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Induced isometric representations

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Abstract. Let σ be an isometric representation of $\mathbb{N}^d$ on a Hilbert space $\mathcal{H}$. We induce σ to an isometric representation V of $\mathbb{R}_+^d$ on another Hilbert space $\mathcal{K}$. We show that the map $\sigma \to V$, restricted to strongly pure isometric representations, preserves index and irreducibility. As an application, we show that, for every $k \in \{0, 1, 2, \cdots\} \cup \{\infty\}$, there is a continuum of prime multiparameter CCR flows with index k.

Contents

1. Introduction

Inducing representations and actions from subgroups is a standard method ([9], [10]) to construct new representations and actions and its importance is well established in representation theory, ergodic theory and in many other branches of mathematics. In this paper, we consider the process of inducing isometric representations from subsemigroups. We only examine a toy model where the semigroup involved is $\mathbb{R}_+^d$ and the subsemigroup involved is $\mathbb{N}^d$.

More precisely, let $\sigma : \mathbb{N}^d \to B(\mathcal{H})$ be an isometric representation. With σ, imitating the group case, we associate an isometric representation V of $\mathbb{R}_+^d$ on another Hilbert space $\mathcal{K}$ as follows: Let

$$\begin{aligned}\mathcal{K} :=&\{\xi : [0,\infty)^d \to \mathcal{H} \,|\, \xi \text{ is measurable, square-integrable over compact sets} \\ &\text{and } \xi(\tilde{x}+\tilde{n}) = \sigma(\tilde{n})\xi(\tilde{x}), \forall \tilde{x} \in [0,\infty)^d, \tilde{n} \in \mathbb{N}^d\}.\end{aligned}$$

Define an inner product $\langle \,|\, \rangle$ on $\mathcal{K}$ by

$$\langle \xi|\eta\rangle := \int_0^1 \int_0^1 \dots \int_0^1 \langle \xi(\tilde{x})|\eta(\tilde{x})\rangle d(\tilde{x})$$

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for all $\xi, \eta \in \mathcal{K}$. As usual, we identify two elements of $\mathcal{K}$ if they agree almost everywhere. Then, $\mathcal{K}$ is a Hilbert space under this inner product. For each $\tilde{t} \in [0,\infty)^d$, define $V_{\tilde{t}} : \mathcal{K} \to \mathcal{K}$ by

$$V_{\tilde{t}}\xi(\tilde{x}) := \xi(\tilde{x} + \tilde{t}).$$

Then, $V := \{V_{\tilde{t}}\}_{\tilde{t} \in \mathbb{R}_+^d}$ is a strongly continuous semigroup of isometries which we call the *isometric representation induced by* σ.

The above construction, in a slightly disguised form, has already appeared in the literature. Under the obvious identification of $\mathcal{K}$ with $L^2([0,1)^d) \otimes \mathcal{H}$, the semigroup $V := \{V_{\tilde{t}}\}_{\tilde{t} \in \mathbb{R}_+^d}$ coincides with the one considered in [6] for $d = 1$ (Lemma 2.4 of [6]) and in [7] for $d \geq 2$. In [7], the author considers the technique of constructing V as an interpolation technique attributed to Bhat and Skeide ([6]). We prefer to refer to the construction as an induced construction as it is exactly analogous to the usual induced construction for group representations.

Before explaining our results, we recall the definition of the index of a semigroup of isometries. Let P be a semigroup. Suppose $V := \{V_x\}_{x \in P}$ is a semigroup of isometries on a Hilbert space $\mathcal{H}$. This means that for every $x \in P$, V_x is an isometry and $V_x V_y = V_{xy}$ for $x, y \in P$. A map $\xi : P \to \mathcal{H}$ is called an additive cocycle of V if

(1) for every $x \in P$, $V_x^* \xi_x = 0$, and
(2) for $x, y \in P$, $\xi_{xy} = \xi_x + V_x \xi_y$.

The space of additive cocycles of V is denoted by $\mathcal{A}(V)$ whose dimension we call the index of V, and we denote it by $Ind(V)$.

Let σ be an isometric representation of $\mathbb{N}^d$, and let V be the corresponding induced representation. We prove that a few properties are preserved when we pass from σ to V. In particular, we show the following.

(1) The index of σ coincides with the index of V.
(2) Let $\{e_1, e_2, \cdots, e_d\}$ be the standard basis for $\mathbb{N}^d$. Suppose $\sigma(e_i)$ is a pure isometry for every $i = 1, 2, \cdots, d$. (Let us agree to call such isometric representations *strongly pure*.) Then, σ is irreducible if and only if V is irreducible.
(3) Let σ_1, σ_2 be two strongly pure irreducible isometric representations of $\mathbb{N}^d$. Denote by V_i the isometric representation induced by σ_i. Then, σ_1 and σ_2 are unitarily equivalent if and only if V_1 and V_2 are unitarily equivalent.

The takeaway from (2) and (3) is that 'enumerating' irreducible isometric representations of $\mathbb{R}_+^d$ is at least as hard as enumerating irreducible isometric representations of $\mathbb{N}^d$. For $d = 2$, it is known from [5] that irreducible isometric representations of $\mathbb{N}^2$, except the one-dimensional ones, are in one-one correspondence with the irreducible unitary representations of the group $\mathbb{Z}_2 * \mathbb{Z}$ whose associated group C^*-algebra is not type I, and consequently its representation theory is quite complicated. Thus, unlike the 1-parameter case, the classification problem of isometric representations of $\mathbb{R}_+^d$, when $d \geq 2$, is quite hard.

Why consider induced isometric representations? The motivation for us comes from a problem in the multiparameter theory of E_0-semigroups which we next explain. Let us recall a few definitions regarding E_0-semigroups. Let P be a closed, convex cone in $\mathbb{R}^d$ that spans $\mathbb{R}^d$ and is pointed, i.e. $P \cap -P = \{0\}$. Let $\mathcal{H}$ be a separable Hilbert space. By an E_0-semigroup, over P, on $B(\mathcal{H})$, we mean a semigroup $\alpha := \{\alpha_x\}_{x \in P}$ of unital normal $*$-endomorphisms of $B(\mathcal{H})$ such that the map

$$P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$$

is continuous for every $A \in B(H)$ and for every $\xi, \eta \in \mathcal{H}$. The equivalence relation on 'the set of E_0-semigroups' that we consider is that of cocycle conjugacy. The first numerical invariant for E_0-semigroups, in the 1-parameter case, is due to Arveson ([3], [4]) and is called the index. Arveson proved that index is a complete invariant for 1-parameter CCR flows. Moreover, index is only relevant for spatial E_0-semigroups.

An E_0-semigroup $\alpha := \{\alpha_x\}_{x \in P}$ on $B(\mathcal{H})$ is said to be *spatial* if it has a unit, by which we mean a strongly continuous semigroup of bounded operators $u := \{u_x\}_{x \in P}$ on $\mathcal{H}$ such that

(1) for $x \in P$, $u_x \neq 0$, and
(2) for $A \in B(\mathcal{H})$, $x \in P$, $\alpha_x(A)u_x = u_x A$.

Following Arveson, in [11] the authors, in the multiparameter case, defined a numerical invariant for a spatial E_0-semigroup α called the index of α which we denote by $Ind(\alpha)$. Roughly, $Ind(\alpha)$ measures 'the number of units' of α.

Arveson, in the 1-parameter case, proved the remarkable fact that index is a 'homomorphism', i.e.

$$Ind(\alpha \otimes \beta) = Ind(\alpha) + Ind(\beta). \tag{1}$$

We mention here that Arveson's proof, without any modification, works in the multiparameter case as well. In view of the above equation, it is quite natural to ask the following question.

Question: Does there exist a prime E_0-semigroup that has index at least two?

Recall that an E_0-semigroup α is said to be *prime* if whenever α is cocycle conjugate to $\beta \otimes \gamma$, where β and γ are E_0-semigroups, then either β or γ is an automorphism group. Affirmative answer to the above question, when $P = [0, \infty)$, was given by Liebscher ([8]) who constructed such examples by probabilistic means. In the multiparameter case, we show that such examples exist even within the class of CCR flows (which are probably the first examples studied in the theory of E_0-semigroups); a total contrast to the one parameter case.

Let us recall the definition of CCR flows. Let $V = \{V_x\}_{x \in P}$ be an isometric representation of P on a Hilbert space $\mathcal{H}$. Let $\Gamma(\mathcal{H})$ denote the symmetric Fock space of $\mathcal{H}$. There exists a unique E_0-semigroup, denoted α^V and called the CCR flow associated to V, on $B(\Gamma(\mathcal{H}))$, such that for all $x \in P$ and $\xi \in \mathcal{H}$,

$$\alpha_x^V(W(\xi)) = W(V_x\xi)$$

where $\{W(\xi)|\xi \in \mathcal{H}\}$ is the collection of Weyl operators on $\Gamma(\mathcal{H})$. We call α^V the *CCR flow associated to V*.

The following facts were proved in [12] and in [11].

(1) For two pure isometric representations V_1 and V_2, the CCR flows α^{V_1} and α^{V_2} are cocycle conjugate if and only if V_1 and V_2 are unitarily equivalent (Thm. 5.2 of [12]).
(2) Let $V = \{V_x\}_{x \in P}$ be a pure isometric representation of P on a Hilbert space $\mathcal{H}$.
   (a) The CCR flow α^V is prime if and only if the representation V is irreducible, i.e. it has no non-trivial closed reducing subspaces (Thm. 7.2 of [12]).
   (b) The index of α^V coincides with that of V (Prop. 2.7 of [11]).

Thus, the problem of constructing prime CCR flows over P with a given index k is equivalent to the problem of constructing irreducible isometric representations of P with index k. Here is where induced isometric representations come into picture. For the discrete semigroup $\mathbb{N}^2$, such examples are available in the literature ([1]), albeit in a slightly different form. We also construct alternate examples. We induce such discrete semigroups of isometries to construct the desired isometric representations, and we prove the following theorem first for $P = \mathbb{R}_+^d$, and show that the general case can be derived from the case of $\mathbb{R}_+^d$.

Theorem 1.1. *Let P be a closed convex cone in $\mathbb{R}^d$ which is spanning and pointed. Suppose that $d \geq 2$. Then, for each $k \in \{0, 1, 2, \cdots, \} \cup \{\infty\}$, there is a continuum of irreducible isometric representations of P that has index k.*

The following theorem is now immediate.

Theorem 1.2. *Let P be a closed convex cone in $\mathbb{R}^d$ which is pointed and spanning. Suppose that $d \geq 2$. Then, for each $k \in \{0, 1, 2, \cdots, \} \cup \{\infty\}$, there is a continuum of prime CCR flows with index k.*

We end this introduction by mentioning that for $k \in \{0, 1\}$, the above theorem is known. For $k = 0$, the CCR flows considered in [2] provide such examples. For $k = 1$, the authors in [11] constructed such examples. We must mention here that the examples constructed in [11] are the first 'genuine' examples of CCR flows/E_0-semigroups, in the multiparameter case, that are type one (which roughly means that there is abundance of units). In [11], the focus was to construct type one examples with index one. On taking tensor product of such examples, we can easily construct type one examples with index greater than one. But this is clearly tautological and this motivated us to seek examples of prime CCR flows with index greater than one.

Notation:- For us, $\mathbb{N}$ stands for the set of natural numbers together with 0. We denote $[0, \infty)$ by $\mathbb{R}_+$. Our convention is that inner products are linear in the first variable.

2. Preliminaries

First, we recall a few definitions that we need. Let G be a locally compact, abelian, second countable, Hausdorfff topological group, and let $P \subseteq G$ be a closed semigroup containing 0 such that $P - P = G$ and $\overline{Int(P)} = P$. Let $\mathcal{H}$ be a separable Hilbert space. Let $V = \{V_x\}_{x \in P}$ be a strongly continuous semigroup of isometries on $\mathcal{H}$. Such a family is also called an *isometric representation* of P on $\mathcal{H}$. We call V a pure isometric representation of P if $\bigcap_{x \in P} Ran(V_x) = \{0\}$. The representation V is said to be *irreducible* if the only closed subspaces of $\mathcal{H}$ invariant under $\{V_x, V_x^* | x \in P\}$ are $\{0\}$ and $\mathcal{H}$.

Let $V = \{V_x\}_{x \in P}$ be an isometric representation of P on a Hilbert space $\mathcal{H}$. Recall that a map $\xi : P \to \mathcal{H}$ is called an *additive cocycle of* V if

(a) for all x in P, $\xi_x \in ker(V_x^*)$, and
(b) for all x, y in P, $\xi_{x+y} = \xi_x + V_x \xi_y$.

The vector space of all additive cocycles of V is denoted by $\mathcal{A}(V)$.

Remark 2.1. *Let $V : P \to B(\mathcal{H})$ be an isometric representation, and let $\xi : P \to \mathcal{H}$ be an additive cocycle. Then, ξ is norm continuous. To see this, let $(s_n)_{n \geq 1}$ be a cofinal sequence in $Int(P)$. Set*

$$E_n := \{x \in P : s_n - x \in Int(P)\}.$$

Note that for $x \in E_n$,

$$\xi_x = (1 - V_x V_x^*) \xi_{s_n}.$$

It follows from the above equality and the fact that $(E_n)_{n \geq 1}$ is an open cover of P that ξ is norm continuous.

Definition 2.2. *For an isometric representation V of P, we define the index of V, denoted $Index(V)$, as the dimension of $\mathcal{A}(V)$.*

Note that if $V = V_1 \oplus V_2$, then:

$$Index(V) = Index(V_1) + Index(V_2).$$

Notation: We define $\mathcal{M}(V) := C^*(\{V_x, V_x^* | x \in P\})$.

Proposition 2.3. *Let G be a locally compact abelian group, and let P be a closed semigroup of G containing 0 such that $P - P = G$. Let $V : P \to B(\mathcal{H})$ be a pure isometric representation on a separable Hilbert space $\mathcal{H}$. Let Q be another closed subsemigroup of G containing 0 such that $Q - Q = G$ and $Q \subset P$. Denote the restriction of V to Q by W. Then,*

(1) *W is pure,*
(2) *$dim(\mathcal{A}(W)) = dim(\mathcal{A}(V))$, and*
(3) *$\mathcal{M}(W)' = \mathcal{M}(V)'$.*

Suppose $V^{(1)}$ and $V^{(2)}$ are two pure isometric representations of P acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Denote the restrictions of $V^{(i)}$ to Q by $W^{(i)}$. Then, $V^{(1)}$ and $V^{(2)}$ are unitarily equivalent if and only if $W^{(1)}$ and $W^{(2)}$ are unitarily equivalent.

Proof. To see that W is pure, let $\xi \in \bigcap_{x\in Q} Ran(W_x)$. Let $a \in P$. Since Q is spanning in G, there exist $x, y \in Q$ such that $a = x - y$, i.e $a + y = x$. Now, there exists η such that $\xi = W_x\eta = V_x\eta = V_a(V_y\eta)$. This implies $\xi \in Ran(V_a)$, for all $a \in P$. However V is pure, and thus $\xi = 0$. Hence, W is pure.

Let $\xi = \{\xi_x\}_{x\in P}$ be an additive cocycle of V. Define, for $y \in Q$,

$$\eta^{\xi}_y := \xi_y.$$

It is straightforward to see that $\eta^{\xi} = \{\eta^{\xi}_y\}_{y\in Q} \in \mathcal{A}(W)$. We claim that the map

$$\mathcal{A}(V) \ni \xi \mapsto \eta^{\xi} \in \mathcal{A}(W)$$

is an isomorphism. To see that it is injective, suppose $\xi \in \mathcal{A}(V)$ is such that $\xi_y = 0$ for all $y \in Q$. Then, for $a \in P$, write $a = x - y$ with $x, y \in Q$, and calculate to observe that

$$0 = \xi_x = \xi_{a+y} = \xi_a + V_a\xi_y = \xi_a.$$

Thus, $\xi = 0$, and hence the map is injective.

For proving it is a surjection, let η be an additive cocycle of W. Note that for any $c, d, \alpha \in Q$, and $b \in P$ such that $b = c - d$,

$$\begin{aligned}
\eta_{c+\alpha} - V_b\eta_{d+\alpha} &= \eta_c + W_c\eta_\alpha - V_b\eta_d - V_bW_d\eta_\alpha \\
&= \eta_c + V_c\eta_\alpha - V_b\eta_d - V_c\eta_\alpha \\
&= \eta_c - V_b\eta_d.
\end{aligned}$$

Thus, for $c, d, \alpha \in Q$, and $b \in P$, if $b = c - d$, then

$$\eta_{c+\alpha} - V_b\eta_{d+\alpha} = \eta_c - V_b\eta_d. \tag{2}$$

Let $a \in P$. Then, there exist $x, y \in Q$ such that $a = x - y$. Define

$$\xi_a := \eta_x - V_a\eta_y.$$

Say there also exist $u, v \in Q$ such that $a = u - v$. This implies $x + v = y + u$, and applying Eq. 2 twice, we get

$$\eta_x - V_a\eta_y = \eta_{x+v} - V_a\eta_{y+v} = \eta_{u+y} - V_a\eta_{v+y} = \eta_u - V_a\eta_v.$$

Thus, ξ_a is well-defined. Also,

$$\begin{aligned}
V_a^*\xi_a &= V_a^*\eta_x - \eta_y \\
&= V_x^*V_y\eta_x - \eta_y = V_x^*(\eta_{x+y} - \eta_y) - \eta_y \\
&= V_x^*(\eta_x + V_x\eta_y - \eta_y) - \eta_y \\
&= \eta_y - V_x^*\eta_y - \eta_y \\
&= -V_x^*\eta_y \\
&= -V_a^*V_y^*\eta_y = 0.
\end{aligned}$$

This shows that $\xi_a \in ker(V_a^*)$. To prove the cocycle nature, let $a, b \in P$. There exist x, y, z such that $a = x - y$ and $b = y - z$. Then,

$$\xi_{a+b} = \eta_x - V_{a+b}\eta_z$$

$$
\begin{aligned}
&= \eta_x - V_a\eta_y + V_a\eta_y - V_{a+b}\eta_z \\
&= \xi_a + V_a(\eta_y - V_b\eta_z) \\
&= \xi_a + V_a\xi_b.
\end{aligned}
$$

Therefore, $\xi = \{\xi_x\}_{x\in P} \in \mathcal{A}(V)$. Also, for $y \in Q$, note that $\xi_y = \eta_y$. Thus, $\eta^\xi = \eta$, and the map is hence a bijection.

Note that for $a \in P$ if $a = x - y$ with $x, y \in Q$, $V_yV_a = V_x$ which in turn implies that $V_a = V_y^*V_x = W_y^*W_x$. Thus, the C^*-algebras generated by $\{V_a | a \in P\}$ and $\{W_x | x \in Q\}$ are the same and hence, $\mathcal{M}(W)' = \mathcal{M}(V)'$.

Let $V^{(1)}$ and $V^{(2)}$ be two pure isometric representations of P acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Denote the restrictions of $V^{(i)}$ to Q by $W^{(i)}$. If $V^{(1)}$ is unitarily equivalent to $V^{(2)}$, it is clear that $W^{(1)}$ is uniatrily equivalent to $W^{(2)}$. Conversely, if $W^{(1)}$ is unitarily equivalent to $W^{(2)}$, there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $W_x^{(1)}U = UW_x^{(2)}$, for all $x \in Q$. Let $a \in P$; there exist $x, y \in Q$ such that $a = x - y$. Then,

$$
\begin{aligned}
V_a^{(1)}U &= W_y^{(1)*}W_x^{(1)}U \\
&= W_y^{(1)*}UW_x^{(2)} \\
&= UW_y^{(2)*}W_x^{(2)} \\
&= UV_a^{(2)}.
\end{aligned}
$$

This proves that $V^{(1)}$ is unitarily equivalent to $V^{(2)}$ iff $W^{(1)}$ is unitarily equivalent to $W^{(2)}$. □

3. Induced isometric representations

Let us recall once again the construction of the induced isometric representation mentioned in the introduction. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Let $d \geq 1$ be an integer. Let $\sigma : \mathbb{N}^d \to B(\mathcal{H})$ be an isometric representation. Let

$$
\begin{aligned}
\mathcal{K} :=\{\xi : [0,\infty)^d \to \mathcal{H} | \xi &\text{ is measurable, square-integrable over compact sets} \\
&\text{and } \xi(\tilde{x}+\tilde{n}) = \sigma(\tilde{n})\xi(\tilde{x}), \forall \tilde{x} \in [0,\infty)^d, \tilde{n} \in \mathbb{N}^d\}
\end{aligned}
$$

Define an inner product $\langle\ |\ \rangle$ on $\mathcal{K}$ by

$$
\langle \xi | \eta \rangle := \int_0^1 \int_0^1 \dots \int_0^1 \langle \xi(\tilde{x}) | \eta(\tilde{x}) \rangle d(\tilde{x})
$$

for all $\xi, \eta \in \mathcal{K}$. It goes without saying that we identify two elements of $\mathcal{K}$ if they agree almost everywhere. Then, $\mathcal{K}$ is a Hilbert space under this inner product. For each $\tilde{t} \in [0,\infty)^d$, define $V_{\tilde{t}} : \mathcal{K} \to \mathcal{K}$ by

$$
V_{\tilde{t}}\xi(\tilde{x}) := \xi(\tilde{x}+\tilde{t}).
$$

It is routine to check that $V := \{V_{\tilde{t}}\}_{\tilde{t}\in\mathbb{R}_+^d}$ is a strongly continuous semigroup of isometries. The representation $V : [0,\infty)^d \to B(\mathcal{K})$ thus obtained is an isometric representation, and we shall call it the *isometric representation induced by* σ.

Let $(t_1,\cdots,t_d) \in \mathbb{R}_+^d$. There exist $n_1,\cdots,n_d \in \mathbb{N}$ such that, for each k, $n_k \leq t_k < n_k + 1$. For each k, let $i_k \in \{n_k, n_k+1\}$. Define

$$I_{i_k} := \begin{cases} [t_k - n_k, 1), & i_k = n_k, \\ [0, t_k - n_k), & i_k = n_k + 1. \end{cases}$$

and

$$J_{i_k} := \begin{cases} [0, 1 - t_k + n_k), & i_k = n_k, \\ [1 - t_k + n_k, 1), & i_k = n_k + 1. \end{cases}$$

Then, one can check that, for any $\xi \in \mathcal{K}$,

$$V^*_{(t_1,\cdots,t_d)}\xi(x_1,\cdots,x_d) = \sigma(i_1,i_2,\cdots,i_d)^*\xi(x_1+i_1-t_1,\cdots,x_d+i_d-t_d),$$

whenever $(x_1,x_2,\cdots,x_d) \in I_{i_1}\times I_{i_2}\times\cdots\times I_{i_d}$.

Also, if $\xi \in ker(V^*_{(t_1,t_2,\cdots,t_d)})$, then $\xi(x_1,x_2,\cdots,x_d) \in ker(\sigma(i_1,i_2,\cdots,i_d)^*)$ whenever $(x_1,x_2,\cdots,x_d) \in J_{i_1}\times J_{i_2}\times\cdots\times J_{i_d}$. Conversely, suppose that $\xi \in L^2([0,1)^d,\mathcal{H})$ is such that $\xi(x_1,x_2,\cdots,x_d) \in ker(\sigma(i_1,i_2,\cdots,i_d)^*)$ whenever $(x_1,x_2,\cdots,x_d) \in J_{i_1}\times J_{i_2}\times\cdots\times J_{i_d}$. Then, $\xi \in ker(V_{(t_1,t_2,\cdots,t_d)})$.

The space $\mathcal{K}$ can be identified with $L^2([0,1)^d)\otimes\mathcal{H}$ via the map

$$\mathcal{K} \ni \xi \mapsto \sum_{n\in\mathbb{N}}(\xi_n\otimes e_n) \in L^2([0,1)^d)\otimes\mathcal{H},$$

where $\xi_n : [0,1)^d \to \mathbb{C}$ is defined by $\xi_n(\tilde{x}) = \langle\xi(\tilde{x})|e_n\rangle$ for $\tilde{x}\in[0,1)^d, n\in\mathbb{N}$. We always use this identification. Under this identification, we obtain

$$V_{\tilde{n}} = 1_{L^2([0,1)^d)}\otimes\sigma(\tilde{n}), \text{for all } \tilde{n}\in\mathbb{N}^d.$$

Remark 3.1. (1) *Note that if σ is a pure isometric representation on $\mathcal{H}$, then V too is a pure isometric representation on $\mathcal{K}$. To see this, consider a pure isometric representation $\sigma : \mathbb{N}^d \to B(\mathcal{H})$ and the corresponding induced representation $V : [0,\infty)^d \to B(\mathcal{K})$. This implies, for all $\tilde{n}\in\mathbb{N}^d$, $V_{\tilde{n}} = 1_{L^2([0,\infty)^d)}\otimes\sigma(\tilde{n})$. Since σ is pure, $\bigcap_{\tilde{n}\in\mathbb{N}^d} Ran(\sigma(\tilde{n})) = \{0\}$. Thus,*

$$\bigcap_{\tilde{t}\in[0,\infty)^d} Ran(V_{\tilde{t}}) \subseteq \bigcap_{\tilde{n}\in\mathbb{N}^d} Ran(V_{\tilde{n}}) = \{0\}.$$

Hence, V is a pure isometric representation.

(2) *Suppose $\sigma^{(1)}$ and $\sigma^{(2)}$ are two pure irreducible isometric representations of $\mathbb{N}^d$ acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and let $V^{(1)}$ and $V^{(2)}$ be the corresponding induced representations. Then, $\sigma^{(1)}$ and $\sigma^{(2)}$ are unitarily equivalent if and only if $V^{(1)}$ and $V^{(2)}$ are unitarily equivalent. This follows from Schur's lemma and the fact that for $i = 1,2$, and $\tilde{n}\in\mathbb{N}^d$, $V^{(i)}_{\tilde{n}} = 1\otimes\sigma^{(i)}_{\tilde{n}}$.*

Let us consider the case when $d = 1$. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Let $\sigma : \mathbb{N} \to B(\mathcal{H})$ be a pure isometric representation on $\mathcal{H}$. Let V be the isometric representation induced by σ on $\mathcal{K}$. Recall that

$$\begin{aligned}\mathcal{K} = \{\xi : [0,\infty) \to \mathcal{H} \,|\, &\xi \text{ is measurable, square-integrable over compact sets,}\\ &\xi(t+n) = \sigma(n)\xi(t), \forall t \geq 0, n \in \mathbb{N}\},\end{aligned}$$

where the inner product is defined by

$$\langle \xi|\eta\rangle = \int_0^1 \langle \xi(t)|\eta(t)\rangle dt.$$

We usually identify $\mathcal{K}$ with $L^2([0,1)) \otimes \mathcal{H}$ and 'move' between the two spaces freely whenever convenient. Recall that, for each $t \geq 0$, $V_t : \mathcal{K} \to \mathcal{K}$ is given by,

$$V_t\xi(x) := \xi(x+t).$$

The representation $V = \{V_t\}_{t\geq 0}$ is pure, since σ is pure. We first calculate V_t^*. Let $t \geq 0$, there exists $n \in \mathbb{N}$ such that $n \leq t < n+1$. Let $\xi, \eta \in \mathcal{K}$. Then,

$$\begin{aligned}
&\langle V_t^*\xi|\eta\rangle = \langle \xi|V_t\eta\rangle\\
&= \int_0^1 \langle \xi(x)|\eta(x+t)\rangle dx\\
&= \int_0^1 \langle \sigma(n)^*\xi(x)|\eta(x+t-n)\rangle dx\\
&= \int_{t-n}^{t-n+1} \langle \sigma(n)^*\xi(x-t+n)|\eta(x)\rangle dx\\
&= \int_{t-n}^{1} \langle \sigma(n)^*\xi(x-t+n)|\eta(x)\rangle dx + \int_1^{1+t-n} \langle \sigma(n)^*\xi(x-t+n)|\eta(x)\rangle dx\\
&= \int_{t-n}^{1} \langle \sigma(n)^*\xi(x-t+n)|\eta(x)\rangle dx\\
&\quad + \int_1^{1+t-n} \langle \sigma(n+1)^*\xi(x-t+n)|\eta(x-1)\rangle dx\\
&= \int_{t-n}^{1} \langle \sigma(n)^*\xi(x-t+n)|\eta(x)\rangle dx\\
&\quad + \int_0^{t-n} \langle \sigma(n+1)^*\xi(x-t+n+1)|\eta(x)\rangle dx
\end{aligned}$$

Thus, we have

$$V_t^*\xi(x) = \begin{cases}\sigma(n+1)^*\xi(x-t+n+1), \text{ for } x \in [0, t-n),\\ \sigma(n)^*\xi(x-t+n), \text{ for } x \in [t-n, 1).\end{cases} \tag{3}$$

Let $0 \leq t < 1$. Note that Eqn. 3 implies that

$$ker(V_t^*) = \Big\{\xi \in L^2((0,1],\mathcal{H}) \,\Big|\, \xi|_{[0,1-t)} = 0 \tag{4}$$

$$\text{and } \xi(x) \in ker(\sigma(1)^*),\ x \in [1-t,1)\Big\} \tag{5}$$

Also, keeping in mind the correspondence $\mathcal{K} \cong L^2((0,1],\mathcal{H})$, for $n \leq t < n+1$ and $\xi \in \mathcal{K}$,

$$(V_t\xi)(x) = \begin{cases} \sigma(n)\xi(x+t-n),\ x \in [0,1-t+n), \\ \sigma(n+1)\xi(x+t-n-1),\ x \in [1-t+n,1). \end{cases} \tag{6}$$

Lemma 3.2. *Let $\eta : \mathbb{N} \to \mathcal{H}$ be a map. Define, for $n \leq t < n+1$, $\xi_t^\eta : [0,1) \to \mathcal{H}$,*

$$\xi_t^\eta(x) := \begin{cases} \eta_n,\ x \in [0,1-t+n), \\ \eta_{n+1},\ x \in [1-t+n,1). \end{cases}$$

Then, $\xi^\eta = \{\xi_t^\eta\}_{t\geq 0}$ is an additive cocycle of V iff $\eta = \{\eta_k\}_{k\in\mathbb{N}}$ is an additive cocycle of σ.

Proof. Assume that ξ^η is an additive cocycle of V. Since $\xi_k^\eta \in ker(V_k^*)$,

$$\sigma(k)^*\eta_k = \sigma(k)^*\xi_k^\eta(x) = V_k^*\xi_k^\eta(x) = 0.$$

Thus, $\eta_k \in ker(\sigma(k)^*)$, for all $k \in \mathbb{N}$. Also,

$$\eta_{k+m} = \xi_{k+m}^\eta(x) = \xi_k^\eta(x) + V_k\xi_m^\eta(x) = \eta_k + \sigma(k)\eta_m.$$

Thus, $\eta = \{\eta_k\}_{k\in\mathbb{N}}$ is an additive cocycle of σ.

Suppose η is an additive cocycle of σ. Then, for $n \leq t < n+1$,

$$\begin{aligned} V_t^*\xi_t^\eta(x) &= \begin{cases} \sigma(n+1)^*\xi_t^\eta(x-t+n+1),\ \text{whenever } x \in [0,t-n), \\ \sigma(n)^*\xi_t^\eta(x-t+n),\ \text{whenever } x \in [t-n,1). \end{cases} \\ &= \begin{cases} \sigma(n+1)^*\eta_{n+1},\ \text{whenever } x \in [0,t-n), \\ \sigma(n)^*\eta_n,\ \text{whenever } x \in [t-n,1). \end{cases} \\ &= 0. \end{aligned}$$

This shows $\xi_t^\eta \in ker(V_t^*)$, for all $t \geq 0$.

To prove the cocycle nature, let $s, t \geq 0$ be given. Choose $m, n \in \mathbb{N}$ such that $m \leq s < m+1$ and $n \leq t < n+1$.Then,

Case (i) $m+n \leq s+t < m+n+1$,
Under this condition,

$$\xi_{s+t}^\eta(x) = \begin{cases} \eta_{m+n},\ x \in [0,1-s-t+m+n), \\ \eta_{m+n+1},\ x \in [1-s-t+m+n,1). \end{cases}$$

Now,

$$\xi_s^\eta(x) + V_s\xi_t^\eta(x)$$

$$= \begin{cases} \xi_s^\eta(x) + \sigma(m)\xi_t^\eta(x+s-m), \ x \in [0, 1-s+m), \\ \xi_s^\eta(x) + \sigma(m+1)\xi_t^\eta(x+s-m-1), \ x \in [1-s+m, 1). \end{cases}$$

$$= \begin{cases} \eta_m + \sigma(m)\eta_n, \ x \in [0, 1-s+m) \cap [0, 1-s-t+m+n), \\ \eta_m + \sigma(m)\eta_{n+1}, \ x \in [0, 1-s+m) \cap [1-s-t+m+n, 1), \\ \eta_{m+1} + \sigma(m+1)\eta_n, \ x \in [1-s+m, 1) \cap [0, 1). \end{cases}$$

$$= \begin{cases} \eta_{m+n}, \ x \in [0, 1-s-t+m+n), \\ \eta_{m+n+1}, \ x \in [1-s-t+m+n, 1). \end{cases}$$

$$= \xi_{s+t}^\eta(x).$$

Case (ii) $m+n+1 \leq s+t < m+n+2$. The verification in this case is similar.

Thus, $\xi^\eta = \{\xi_t^\eta\}_{t \geq 0}$ is an additive cocycle of V whenever η is an additive cocycle of σ. □

Let $\mathcal{A}(\sigma)$ and $\mathcal{A}(V)$ denote the space of additive cocycles of σ and V respectively.

Proposition 3.3. *The map*

$$\mathcal{A}(\sigma) \ni \eta \mapsto \xi^\eta \in \mathcal{A}(V)$$

is an isomorphism. Hence, $Ind(\sigma) = Ind(V)$.

Proof. The map is clearly injective. To see that it is onto, let $\xi = \{\xi_t\}_{t \geq 0}$ be a non-zero additive cocycle of V. Recall that V acts on $\mathcal{K}$ where $\mathcal{K}$ is given by

$$\mathcal{K} = \{\xi : [0, \infty) \to \mathcal{H} | \xi \text{ is measurable, square-integrable over compact sets,} \\ \xi(t+n) = \sigma(n)\xi(t), \forall t \geq 0, n \in \mathbb{N}\},$$

The fact that ξ is an additive cocylce implies that for all $s, t \geq 0$,

$$\xi_{s+t}(x) = \xi_s(x) + \xi_t(x+s)$$

for almost all x. By Theorem 5.3.2 of [4], there exists $f \in L^2_{loc}([0, \infty), \mathcal{H})$, such that for every t,

$$\xi_t(x) = f(x+t) - f(x), \tag{7}$$

for almost all x. However, $\xi_t \in ker(V_t^*)$, and hence, by Eqn. 4, for $0 \leq t < 1$, $\xi_t(x) = 0$, for almost all $x \in [0, 1-t)$. This implies, for every $t \in [0, 1)$, $f(x+t) = f(x)$, for almost all $x \in [0, 1-t)$. Define $f_n : [0, \infty) \to \mathbb{C}$ by $f_n(x) := \langle f(x) | e_n \rangle$, for all $n \in \mathbb{N}$. Then, for every $t \in [0, 1)$, $f_n(x+t) = f_n(x)$ for almost all $x \in [0, 1-t)$. Thus, the distributional derivative of f_n is zero on $(0, 1)$. Hence, the function f_n is constant on the interval $[0, 1)$, and thereby, so is the function f

Therefore, there exists a vector $\gamma \in \mathcal{H}$, such that

$$f(x) = \gamma \tag{8}$$

for almost all $x \in [0, 1)$.

Let $\phi : [0,\infty) \to \mathcal{H}$ be defined by

$$\phi(x) = f(x+1) - \sigma(1)f(x).$$

For every $t > 0$, since $\xi_t(x+1) = \sigma(1)\xi_t(x)$ for almost all x, it follows that given $t > 0$,

$$f(x+t+1) - f(x+1) = \sigma(1)(f(x+t) - f(x))$$

for almost all $x \in [0,\infty)$. In other words, given $t > 0$, $\phi(x+t) = \phi(x)$ for almost all $x \in [0,\infty)$. This forces that ϕ is constant. Let $\widetilde{\gamma} \in \mathcal{H}$ be such that $\phi(x) = \widetilde{\gamma}$ for almost all $x \in [0,\infty)$. It follows from Eq. 8, that for almost all $x \in [0,1)$,

$$f(x+1) = \sigma(1)f(x) + \widetilde{\gamma} = \sigma(1)\gamma + \widetilde{\gamma} \tag{9}$$

for almost all $x \in [0,1)$. Set $\eta_1 := \sigma(1)\gamma + \widetilde{\gamma} - \gamma$.

Combining Eq. 7, Eq. 8 and Eq. 9, we have, for $0 \le t < 1$,

$$\xi_t(x) = \begin{cases} 0, & x \in [0,1-t), \\ \eta_1, & x \in [1-t,1). \end{cases}$$

By Eqn. 4, we have $\eta_1 \in ker(\sigma(1)^*)$. Set $\eta_{k+1} = \eta_k + \sigma(k)\eta_1$, for all $k \ge 1, k \in \mathbb{N}$, and $\eta_0 = 0$. This makes $\eta = \{\eta_k\}_{k\in\mathbb{N}}$ an additive cocycle of σ. Therefore, ξ^η is an additive cocycle of V. Also,

$$\xi_t = \xi_t^\eta$$

for $0 \le t < 1$. Since ξ and ξ^η are additive cocycles, $\xi = \xi^\eta$. This proves that the map is indeed a bijection. □

Remark 3.4. *Let $W = \{W_t\}_{t\ge 0}$ be a pure isometric representation on a separable Hilbert space $\mathcal{L}$. Then, $\{W_t\}_{t\ge0}$ is unitarily equivalent to $\{S_t \otimes 1\}_{t\ge0}$ on $L^2([0,\infty)) \otimes \mathcal{H}_0$, for some separable Hilbert space $\mathcal{H}_0$. Here, $\{S_t\}_{t\ge0}$ is the shift semigroup on $L^2([0,\infty))$. Moreover, $dim(\mathcal{A}(W)) = dim(\mathcal{H}_0)$. The last equality can be proved directly, or by appealing to the index computation, due to Arveson, of 1-parameter CCR flows and the fact that for CCR flows, $Ind(\alpha^W) = dim(\mathcal{A}(W))$. Thus, W is irreducible if and only if $dim(\mathcal{A}(W)) = 1$.*

Recall that

$$\mathcal{M}(\sigma) = C^*(\{\sigma(1)\}),$$
$$\mathcal{M}(V) = C^*(\{V_t | t \ge 0\})$$

and denote their commutants by $\mathcal{M}(\sigma)'$ and $\mathcal{M}(V)'$.

Proposition 3.5. *With the foregoing notation, $\mathcal{M}(V)' = 1_{L^2([0,1))} \otimes \mathcal{M}(\sigma)'$.*

Proof. Define $\tilde{\sigma} : \mathbb{N} \to B(\ell^2(\mathbb{N}))$ by $\tilde{\sigma}(n) = S^n$, where $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is the unilateral shift operator. Then, $\tilde{\sigma}$ is a pure isometric representation on $\ell^2(\mathbb{N})$. Let $\tilde{V} : [0,\infty) \to B(L^2([0,1)) \otimes \ell^2(\mathbb{N}))$ be the pure isometric representation induced by $\tilde{\sigma}$. Clearly, $dim(\mathcal{A}(\tilde{\sigma})) = 1$. By Prop. 3.3, we have $dim(\mathcal{A}(\tilde{V})) = 1$, and consequently, $\tilde{V}$ is irreducible, and

$$\mathcal{M}(\tilde{V})' = \mathbb{C}1_{L^2([0,1))} \otimes 1_{\ell^2(\mathbb{N})}. \tag{10}$$

Let $\sigma : \mathbb{N} \to B(\mathcal{H})$ be a pure isometric representation. Since σ is pure, by Wold decomposition, there exists a Hilbert space $\mathcal{H}_0$ and a unitary $U : \ell^2(\mathbb{N}) \otimes \mathcal{H}_0 \to \mathcal{H}$ such that

$$\sigma(1) = U(S \otimes 1)U^*.$$

So, we can assume that up to a unitary equivalence, $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \mathcal{H}_0$, for some Hilbert space $\mathcal{H}_0$, and $\sigma(1) = S \otimes 1 = \tilde{\sigma}(1) \otimes 1$, where $1 \in B(\mathcal{H}_0)$ is the identity on $\mathcal{H}_0$. Let V be the induced representation due to σ. We thus have

$$\begin{aligned} \mathcal{M}(\sigma)' &= \{1_{\ell^2(\mathbb{N})} \otimes T | T \in B(\mathcal{H}_0)\}, \\ V_t &= \tilde{V}_t \otimes 1, \text{ and} \\ \mathcal{M}(V)' &= \{1_{L^2([0,1))} \otimes 1_{\ell^2(\mathbb{N})} \otimes T | T \in B(\mathcal{H}_0)\} \text{ (by Eq. 10)} \\ &= 1_{L^2([0,1))} \otimes \mathcal{M}(\sigma)'. \end{aligned}$$

The proof is complete. □

Now, let $d \geq 2$. Let $\sigma : \mathbb{N}^d \to B(\mathcal{H})$ be an isometric representation, and let V be the associated induced isometric representation. For a map $\eta : \mathbb{N}^d \to \mathcal{H}$, we define $\xi^\eta : \mathbb{R}_+^d \to L^2([0,1)^d, \mathcal{H})$ in the following manner: let $(t_1, t_2, \cdots, t_d) \in \mathbb{R}_+^d$. Then, there exist $n_1, n_2, \cdots, n_d \in \mathbb{N}$ such that $n_k \leq t_k < n_k + 1$, for all $1 \leq k \leq d$. Define,

$$\xi^\eta_{(t_1,\cdots,t_d)}(x_1, \cdots, x_d) := \eta_{(i_1,\cdots,i_d)},$$

whenever $(x_1, x_2, \cdots, x_d) \in J_{i_1} \times J_{i_2} \times \cdots \times J_{i_d}$. Recall that

$$J_{i_k} := \begin{cases} [0, 1 - t_k + n_k), \ i_k = n_k, \\ [1 - t_k + n_k, 1), \ i_k = n_k + 1. \end{cases}$$

With the foregoing notation, we have the following theorem.

Theorem 3.6. *Assume that σ is strongly pure. Then,*

(1) *The representation V is pure.*
(1) *The map $\mathcal{A}(\sigma) \ni \eta \mapsto \xi^\eta \in \mathcal{A}(V)$ is an isomorphism.*
(2) $\mathcal{M}(V)' = 1_{L^2([0,1)^d)} \otimes \mathcal{M}(\sigma)'$.

In particular, $Index(\sigma) = Index(V)$. Also, V is irreducible if and only if σ is irreducible.

Proof. We prove this by induction. The case $d = 1$ follows from Prop. 3.3 and Prop. 3.5. Assume that the theorem holds for d. Let $\sigma : \mathbb{N}^{d+1} \to B(\mathcal{H})$ be a strongly pure isometric representation. Define

$$\sigma^{(1)} : \mathbb{N}^d \to B(\mathcal{H}), \sigma^{(1)}(\tilde{m}) := \sigma(\tilde{m}, 0),$$

$$\sigma^{(2)} : \mathbb{N} \to B(\mathcal{H}), \sigma^{(2)}(n) := \sigma(\tilde{0}, n).$$

Then, for all $(\tilde{m}, n) \in \mathbb{N}^{d+1}$, $\sigma(\tilde{m}, n) = \sigma^{(1)}(\tilde{m})\sigma^{(2)}(n) = \sigma^{(2)}(n)\sigma^{(1)}(\tilde{m})$. Note that $\sigma^{(1)}$ and $\sigma^{(2)}$ are strongly pure. Let $V^{(1)} : \mathbb{R}_+^d \to B(L^2([0,1)^d, \mathcal{H}))$ and $V^{(2)} : \mathbb{R}_+ \to B(L^2([0,1), \mathcal{H}))$ be the respective induced isometric representation. Define

$$U : L^2([0,1)^{d+1}, \mathcal{H}) \to L^2([0,1)^{d+1}, \mathcal{H}),$$

$$U\xi(x_1, x_2, \cdots, x_{d+1}) := \xi(x_2, x_3, \cdots, x_{d+1}, x_1)$$

for all $\xi \in L^2([0,1)^{d+1}, \mathcal{H})$. This implies,

$$U(1_{L^2([0,1))} \otimes V^{(1)}_{\tilde{s}})U^* = V_{(\tilde{s},0)},\ \forall \tilde{s} \in \mathbb{R}^d_+,$$

$$1_{L^2([0,1)^d)} \otimes V^{(2)}_t = V_{(\tilde{0},t)},\ \forall t \in \mathbb{R}_+$$

Thus, by our assumption, we have

$$\begin{aligned} C^*\{V_{(\tilde{s},0)} | \tilde{s} \in \mathbb{R}^d_+\}' &= U(B(L^2([0,1))) \otimes 1_{L^2([0,1)^d)} \otimes \mathcal{M}(\sigma^{(1)})')U^* \\ &= 1_{L^2([0,1)^d)} \otimes B(L^2([0,1))) \otimes \mathcal{M}(\sigma^{(1)})' \\ C^*\{V_{(\tilde{0},t)} | t \geq 0\}' &= B(L^2([0,1)^d)) \otimes 1_{L^2([0,1))} \otimes \mathcal{M}(\sigma^{(2)})'. \end{aligned}$$

However, $\mathcal{M}(V)' = C^*\{V_{(\tilde{s},0)} | \tilde{s} \in \mathbb{R}^d_+\}' \cap C^*\{V_{(\tilde{0},t)} | t \geq 0\}'$. Therefore,

$$\begin{aligned} \mathcal{M}(V)' &= 1_{L^2([0,1)^d)} \otimes 1_{L^2([0,1))} \otimes (\mathcal{M}(\sigma^{(1)})' \cap \mathcal{M}(\sigma^{(2)})') \\ &= 1_{L^2([0,1)^{d+1})} \otimes \mathcal{M}(\sigma)'. \end{aligned}$$

Let $\eta : \mathbb{N}^{d+1} \to \mathcal{H}$ be a map. Recall that $\xi^\eta : \mathbb{R}^{d+1}_+ \to L^2([0,1)^{d+1}, \mathcal{H})$ is defined as follows: let $(t_1, t_2, \cdots, t_{d+1}) \in \mathbb{R}^{d+1}_+$, and let $n_1, n_2, \cdots, n_{d+1}$ be non-negative integers such that $n_k \leq t_k < n_k + 1$, for all $1 \leq k \leq d+1$. Then,

$$\xi^\eta_{(t_1,\cdots,t_{d+1})}(x_1, \cdots, x_{d+1}) := \eta_{(i_1,\cdots,i_{d+1})},$$

whenever $(x_1, x_2, \cdots, x_{d+1}) \in J_{i_1} \times J_{i_2} \times \cdots \times J_{i_{d+1}}$. Then, ξ^η is an additive cocycle of V iff η is an additive cocycle of σ and the proof is similar to Lemma 3.2.

We now claim that the map

$$\mathcal{A}(\sigma) \ni \eta \mapsto \xi^\eta \in \mathcal{A}(V)$$

is an isomorphism. This map is clearly injective.

Let $\xi \in \mathcal{A}(V)$. Then, $\{\xi_{(\tilde{0},t)}\}_{t\geq 0}$ is an additive cocycle of the 1-parameter isometric representation $\{V_{(\tilde{0},t)}\}_{t\geq 0} = \{1 \otimes V^{(2)}_t\}_{t \geq 0}$. Thus, there exists $f \in L^2([0,1)^d)$ and an additive cocycle $\xi^{(2)}$ of $V^{(2)}$ such that $\xi_{(\tilde{0},t)} = f \otimes \xi^{(2)}_t$, i.e.

$$\xi_{(\tilde{0},t)}(x_1, \cdots, x_{d+1}) = f(x_1, \cdots, x_d)\xi^{(2)}_t(x_{d+1}).$$

By Prop. 3.3, there exists a unique $\eta^{(2)} \in \mathcal{A}(\sigma^{(2)})$ such that $\xi^{(2)} = \xi^{(\eta^{(2)})}$. Therefore, for $n \leq t < n+1$,

$$\xi_{(\tilde{0},t)}(x_1, \cdots, x_{d+1}) = \begin{cases} f(x_1, \cdots, x_d)\eta^{(2)}_n, & \text{if } x_{d+1} \in [0, 1-t+n), \\ f(x_1, \cdots, x_d)\eta^{(2)}_{n+1}, & \text{if } x_{d+1} \in [1-t+n, 1). \end{cases}$$

Similarly, $\{\xi_{(\tilde{s},0)}\}_{\tilde{s}\in\mathbb{R}^d_+}$ is an additive cocycle of the isometric representation $\{V_{(\tilde{s},0)}\}_{\tilde{s}\in\mathbb{R}^d_+} = \{U(1_{L^2([0,1))} \otimes V^{(1)}_{\tilde{s}})U^*\}_{\tilde{s}\in\mathbb{R}^d_+}$. This implies that $\{U^*\xi_{(\tilde{s},0)}\}_{\tilde{s}\in\mathbb{R}^d_+}$ is an additive cocycle of $\{1_{L^2([0,1))} \otimes V^{(1)}_{\tilde{s}}\}_{\tilde{s}\in\mathbb{R}^d_+}$. Therefore, there exists $g \in L^2([0,1))$ and $\xi^{(1)} \in \mathcal{A}(V^{(1)})$ such that

$$U^*\xi_{(\tilde{s},0)}(x_1, \cdots, x_{d+1}) = (g \otimes \xi^{(1)}_{\tilde{s}})(x_1, \cdots, x_{d+1}),$$

and hence, $\xi_{(\tilde{s},0)}(x_1,\cdots,x_{d+1}) = g(x_{d+1})\xi^{(1)}_{\tilde{s}}(x_1,\cdots,x_d)$.

By the induction hypothesis, there exists $\eta^{(1)} \in \mathcal{A}(\sigma^{(1)})$ such that, for $m_k \leq s_k < m_k + 1$,

$$\xi_{(s_1,\cdots,s_d,0)}(x_1,\cdots,x_{d+1}) = g(x_{d+1})\eta^{(1)}_{(i_1,\cdots,i_d)}$$

whenever $(x_1,\cdots,x_d) \in J_{i_1}\times\cdots\times J_{i_d}$ and $x_{d+1} \in [0,1)$, where $i_k \in \{m_k, m_k+1\}$.

Let $0 \leq t_k < 1$, for $1 \leq k \leq d+1$. Then, $i_k \in \{0,1\}$, for $1 \leq k \leq d+1$.

$$\xi_{(t_1,\cdots,t_d,0)}(x_1,\cdots,x_{d+1}) = g(x_{d+1})\eta^{(1)}_{(i_1,\cdots,i_d)},$$

whenever $(x_1,\cdots,x_d) \in J_{i_1}\times\cdots\times J_{i_d}$ and $x_{d+1} \in [0,1)$.

$$\xi_{(0,\cdots,0,t_{d+1})}(x_1,\cdots,x_{d+1}) = \begin{cases} 0, \ (x_1,\cdots,x_{d+1}) \in [0,1)^d \times [0,1-t_{d+1}), \\ f(x_1,\cdots,x_d)\eta^{(2)}_1, \ (x_1,\cdots,x_{d+1}) \in [0,1)^d \times [1-t_{d+1},1). \end{cases}$$

Since ξ is an additive cocycle, we get the following.

$$\begin{aligned} &\xi_{(t_1,\cdots,t_{d+1})}(x_1,\cdots,x_{d+1}) \\ &= \xi_{(t_1,\cdots,t_d,0)}(x_1,\cdots,x_{d+1}) + V_{(t_1,\cdots,t_d,0)}\xi_{(0,\cdots,0,t_{d+1})}(x_1,\cdots,x_{d+1}) \\ &= \begin{cases} g(x_{d+1})\eta^{(1)}_{(i_1,\cdots,i_d)}, \ (x_1,\cdots,x_d,x_{d+1}) \in J_{i_1}\times\cdots\times J_{i_d}\times J_0, \\ g(x_{d+1})\eta^{(1)}_{(i_1,\cdots,i_d)} + \sigma(i_1,\cdots,i_d,0)f(x_1+t_1-i_1,\cdots,x_d+t_d-i_d)\eta^{(2)}_1, \\ (x_1,\cdots,x_d,x_{d+1}) \in J_{i_1}\times\cdots\times J_{i_d}\times J_1, \end{cases} \end{aligned}$$

Again,

$$\begin{aligned} &\xi_{(t_1,\cdots,t_{d+1})}(x_1,\cdots,x_{d+1}) \\ &= \xi_{(0,\cdots,0,t_{d+1})}(x_1,\cdots,x_{d+1}) + V_{(0,\cdots,0,t_{d+1})}\xi_{(t_1,\cdots,t_d,0)}(x_1,\cdots,x_{d+1}) \\ &= \begin{cases} g(x_{d+1}+t_{d+1})\eta^{(1)}_{(i_1,\cdots,i_d)}, \ (x_1,\cdots,x_d,x_{d+1}) \in J_{i_1}\times\cdots\times J_{i_d}\times J_0, \\ \sigma(0,\cdots,0,1)g(x_{d+1}+t_{d+1}-1)\eta^{(1)}_{(i_1,\cdots,i_d)} + f(x_1,\cdots,x_d)\eta^{(2)}_1, \\ (x_1,\cdots,x_d,x_{d+1}) \in J_{i_1}\times\cdots\times J_{i_d}\times J_1, \end{cases} \end{aligned}$$

Combining both, we get, for each $(i_1,\cdots,i_d)$, given $0 \leq t_{d+1} < 1$,

$$g(y)\eta^{(1)}_{(i_1,\cdots,i_d)} = g(y+t_{d+1})\eta^{(1)}_{(i_1,\cdots,i_d)},$$

for all almost all $y \in [0,1-t_{d+1})$. Similarly, when $(i_1,\cdots,i_d) = (0,\cdots,0)$, given $t_1,\cdots,t_d \in [0,1)$, we have, for almost all $x_k \in [0,1-t_k)$,

$$f(x_1,\cdots,x_d)\eta^{(2)}_1 = f(x_1+t_1,\cdots,x_d+t_d)\eta^{(2)}_1,$$

Thus, $f\eta^{(2)}_1$ is constant on $[0,1)^d$ and, for each $i_1,\cdots,i_d) \in \{0,1\}$, $g\eta^{(1)}_{(i_1,\cdots,i_d)}$ is constant on $[0,1)$, i.e, there exist $c_{(i_1,\cdots,i_d)}, d_0 \in \mathbb{C}$ such that $f(x_1,\cdots,x_d)\eta^{(2)}_1 =$

$d_0\eta_1^{(2)}$ almost everywhere on $[0,1)^d$, and $g(y)\eta^{(1)}_{(i_1,\cdots,i_d)} = c_{(i_1,\cdots,i_d)}\eta^{(1)}_{(i_1,\cdots,i_d)}$ for almost all $y \in [0,1)$. Hence, for $n_k \leq t_k < n_k+1$, $k = 1,2,\cdots,d+1$, we have, for $(x_1,x_2,\cdots,x_{d+1}) \in J_{i_1} \times \cdots \times J_{i_d} \times [0,1)$,

$$\xi_{(t_1,\cdots,t_d,0)}(x_1,\cdots,x_{d+1}) = c_{(i_1-n_1,\cdots,i_d-n_d)}\eta^{(1)}_{(i_1,\cdots,i_d)}.$$

Also,

$$\xi_{(0,\cdots,0,t)}(x_1,\cdots,x_{d+1}) = \begin{cases} d_0\eta^{(2)}_{n_{d+1}}, & (x_1,\cdots,x_{d+1}) \in [0,1)^d \times J_{n_{d+1}}, \\ d_0\eta^{(2)}_{n_{d+1}+1}, & (x_1,\cdots,x_{d+1}) \in [0,1)^d \times J_{n_{d+1}+1}. \end{cases}$$

Denote $c_0 = c_{(0,\cdots,0)}$ and define

$$\eta_{(n_1,\cdots,n_{d+1})} := c_0\eta^{(1)}_{(n_1,\cdots,n_d)} + \sigma(n_1,\cdots,n_d,0)d_0\eta^{(2)}_{n_{d+1}},$$

for all $n_i \in \mathbb{N}$. It is clear that $\eta_{(n_1,\cdots,n_{d+1})} \in ker(\sigma(n_1,\cdots,n_{d+1})^*)$. For any $\tilde{m} \in \mathbb{N}^d$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\eta_{(\tilde{m},0)} + \sigma(\tilde{m},0)\eta_{(\tilde{0},n)} \\
&= c_0\eta^{(1)}_{\tilde{m}} + \sigma(\tilde{0},n)d_0\eta^{(2)}_0 + \sigma(\tilde{m},0)(c_0\eta^{(1)}_{\tilde{0}} + \sigma(\tilde{0},0)d_0\eta^{(2)}_n) \\
&= c_0\eta^{(1)}_{\tilde{m}} + \sigma(\tilde{m},0)d_0\eta^{(2)}_n \\
&= \xi_{(\tilde{m},0)}(x_1,\cdots,x_{d+1}) + \sigma(\tilde{m},0)\xi_{(\tilde{0},n)}(x_1,\cdots,x_{d+1}) \\
&= \xi_{(\tilde{m},n)}(x_1,\cdots,x_{d+1}) \\
&= \xi_{(\tilde{0},n)}(x_1,\cdots,x_{d+1}) + \sigma(\tilde{0},n)\xi_{(\tilde{m},0)}(x_1,\cdots,x_{d+1}) \\
&= d_0\eta^{(2)}_n + \sigma(\tilde{0},n)c_0\eta^{(1)}_{\tilde{m}} \\
&= (c_0\eta^{(1)}_{\tilde{0}} + \sigma(\tilde{0},0)d_0\eta^{(2)}_n) + \sigma(\tilde{0},n)(c_0\eta^{(1)}_{\tilde{m}} + \sigma(\tilde{m},0)d_0\eta^{(2)}_0) \\
&= \eta_{(\tilde{0},n)} + \sigma(\tilde{0},n)\eta_{(\tilde{m},0)}.
\end{aligned}$$

This suffices to prove that η is an additive cocycle of σ. But, this also implies that ξ^η is an additive cocycle of V. Since ξ and ξ^η are additive cocycles, and $\xi_{(\tilde{s},0)} = \xi^\eta_{(\tilde{s},0)}$, $\xi_{(\tilde{0},t)} = \xi^\eta_{(\tilde{0},t)}$ for $\tilde{s} \in \mathbb{R}^d_+, t \geq 0$, it follows that $\xi = \xi^\eta$. Thus, the map $\mathcal{A}(\sigma) \ni \eta \to \xi^\eta \in \mathcal{A}(V)$ is a bijection and that concludes our proof. □

Remark 3.7. *Let $\sigma : \mathbb{N}^d \to B(\mathcal{H})$ be an isometric representation. For $k \in \{1,2,\cdots,d\}$, define isometric representations $\sigma^{(k)}$ of $\mathbb{N}$ on $\mathcal{H}$ by setting*

$$\sigma^{(k)}(m) := \sigma(0,\cdots,0,m,0,\cdots,0),$$

with m in the k^{th} position. Recall that the representation σ is called strongly pure if $\sigma^{(k)}$ is a pure isometric representation, for all $1 \leq k \leq d$.

Suppose σ is a pure isometric representation, not necessarily strongly pure, of $\mathbb{N}^d$ on $\mathcal{H}$. Let $a_1,a_2,\cdots,a_d \in \mathbb{N}^d$ be order units for $\mathbb{Z}^d$, i.e., for every $x \in \mathbb{Z}^d$, there exist $m_1,m_2,\cdots,m_d \in \mathbb{N}$ such that $m_ka_k - x \in \mathbb{N}^d$, for all k. Define $\tilde{\sigma}$:

$\mathbb{N}^d \to B(\mathcal{H})$ *by setting* $\tilde{\sigma}(0,\cdots,0,1,0,\cdots,0) := \sigma(a_k)$ *whenever* 1 *is at the* k^{th} *position. Since* σ *is a pure isometric representation and* a_k *is an order unit,*

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} Ran(\tilde{\sigma}(0,\cdots,n,\cdots,0)) &= \bigcap_{n\in\mathbb{N}} Ran(\sigma(na_k)) \\ &= \bigcap_{(n_1,\cdots,n_d)\in\mathbb{N}^d} Ran(\sigma(n_1,\cdots,n_d)) \\ &= \{0\}.\end{aligned}$$

Therefore, $\tilde{\sigma}^{(k)}$ *is a pure isometric representation of* $\mathbb{N}$*, for all* $1 \leq k \leq d$*. Thus,* $\tilde{\sigma}$ *is strongly pure.*

We can choose the order units $a_1,\cdots,a_d$ *such that they span* $\mathbb{Z}^d$*. Then, due to Prop.* 2.3*,* $\dim(\mathcal{A}(\sigma)) = dim(\mathcal{A}(\tilde{\sigma}))$ *and* $\mathcal{M}(\sigma)' = \mathcal{M}(\tilde{\sigma})'$*.*

4. Examples

In this section, we prove Thm. 1.1. First, we explain that Thm. 1.1 follows if we prove the analogous two parameter discrete version. The reduction is explained below. Let P be a closed, convex cone in $\mathbb{R}^d$ that is spanning and pointed. Assume that $d \geq 2$.

(1) Since P is spanning and pointed, without loss of generality, we can assume that $P \subset \mathbb{R}^d_+$. Thanks to Prop. 2.3, it suffices to prove Thm. 1.1 when $P = \mathbb{R}^d_+$. Hereafter, assume that $P = \mathbb{R}^d_+$.

(2) Suppose $V : \mathbb{R}^2_+ \to B(\mathcal{H})$ is an isometric representation of $\mathbb{R}^2_+$. Let $W : \mathbb{R}^d_+ \to B(\mathcal{H})$ be defined by

$$W_{(t_1,t_2,\cdots,t_d)} := V_{(t_1,t_2)}.$$

Then, it is not difficult to show that $\mathcal{A}(W) \cong \mathcal{A}(V)$. Consequently, $Ind(W) = Ind(V)$. Clearly, W is irreducible if and only if V is irreducible. To denote the dependence of W on V, we denote W by W^V.

Moreover, for isometric representations V_1 and V_2 of $\mathbb{R}^2_+$, W^{V_1} and W^{V_2} are unitarily equivalent if and only if V_1 and V_2 are unitarily equivalent. Thus, it suffices to prove Thm. 1.1 under the assumption that $P = \mathbb{R}^2_+$.

(3) Thanks to Thm. 3.6, to prove Thm. 1.1 when $P = \mathbb{R}^2_+$, it suffices to produce, for any given k, a continuum of strongly pure irreducible isometric representations of $\mathbb{N}^2$ with index k. Remark 3.7 allows us to drop the requirement that the desired irreducible isometric representations of $\mathbb{N}^2$ need to be strongly pure.

With the discussion above, the problem now boils down to finding pure isometric representations of $\mathbb{N}^2$ that are irreducible and whose space of additive cocycles have dimension k for $k \in \{0,1,2,3,\cdots,\} \cup \{\infty\}$.

Proposition 4.1. *Let* $d \in \{1,2,\cdots\}\cup\{\infty\}$*, and let* $\mathcal{H}$ *be a separable Hilbert space. Let* $\{P_i | 1 \leq i \leq d\}$ *be a family of mutually orthogonal projections on* $\mathcal{H}$ *such that*

$\sum_{i=1}^{d} P_i = 1$. *For each* $k \geq 1$, *define* $Q_k := 1 - \sum_{i=1}^{k} P_i$. *Let* U *be a unitary on* $\mathcal{H}$. *Define an isometric representation* $\sigma : \mathbb{N}^2 \to B(\mathcal{H} \otimes \ell^2(\mathbb{N}))$ *by*

$$\sigma(1,0) := 1 \otimes S,\ \sigma(0,1) := \sum_{i=1}^{d} UP_i \otimes S^{i-1},$$

where S *is the usual shift operator on* $\ell^2(\mathbb{N})$. *Then,*

(1) $dim(\mathcal{A}(\sigma)) = dim(\{x \in \mathcal{H} | x \in ker(U-1)$ *and* $\sum_{i=1}^{d} ||Q_i x||^2 < \infty\})$, *and*
(2) $\mathcal{M}(\sigma)' = C^*(\{U, P_i | 1 \leq i \leq d\})' \otimes 1$.

Proof. Let $\{\delta_i\}_{i\geq 0}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$. Suppose $\xi = \{\xi_{(m,n)}\}_{(m,n)\in\mathbb{N}^2}$ is an additive cocycle of σ. Since $\xi_{(1,0)} \in ker(\sigma(1,0)^*)$, $\xi_{(1,0)} = x \otimes \delta_0$, for some $x \in \mathcal{H}$. Let $\xi_{(0,1)} = \sum_{j\geq 0} y_j \otimes \delta_j$. As $\sigma(0,1)^*\xi_{(0,1)} = 0$,

$$\sum_{k\geq 0}(\sum_{j\geq k} P_{j-k+1}U^* y_j) \otimes \delta_k = 0.$$

This gives us that for each $j \geq 0$,

$$U^* y_j \in ker(\sum_{i=1}^{j+1} P_i) = Ran(Q_{j+1}).$$

Now, since ξ is an additive cocycle, it satisfies

$$\xi_{(1,0)} + \sigma(1,0)\xi_{(0,1)} = \xi_{(0,1)} + \sigma(0,1)\xi_{(1,0)},$$

which in turn implies

$$U^* y_j = U^* x - (\sum_{i=1}^{j+1} P_i)x, \forall j \geq 0.$$

Since $U^* y_j \in Ran(Q_{j+1})$, we get $P_i x = P_i U^* x$ for every i, and consequently, for every j, $U^* y_j = Q_{j+1} U^* x$, i.e $y_j = UQ_{j+1}U^* x$. Since $P_i(x - U^* x) = 0$, for all i, and $\sum_i P_i = 1$, we have $U^* x = x$. Hence, $y_j = UQ_{j+1}x$. The fact that $\sum_{j\geq 0} y_j \otimes \delta_j \in \mathcal{H} \otimes \ell^2(\mathbb{N})$ implies $\sum_{i=1}^{d} ||Q_i x||^2 < \infty$.

Conversely, choose $x \in ker(U-1)$ such that $\sum_{i=1}^{d} ||Q_i x||^2 < \infty$. Let $\eta_{(1,0)} = x \otimes \delta_0$ and $\eta_{(0,1)} = \sum_{j\geq 0} UQ_{j+1}x \otimes \delta_j$. It is routine to check that

$$\eta_{(1,0)} + \sigma(1,0)\eta_{(0,1)} = \eta_{(0,1)} + \sigma(0,1)\eta_{(1,0)}.$$

Therefore, there exists an additive cocycle $\xi := \{\xi_{(m,n)}\}_{(m,n)\in\mathbb{N}^2}$ such that $\xi_{(1,0)} = x \otimes \delta_0$ and $\xi_{(0,1)} = \sum_{j\geq 0} UQ_{j+1}x \otimes \delta_j$.

Define a map $\{x \in \mathcal{H} | x \in ker(U-1)$ and $\sum_{i=1}^{d} ||Q_i x||^2 < \infty\} \to \mathcal{A}(\sigma)$ by

$$x \mapsto \xi^x,$$

where $\xi^x_{(1,0)} := x \otimes \delta_0$ and $\xi^x_{(0,1)} := \sum_{j\geq 0} UQ_{j+1}x \otimes \delta_j$. It is now obvious that this map is an isomorphism. Therefore,

$$dim(\mathcal{A}(\sigma)) = dim(\{x \in \mathcal{H} | x \in ker(U-1) \text{ and } \sum_{i=1}^{d} ||Q_i x||^2 < \infty\}).$$

That concludes the first part of the proof.

Let $T \in \mathcal{M}(\sigma)'$. Since, $T \in C^*\{\sigma(1,0) = 1 \otimes S\}'$, T is of the form $T = T_0 \otimes 1$, for some $T_0 \in B(\mathcal{H})$. Also, $T\sigma(0,1) = \sigma(0,1)T$. Thus,

$$\sum_{i\geq 1} T_0 UP_i \otimes S^{i-1} = \sum_{i\geq 1} UP_i T_0 \otimes S^{i-1}$$

Hence, $T_0 UP_i = UP_i T_0$, for all $i \geq 1$. Summing over all i, we get

$$T_0 U = UT_0,$$

and that implies, for all $i \geq 1$,

$$T_0 P_i = P_i T_0.$$

Thus, $\mathcal{M}(\sigma)' \subseteq C^*(\{U, P_i | 1 \leq i \leq d\})' \otimes 1$. Clearly, the reverse side of the inclusion holds as well, and we get

$$\mathcal{M}(\sigma)' = C^*(\{U, P_i | 1 \leq i \leq d\})' \otimes 1.$$

□

Remark 4.2. *Note that, in Prop.* 4.1*, when $\mathcal{H}$ is finite-dimensional,*

$$Ind(\sigma) = dim(\mathcal{A}(\sigma)) = dim(ker(U-1)).$$

We use Prop. 4.1 to produce concrete examples of irreducible isometric representations of $\mathbb{N}^2$ with any given index.

Example 1: For this example, we refer to the work of Albeverio and Rabanovich [1]. Let B_3 denote Artin's braid group,

$$B_3 = \langle S, J | S^2 = J^3 \rangle$$

Let m be a positive integer. Theorem 5 of [1] asserts that there exists a non-empty open set Ω in a Euclidean space and a family of irreducible unitary representations $\{\pi_h\}_{h\in\Omega}$ of B_3 on $\mathbb{C}^{6m}$ such that

(1) for $h \neq k$, π_h and π_k are not unitarily equivalent, and
(2) for $h \in \Omega$, $dim(ker(\pi_h(J) - 1)) = 2m$.

For the explicit expression of the representation π_h, the reader is referred to Section 3 of [1]. For $h \in \Omega$, set $P_h := \frac{1+\pi_h(S)}{2}$, and $U_h := \pi_h(J)$, and define an isometric representation $\sigma_h : \mathbb{N}^2 \to B(\mathbb{C}^{6m} \otimes \ell^2(\mathbb{N}))$ by setting

$$\sigma_h(1,0) = 1 \otimes S; \ \sigma_h(0,1) = U_h P_h \otimes 1 + U_h(1 - P_h) \otimes S.$$

Let $h \in \Omega$. It follows from (2) and from Prop. 4.1 that $Ind(\sigma_h) = 2m$ for every $h \in \Omega$. By Prop. 4.1 and the fact that π_h is irreducible, it follows that σ_h is irreducible.

Let $h, k \in \Omega$ be given. Suppose T is a unitary operator that intertwines σ_h and σ_k, i.e $T\sigma_h(\cdot) = \sigma_k(\cdot)T$. Then, $T(1 \otimes S) = (1 \otimes S)T$. Hence, T is of the form $T = T_0 \otimes 1$ for some unitary operator T_0 on $\mathbb{C}^{2m}$. The equality

$$(T_0 \otimes 1)\sigma_h(0,1) = \sigma_k(0,1)(T_0 \otimes 1)$$

leads to the conclusion $T_0 U_h = U_k T_0$ and $T_0 P_h = P_k T_0$. In other words, π_h and π_k are unitarily equivalent. Thus, the isometric representations σ_h and σ_k are not unitarily equivalent if $h \neq k$. This gives a plethora of irreducible examples with even index that are not unitarily equivalent.

It is not difficult to construct other examples as the following two classes of examples show.

Example 2: Let $\mathcal{H}$ be a finite-dimensional Hilbert space with an orthonormal basis $\{e_1, e_2, ..., e_n\}$. Let $P_i \in B(\mathcal{H})$ be the projection onto the subspace spanned by e_i, for $i = 1, 2, ..., n$. Choose $a \in \mathcal{H}$ such that $\langle a|e_i\rangle \neq 0$ for every $i = 1, 2, ..., n$. Let $P_a \in B(\mathcal{H})$ be the projection onto the subspace spanned by a, i.e. $P_a(x) := \langle x|a\rangle a$, for all $x \in \mathcal{H}$. Define $U_a \in B(\mathcal{H})$ by $U_a = 1 - 2P_a$. Then, U_a is a self-adjoint unitary on $\mathcal{H}$ and $dim(ker(U_a - 1)) = dim(ker(P_a)) = n-1$. Also, $C^*(\{U_a, P_1, ..., P_n\})' = \mathbb{C}$. Define an isometric representation $\sigma^{(a)} : \mathbb{N}^2 \to B(\mathcal{H} \otimes \ell^2(\mathbb{N}))$ by

$$\sigma^{(a)}(1,0) := 1 \otimes S, \ \sigma^{(a)}(0,1) := \sum_{i=1}^{n} U_a P_i \otimes S^{i-1}.$$

It follows from Prop. 4.1 that $Ind(\sigma^{(a)}) = n - 1$, and $\sigma^{(a)}$ is irreducible

Let $a, b \in \mathcal{H}$ be such that $\langle a|e_i\rangle \neq 0$, $\langle b|e_i\rangle \neq 0$, for all $1 \leq i \leq n$, and there exists $k \in \mathbb{N}$, $1 \leq k \leq n$ such that $|\langle a|e_k\rangle| \neq |\langle b|e_k\rangle|$. We claim that $\sigma^{(a)}$ and $\sigma^{(b)}$ are not unitarily equivalent. Let us assume there exists a unitary $T : \mathcal{H} \otimes \ell^2(\mathbb{N}) \to \mathcal{H} \otimes \ell^2(\mathbb{N})$ such that

$$T\sigma^{(a)}(m,n) = \sigma^{(b)}(m,n)T$$

for $(m,n) \in \mathbb{N}^2$.

Since T commutes with $\sigma^{(a)}(1,0) = \sigma^{(b)}(1,0) = 1 \otimes S$, it follows that there exists a unitary $T_0 \in B(\mathcal{H})$ such that $T = T_0 \otimes 1$. Also, the equality $(T_0 \otimes 1)\sigma^{(a)}(0,1) = \sigma^{(b)}(0,1)(T_0 \otimes 1)$ gives

$$T_0 U_a P_i = U_b P_i T_0, \text{ for all } 1 \leq i \leq n. \tag{11}$$

Adding them, we get

$$T_0 U_a = U_b T_0. \tag{12}$$

Eq. 12 and Eq. 11 imply that $T_0 P_i = P_i T_0$ for every i. Thus,

$$T_0 e_i = \lambda_i e_i$$

for some $\lambda_i \in \mathbb{T}$, for all $1 \leq i \leq n$. Eq. 12 implies $T_0 P_a e_k = P_b T_0 e_k$ for all k, which simplifies to

$$\lambda_i \langle e_k|a\rangle\langle a|e_i\rangle = \lambda_k \langle e_k|b\rangle\langle b|e_i\rangle, \ \forall i$$

Thus, we get $|\langle a|e_k\rangle| = |\langle b|e_k\rangle|$ for all k, which is a contradiction. Therefore, $\sigma^{(a)}$ and $\sigma^{(b)}$ are not unitarily equivalent whenever there exists k such that $|\langle a|e_k\rangle| \neq |\langle b|e_k\rangle|$.

Example 3: Let $\mathcal{H}$ be an infinite dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Define $P_i : \mathcal{H} \to \mathcal{H}$ by $P_i e_j = \delta_{ij} e_i$, for all $i \in \mathbb{N}$. Choose any $a \in \mathcal{H}$ with $\langle a|e_i\rangle \neq 0$ for all $i \in \mathbb{N}$, and let P_a be the projection defined by $P_a e_i := \langle e_i|a\rangle a$, for all i. Set $U_a = 1 - 2P_a$. Define an isometric representation $\sigma^{(a)} : \mathbb{N}^2 \to B(\mathcal{H} \otimes \ell^2(\mathbb{N}))$ by

$$\sigma^{(a)}(1,0) := 1 \otimes S,\ \sigma^{(a)}(0,1) := \sum_{i=1}^{\infty} U_a P_i \otimes S^{i-1}.$$

Note that $C^*\{U_a, P_i : i = 1, 2, \cdots\}' = \mathbb{C}$. Hence, by Prop. 4.1 $\sigma^{(a)}$ is an irreducible isometric representation of $\mathbb{N}^2$. By Prop. 4.1, $\sigma^{(a)}$ has index equal to the dimension of the space $\left\{x \in ker(U_a - 1) | \sum_{n\geq 2}(n-1)||P_n x||^2 < \infty\right\}$ and the latter has infinite dimension. Thus, $Ind(\sigma^{(a)}) = \infty$. Also, just like in the finite-dimensional case, if we choose $a, b \in \mathcal{H}$ such that for some $k \in \mathbb{N}$, $|\langle a|e_k\rangle| \neq |\langle b|e_k\rangle|$, then $\sigma^{(a)}$ and $\sigma^{(b)}$ are not unitarily equivalent.

We encompass all of the above in the following theorem.

Theorem 4.3. *For each $k \in \{0, 1, 2, \cdots,\} \cup \{\infty\}$, there is a continuum of irreducible isometric representations of $\mathbb{N}^2$ that has index k.*

Note that Thm. 1.1 is now immediate from Thm. 4.3 and the discussions made at the beginning of this section. As explained in the introduction, Thm. 1.1 implies Thm. 1.2.

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