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Addendum to "One-parameter isometry groups and inclusions between operator algebras", NYJM 27 (2021), 164–204

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In our original paper [1], in Section 2, we use the Three-Lines Theorem in an imprecise way. We address these deficiencies here, in particular, giving an improvement to Lemma 2.14 which is enough for subsequent applications in our paper. We thank Stefaan Vaes for bringing these issues to our attention.

We use the notation of [1, Section 2]. The Three-Lines Theorem says that given a scalar-valued, *bounded*, regular function $f : S(z) \to \mathbb{C}$, the function $M(y) = \sup_t |f(t+iy)|$ satisfies that $\log M$ is convex. In particular,

$$|f(w)| \leq \max\Big(\sup_t |f(t)|, \sup_t |f(t+z)|\Big) \qquad (w \in S(z)).$$

1. Remark 2.4

Given a norm regular, bounded function $f : S(z) \to E$ with values in a Banach space E, applying the Three-Lines Theorem to functions of the form $\mu \circ f$, as μ varies over the unit ball of E^*, gives a vector-valued version of the Three-Lines Theorem. We tacitly used this in [1, Remark 2.4], but failed to note that the weak*-case is more subtle.

Indeed, let (α_t) be a weak*-continuous isometry group on E, a dual Banach space with predual E_*, again suppose $s > 0$, and let $x \in D(\alpha_{is})$. Then $\|\alpha_t(x)\| = \|x\|$ and $\|\alpha_{t+is}(x)\| = \|\alpha_{is}(x)\|$ for each $t \in \mathbb{R}$. However, a priori, we do not know that $[0,s] \to [0,\infty), r \mapsto \|\alpha_{ir}(x)\|$ is bounded, as this map is not assumed to be continuous in the weak*-case.

A standard corollary of the Uniform Boundedness Theorem, [2, Chapter III, Corollary 14.4], says that $X \subseteq E$ is bounded if (and only if) for each $\mu \in E_*$, we have that $\sup\{|\mu(x)| : x \in X\} < \infty$. We apply this with $X = \{\alpha_{ir}(x) : 0 \leq r \leq s\}$, as for each $\mu \in E_*$, the map $[0,s] \to \mathbb{C}; r \mapsto \mu(\alpha_{ir}(x))$ is continuous, and so bounded. Using that $\|\alpha_{t+ir}(x)\| = \|\alpha_{ir}(x)\|$ for $t \in \mathbb{R}, 0 \leq r \leq s$, it follows that $S(z) \to E; w \mapsto \alpha_w(x)$ is norm bounded, and we can proceed to apply the Three-Lines Theorem to functions of the form $w \mapsto \mu(\alpha_w(x))$, as μ varies over the unit ball of E_*.

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2. Lemma 2.14

A more serious problem applies to [1, Lemma 2.14], as here we apply the Three-Lines Theorem to a semi-norm, and it is far from clear what this actually means.

We shall repair the proof, at least in the situation we are interested in, namely for the σ-strong* topology on a von Neumann algebra M. Similar remarks would apply to the σ-strong topology. Recall that the σ-strong* topology is defined by seminorms of the form $p'_\omega(x) = \omega(xx^* + x^*x)^{1/2}$, for $\omega \in M_*^+$. We recall that the space of linear functionals continuous for the strong*-topology is precisely the predual M_*, see [3, Chapter II, Theorem 2.6] for example. Further, in a general locally convex space with P a defining family of seminorms, a linear functional μ is continuous if and only if there are seminorms $p_1, \cdots, p_n$ in P and $\alpha_1, \cdots, \alpha_n \in [0, \infty)$, with $|\mu(x)| \leq \sum \alpha_i p_i(x)$ for all x, see [2, Chapter IV, Theorem 3.1] for example.

Lemma. *Let $z \in \mathbb{C}$, and let $g : S(z) \to M$ be bounded and weak*-regular. Let $p = p'_\omega$ for some $\omega \in M_*^+$, let $\epsilon > 0$, and suppose that $p(g(t)) \leq \epsilon$ and $p(g(z+t)) \leq \epsilon$ for each $t \in \mathbb{R}$. Then $p(g(w)) \leq \epsilon$ for each $w \in S(z)$.*

Proof. Choose $w \in S(z)$ and set $x = g(w) \in M$. On the one-dimensional subspace $\mathbb{C}x$ define $\mu(\lambda x) = \lambda p(x)$ for $\lambda \in \mathbb{C}$, so μ is linear, and $|\mu(y)| \leq p(y)$ for each y in this subspace. By Hahn-Banach, for example [2, Chapter III, Corollary 6.4], we can extend μ to all of M with $|\mu(y)| \leq p(y)$ for all $y \in M$. Thus μ is continuous for the strong*-topology, and so $\mu \in M_*$. By hypothesis, the map $\mu \circ g : S(z) \to \mathbb{C}$ is bounded and regular, and so the Three-Lines Theorem applies to show that

$$\begin{aligned} |\mu(g(w))| &\leq \max\Big(\sup_t |\mu(g(t))|, |\mu(g(t+z))|\Big) \\ &\leq \max\Big(\sup_t p(g(t)), p(g(t+z))\Big) \leq \epsilon. \end{aligned}$$

Thus $p(g(w)) = p(x) = \mu(x) = \mu(g(w)) \leq \epsilon$ as required. □

This is enough to repair the proof of [1, Lemma 2.14] at least in the case of the strong*-topology, as, in the notation of that lemma, it shows that having $p(g_n - g) \to 0$ uniformly on $\mathbb{R}$ and $\mathbb{R} + z$ is enough to give uniform convergence on all of $S(z)$.

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