbourhood of x, which completes the proof. □

Theorem 3.9. *Theorem 3.4 implies Theorem 3.5.*

Proof. Let $X = L^0(\mathcal{F}, \mathbb{R})^E$, consider now the sets $X_0 = \prod_{x\in E}[-p(-x), p(x)]$ and $X_1 = \{g \in X_0 : -p(-x) \le g(x + y) - g(y) \le p(x)\ \forall x, y \in E$ and $g(z) = f(z)\ \forall z \in M\}$. Clearly, X_0 is a nonempty stably compact L^0-convex subset of X, and X_1 is a nonempty σ-stable closed L^0-convex subset of X_0, thus X_1 is also a nonempty stably compact L^0-convex subset of X.

Define a family $\{T_y : y \in E\}$ of mappings $T_y : X_1 \to X_1$ by

$$T_y(g)(x) = g(x+y) - g(y),\ \forall g \in X_1,\ \forall x \in E.$$

It is easy to check that $\{T_y : y \in E\}$ is a commutative family of $\mathcal{T}_c(\mathcal{P})$-continuous L^0-affine mappings. Then $\{T_y : y \in E\}$ is a commutative family of $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous L^0-affine mappings by Lemma 3.8. According to Theorem 3.4, there exists $f_1 \in X_1$ such that $T_y(f_1) = f_1$ for all $y \in E$, which implies that $f_1(x+y) = f_1(x) + f_1(y)$ for all $x, y \in E$. Let X_2 be the set of common fixed points of $\{T_y : y \in E\}$, namely, $X_2 = \{f \in X_1 : f(x+y) = f(x) + f(y)\ \forall x, y \in E\}$. Then, X_2 is nonempty, stably compact, and L^0-convex.

Define a family $\{S_r : r \in \mathbb{R} \text{ and } r > 0\}$ of mappings $S_r : X_2 \to X_2$ by

$$S_r(f)(x) = \frac{f(rx)}{r},\ \forall f \in X_2,\ x \in E.$$

It is easy to verify that $\{S_r : r \in \mathbb{R} \text{ and } r > 0\}$ is a commutative family of $\mathcal{T}_c(\mathcal{P}_{cc})$-continuous L^0-affine mappings. According to Theorem 3.4, there exists $F \in X_2$ such that $S_r(F) = F$ for all $r \in \mathbb{R}$ with $r > 0$, which implies that $F(rx) = rF(x)$ for all $r \in \mathbb{R}$ with $r > 0$. It follows from $F(\theta) = f(\theta) = 0$ that $F(rx) = rF(x)$ for all $r \in \mathbb{R}$, and then F is a random linear functional. Furthermore, according to Lemma 2.12 of [14], F is also an L^0-linear function. It is known from [14] that the complex case of Theorem 3.5 can be derived from the real case of it, and thus the proof is completed. □

4. A more general strict separation theorem in a random locally convex module

When K is a singleton, Lemma 4.1 below is just Theorem 4.2 of [23] (see also Lemma 2.9 of [21]). The aim of this section is to establish a more general strict separation theorem in a random locally convex module, namely, Theorem 4.2, which includes Theorem 3.16 of [14] as a special case and is called a geometric form of the random Hahn-Banach theorem. Using Theorem 4.2, we can provide another proof for the random Markov-Kakutani fixed point theorem. The main results in this section are Theorems 4.2 and 4.3.

On the proof of Lemma 4.1, we can, in fact, prove that $K - F$ is $\mathcal{T}_c(\mathcal{P}_{cc})$-closed, then Lemma 4.1 can be derived from Theorem 4.2 of [23], but such a proof needs to construct a $B_{\mathcal{F}}$-stable net from an usual net as done in [18]. In order to avoid the introduction of the abstract notion of a $B_{\mathcal{F}}$-stable net, we give a direct proof of Lemma 4.1 as follows.

Lemma 4.1. *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, K and F be two nonempty subsets of E such that K is stably compact and F is σ-stable and closed. If $\tilde{I}_A K \cap \tilde{I}_A F = \emptyset$ for all $A \in \mathcal{F}_+$, then there exists $V \in \mathcal{V}_\theta(\mathcal{P}_{cc})$ such that*

$$\tilde{I}_A(K+V) \cap \tilde{I}_A(F+V) = \emptyset$$

for all $A \in \mathcal{F}_+$.

Proof. For any $V \in \mathcal{V}_\theta(\mathcal{P}_{cc})$, since $\theta+\theta=\theta$ and addition is continuous under $\mathcal{T}_c(\mathcal{P}_{cc})$, there exists $V_1 \in \mathcal{V}_\theta(\mathcal{P}_{cc})$ such that $V_1+V_1 \subset V$. Then, for any x in K, by Theorem 4.2 of [23], there exists $V_x \in \mathcal{V}_\theta(\mathcal{P}_{cc})$ such that

$$\tilde{I}_A(x+V_x+V_x)\cap \tilde{I}_A(F+V_x)=\emptyset \text{ for all } A\in\mathcal{F}_+. \tag{4.1}$$

It is easy to check that $\sigma\{(x+V_x)\cap K : x\in K\}=\{\sum_{n=1}^{\infty}\tilde{I}_{C_n}[(x_n+V_{x_n})\cap K] : \{C_n, n\in\mathbb{N}\}\in\Pi_{\mathcal{F}}, \{x_n, n\in\mathbb{N}\}$ is a sequene in $K\}$ is a σ-stable family of σ-stable open sets of $(K,\mathcal{T}_c(\mathcal{P}_{cc}))$ with $K=\bigcup\sigma\{(x+V_x)\cap K : x\in K\}$. Since K is stably compact, there exist $\{A_m, m\in\mathbb{N}\}\in\Pi_{\mathcal{F}}$ and a sequence $\{\mathcal{E}_m, m\in\mathbb{N}\}$ of nonempty finite subfamilies of $\sigma\{(x+V_x)\cap K : x\in K\}$ such that $K=\bigcup\sum_{m=1}^{\infty}\tilde{I}_{A_m}\sigma(\mathcal{E}_m)$. For sake of convenience, let $\mathcal{E}_m=\{G_{m,l} : l=1\sim k_m\}$ for any $m\in\mathbb{N}$, with $G_{m,l}=\sum_{n=1}^{\infty}\tilde{I}_{C_{m,l,n}}[(x_{m,l,n}+V_{x_{m,l,n}})\cap K]$ for some sequence $\{x_{m,l,n}, n\in\mathbb{N}\}$ in K and some partition $\{C_{m,l,n}, n\in\mathbb{N}\}\in\Pi_{\mathcal{F}}$. Then, we have

$$\begin{aligned}K&=\bigcup\sum_{m=1}^{\infty}\tilde{I}_{A_m}\sigma(G_{m,1},\cdots,G_{m,k_m})\\&=\bigcup\sum_{m=1}^{\infty}\tilde{I}_{A_m}\sigma[\sum_{n=1}^{\infty}\tilde{I}_{C_{m,1,n}}((x_{m,1,n}+V_{x_{m,1,n}})\cap K),\cdots,\\&\qquad\qquad\sum_{n=1}^{\infty}\tilde{I}_{C_{m,k_m,n}}((x_{m,k_m,n}+V_{x_{m,k_m,n}})\cap K)].\end{aligned}\tag{4.2}$$

For each $x_{m,l,n}$, let $V_{x_{m,l,n}}=\{y\in E : \|y\|_{Q_{m,l,n}}<\varepsilon_{m,l,n} \text{ on } \Omega\}$, and further let

$$\begin{aligned}Q_{m,l}&=\sum_{n=1}^{\infty}\tilde{I}_{C_{m,l,n}}Q_{m,l,n},\\ \varepsilon_{m,l}&=\sum_{n=1}^{\infty}\tilde{I}_{C_{m,l,n}}\varepsilon_{m,l,n},\\ Q_m&=\cup_{l=1}^{k_m}Q_{m,l},\\ \varepsilon_m&=\wedge_{l=1}^{k_m}\varepsilon_{m,l},\\ Q&=\sum_{m=1}^{\infty}\tilde{I}_{A_m}Q_m,\\ \varepsilon&=\sum_{m=1}^{\infty}\tilde{I}_{A_m}\varepsilon_m.\end{aligned}$$

Then $Q\in(\mathcal{P}_{cc})_f$ and $\varepsilon\in L^0_{++}(\mathcal{F})$, which implies that $V:=\{y\in E : \|y\|_Q<\varepsilon \text{ on } \Omega\}\in\mathcal{V}_\theta(\mathcal{P}_{cc})$.

We assert that $\tilde{I}_A(K+V)\cap\tilde{I}_A(F+V)=\emptyset$ for all $A\in\mathcal{F}_+$. Otherwise, there exist $x\in K$ and $A\in\mathcal{F}_+$ such that $\tilde{I}_A(x+V)\cap\tilde{I}_A(F+V)\neq\emptyset$. By (4.2), there exists a sequence $\{\{B_{m,l}, l = 1 \sim k_m\}, m \in \mathbb{N}\}$ in $\Pi_{\mathcal{F}}$ such that

$$x \in \sum_{m=1}^{\infty} \tilde{I}_{A_m}\{\sum_{l=1}^{k_m} \tilde{I}_{B_{m,l}}[\sum_{n=1}^{\infty} \tilde{I}_{C_{m,l,n}}((x_{m,l,n} + V_{x_{m,l,n}}) \cap K)]\}.$$

Therefore,

$$\tilde{I}_A(\sum_{m=1}^{\infty} \tilde{I}_{A_m}\{\sum_{l=1}^{k_m} \tilde{I}_{B_{m,l}}[\sum_{n=1}^{\infty} \tilde{I}_{C_{m,l,n}}((x_{m,l,n} + V_{x_{m,l,n}}) \cap K)]\} + V) \cap \tilde{I}_A(F + V) \neq \emptyset.$$

Since $A \in \mathcal{F}_+$, there exist some $A_m, B_{m,l}$ and $C_{m,l,n}$ such that $A \cap A_m \cap B_{m,l} \cap C_{m,l,n} \in \mathcal{F}_+$, then we have

$$\tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}}[(x_{m,l,n} + V_{x_{m,l,n}}) \cap K + V] \cap \tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}}(F + V) \neq \emptyset,$$

which implies that

$$\tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}}(x_{m,l,n} + V_{x_{m,l,n}} + V) \cap \tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}}(F + V) \neq \emptyset.$$

Furthermore, it is easy to check that

$$\tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}} V \subset \tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}} V_{x_{m,l,n}},$$

then we have

$$\tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}}(x_{m,l,n} + V_{x_{m,l,n}} + V_{x_{m,l,n}}) \cap \tilde{I}_{A\cap A_m \cap B_{m,l}\cap C_{m,l,n}}(F + V_{x_{m,l,n}}) \neq \emptyset,$$

which contradicts with (4.1). □

Let E be an $L^0(\mathcal{F}, \mathbb{K})$-module, M and G two nonempty σ-stable subsets of E such that $M \cap G = \emptyset$. Then, $H(M, G) = \Omega \backslash esssup\{A \in \mathcal{F} : \tilde{I}_A M \cap \tilde{I}_A G \neq \emptyset\}$ is called the hereditarily disjoint stratification of M and G, and $P(H(M, G))$ is called the hereditarily disjoint probability of M and G. According to Theorem 3.13 of [14], $H(M, G)$ is unique a.s. and has the following properties:

(1) $P(H(M, G)) > 0$;
(2) $\tilde{I}_A M \cap \tilde{I}_A G = \emptyset$ for all $A \in \mathcal{F}_+$ with $A \subset H(M, G)$;
(3) $\tilde{I}_A M \cap \tilde{I}_A G \neq \emptyset$ for all $A \in \mathcal{F}_+$ with $A \subset \Omega \backslash H(M, G)$.

Just as we have stated on the proof of Lemma 4.1, Theorem 4.2 below can also be derived from Theorem 3.16 of [14], here we give a direct proof based on Lemma 4.1.

Theorem 4.2. *Let $(E, \mathcal{P})$ be an RLC module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, K and C two L^0-convex subsets of E such that K is stably compact and C is σ-stable and closed. If $K \cap C = \emptyset$, then there exists $f \in E_c^*(\mathcal{P}_{cc})$ such that*

$$(Ref)(x) > (Ref)(y) \text{ for all } x \in K \text{ and } y \in C.$$

and

$$(Ref)(x) > (Ref)(y) \text{ on } H(K, C) \text{ for all } x \in K \text{ and } y \in C.$$

Proof. If $H(K,C) = \Omega$, by Lemma 4.1, there exists $V \in \mathcal{V}_\theta(\mathcal{P}_{cc})$ such that $\tilde{I}_A(K+V) \cap \tilde{I}_A(C+V) = \emptyset$ for all $A \in \mathcal{F}_+$. It is easy to check that $K+V$ is a σ-stable L^0-convex $\mathcal{T}_c(\mathcal{P}_{cc})$-open subset of E. Then, by Theorem 3.15 of [14], there exists $f \in E_c^*(\mathcal{P}_{cc})$ such that $(Ref)(x) > (Ref)(y)$ *on* Ω for all $x \in K+V$ and all $y \in C$, which implies that

$$(Ref)(x) > (Ref)(y) \text{ on } \Omega \text{ for all } x \in K \text{ and } y \in C.$$

If $0 < P(H(K,C)) < 1$, let $\Omega' = H(K,C)$, $\mathcal{F}' = \Omega' \cap \mathcal{F} = \{\Omega' \cap F : F \in \mathcal{F}\}$ and $P' : \mathcal{F}' \to [0,1]$ be defined by $P'(\Omega' \cap F) = \frac{P(\Omega' \cap F)}{P(\Omega')}$. Take $E' = \tilde{I}_{\Omega'}E$, $\mathcal{P}' = \{\|\cdot\|_{E'} : \|\cdot\| \in \mathcal{P}\}$, $K' = \tilde{I}_{\Omega'}K$, $C' = \tilde{I}_{\Omega'}C$, then $(E', \mathcal{P}')$ is an *RLC* module with base $(\Omega', \mathcal{F}', P')$ and K' and C' two L^0-convex subsets of E' such that K' is stably compact and C' is σ-stable and closed. As shown in the proof for the case $H(K,C) = \Omega$, there exists $f' \in (E')_c^*(\mathcal{P}'_{cc})$ such that

$$(Ref)(x) > (Ref)(y) \text{ on } \Omega' \text{ for all } x \in K' \text{ and } y \in C'.$$

According to Theorem 3.6, f' has an extension $f'' \in E_c^*(\mathcal{P}_{cc})$. Now, let $f = I_{\Omega'}f''$, then $f \in E_c^*(\mathcal{P}_{cc})$ and we have

$$(Ref)(x) > (Ref)(y) \text{ for all } x \in K \text{ and } y \in C.$$

and

$$(Ref)(x) > (Ref)(y) \text{ on } H(K,C) \text{ for all } x \in K \text{ and } y \in C.$$

□

Using Theorem 4.2, we can provide another proof for the random Markov-Kakutani fixed point theorem.

Theorem 4.3. *Theorem 4.2 implies Theorem 3.4.*

Proof. We only need to show Theorem 4.2 implies Proposition 3.3. We employ a method by contradiction, suppose that the mapping T in Proposition 3.3 has no fixed point in G. Then, its graph $\Gamma := \{(x,Tx) : x \in G\}$ is disjoint from the diagonal $\Delta := \{(x,x) : x \in G\}$. Obviously, Γ is stably compact and L^0-convex, and Δ is σ-stable, closed and L^0-convex.

Let $(E \times E, \tilde{\mathcal{P}})$ be $(E, \mathcal{P}_{cc}) \times (E, \mathcal{P}_{cc})$, then it is easy to check that $(E \times E)_c^*(\tilde{\mathcal{P}}) = E_c^*(\mathcal{P}_{cc}) \times E_c^*(\mathcal{P}_{cc})$, namely, $f \in (E \times E)_c^*(\tilde{\mathcal{P}})$ iff there exist f_1 and f_2 in $E_c^*(\mathcal{P}_{cc})$ such that $f(x,y) = f_1(x) + f_2(y)$ for any $(x,y) \in E \times E$. Now, by Theorem 4.2, there exist $f_1, f_2 \in E_c^*(\mathcal{P}_{cc})$ and $\xi_1, \xi_2 \in L^0_{++}(\mathcal{F})$ such that

$$(Ref_1)(x) + (Ref_2)(x) < \xi_1 < \xi_2 < (Ref_1)(y) + (Ref_2)(T(y))$$

on $H(\Delta, \Gamma)$ for all $x, y \in G$. Then, we have

$$(Ref_2)(T(y)) - (Ref_2)(y) > \xi_2 - \xi_1$$

on $H(\Delta, \Gamma)$ for all $y \in G$. It follows that

$$(Ref_2)(T^n(y)) - (Ref_2)(y) > n(\xi_2 - \xi_1)$$

on $H(\Delta, \Gamma)$ for any $y \in G$ and $n \in \mathbb{N}$, which contradicts with the a.s boundedness of $(Ref_2)(G)$. □

Remark 4.4. *Theorem 3.9 shows that the random Markov-Kakutani fixed point theorem implies Theorem 3.5 (namely, the algebraic form of the random Hahn-Banach theorem), Theorem 3.6 (namely, the analytic form of the random Hahn-Banach theorem) can be derived from Theorem 3.5, Theorem 4.2 (namely, the geometric form of the random Hahn-Banach theorem) can be derived from Theorem 3.6, and Theorem 4.3 shows that Theorem 4.2 implies the random Markov-Kakutani fixed point theorem. Thus, the random Markov-Kakutani fixed point theorem, the algebraic form and geometric form of the random Hahn-Banach theorem are all equivalent.*

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