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A NOTE ON THE TWO CARDINAL PROBLEM

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Let T be a a theory in a countable language L with 1-placed predicate symbol U and let $\mathfrak{A} = \langle A, V, \dots \rangle$ be a model for L, V being interpretation of predicate symbol U. We say that $\mathfrak{A}$ is an (α, β) model if $| A |= \alpha$ and $| V |= \beta$. If T has an (α, β) model, we say that T admits the pair of cardinals (α, β). Given a theory T, the question is which pairs (α, β) does T admit. Theorems 1 through 5 are from (1) (consequtively, Proposition 3.2.11, Theorem 4.3.10, Corollary 4.3.11, Theorem 6.5.11).

THEOREM 1. *Let T be a theory in a countable language L, and let α, β, γ range over infinite cardinals. Then:*

(i) *if T admits (α, β) then T admits (γ, β) for all γ such that $\beta \leq \gamma \leq \alpha$.*

(ii) *if T admits (α, β) then T admits all (γ, γ).*

(iii) *for each $n \in \omega$, there is theory T such that T admits every $(\aleph_n(\alpha), \alpha)$ and T does not admit any $(\aleph_n(\alpha)^+, \alpha)$.*

(iv) *for each $n \in \omega$, there is a theory T such that T admits every $(\aleph_n(\alpha), \alpha)$ and T does not admit any $(\aleph_{n+1}(\alpha), \alpha)$.*

THEOREM 2. *If a countable theory T admits (α, β) with $\alpha > \beta \geq \omega$, then T admits the pair (ω_1, ω).*

THEOREM 3. *If a theory T admits (α, β) and $\beta \geq \omega$, then for all cardinals γ, T admits $(\alpha^\gamma, \beta^\gamma)$. In fact every (α, β) model has an elementary extension, which is an $(\alpha^\gamma, \beta^\gamma)$ model.*

COROLLARY 4. *Assume the* GCH. *Suppose $\alpha \geq \alpha' \geq \beta' \geq \beta \geq \omega$ and $\|L\| \leq \alpha'$. Then every theory T in L which admits (α, β), admits (α', β').*

THEOREM 5. *Let L have a 1-placed predicate symbol U.*

(i) *if $\alpha^\omega \geq \beta' \geq \beta^\omega$ and $\alpha \geq \beta \geq \omega$, then every (α, β) model has a complete extension which is an (α^ω, β') model.*

(ii) *suppose* $\alpha \geq \alpha' \geq \beta' \geq \beta^{\omega}$, $\beta \geq \omega$ *and* $\alpha' \geq \|L\|$. *Then every theory which admits* (α, β), *admits* (α', β').

From the above it is clear that, given a pair (α, β) *which a theory* T *admits, interesting would be to get other pairs* (γ, δ) *admitted by* T, *such that* $\alpha \geq \gamma \geq \delta \geq \beta$ *does not hold, with* γ *and* δ *being as far apart as possible.*

Theorem 3 *is obtained using ultrapowers of* (α, β) *model, over an regular ultrafilter with the cardinal* γ *as index set. Similarly, it follows that if a theory* T *admits the pair* (α, β), *then* T *admits the pair*

$$\left(\left|\prod_D \alpha\right|, \left|\prod_D \beta\right|\right),$$

for any ultrafilter D. *The following theorem is obtained generalizing the proof of Theorem* 3.

THEOREM 6. *If* T *admits pairs* (α_i, β_i), *where* $i \in I$ *and* D *is an ultrafilter over* I *then* T *admits the pair*

$$\left(\left|\prod_D \alpha_i\right|, \left|\prod_D \beta_i\right|\right).$$

Hence, if using ultraproducts in two cardinal problem, the goal would be to keep $|\prod_D \alpha_i|$ *and* $|\prod_D \beta_i|$ *as far apart as possible.*

The following theorem is from (2).

THEOREM 7. *If* D *is* (α, β) *regular and* ν, k *are cardinals such that*

$$\alpha \leq \nu \leq \beta, \ \ \alpha \leq k \leq \beta \text{ and}$$

$k^{\nu} = k$, *then*

(i) $|\prod_D k| \geq 2^{\beta}$ *and*

(ii) $|\prod_D k| = |\prod_D k|^{\nu}$.

COROLLARY 8

(iii) *If* $(cf\gamma)^{\overset{cf\gamma}{\smile}} = cf\gamma$ *then* $|\prod_D cf\gamma|^{cf\gamma} = |\prod_D cf\gamma|$.

(iv) *If* γ *is strongly inaccessible or* ($\gamma = \lambda^+$ *and* $2^{\lambda} = \lambda^+$) *then*

$$\left|\prod_D \gamma\right| = 2^{\gamma}.$$

This indicates that the ultrafilter D above should be as irregular as possible.

To obtain nonregular ultrafilters, large cardinal axioms are used. In fact, the smaller the cardinal over which nonregular ultrafilter is constructed, the stronger large cardinal property is used.

THEOREM 9. *Let k be a measurable cardinal and let D be an normal ultrafiletr over k. Let T be a theory in a countable language.*

(i) *if T admits pairs (α^+, α) for a set of α's in D then T admits (k^+, k).*

(ii) *if T admits $(2^\alpha, \alpha)$ for a set of α's in D, then T admits the pair $(2^k, k)$.*

(iii) *if T admits (α_i, β_i) with α_i being unlimited almost everywhere and β_i being limited almost everywhere, with, say λ then T admits the pair $(2^k, \lambda)$.*

PROOF. Using method of 4.3 of (1).

THEOREM 10. *Let k be a measurable cardinal with normal ultrafilter D and let D^+ be a k complete uniform ultrafilter over k^+; let $f = \langle \alpha_i : i \in k \rangle$ and $g = \langle \beta_i : i \in k \rangle$ be such that $g <_D id \leq_D f$, id being identity function on k. If a theory T admits pairs (α_i, β_i), $i \in k$, then there is $\lambda < k$ such that T admits the pair $(2^{k+}, \lambda)$.*

PROOF. From theorem 9, it follows that T admits $(2^k, \lambda)$. Taking ultrapower modulo filter D^+ (which is (k, k)-regular and (k^+, k^+)-regular), we have, applying Theorem 7, that

$$\left|\prod_{D+} 2^k\right| = \left|\prod_{D+} {k^+}^{k+}\right| = \left|\prod_{D+} k^+ \overset{k+}{\smile}\right| \overset{k+}{\smile}= 2^{k+}.$$

The assertion then follows from Theorem 6 and Theorem IX.

With assumption of huge cardinals Magidor (3) obtained the following result.

THEOREM 11. *If there is a huge cardinal then*

(i) *there is an uniform ultrafilter D over ω_2 such that $|\prod_D \omega| \leq \omega_2$*

(ii) *there is an uniform ultrafilter E over ω_3 which is not (ω_3, ω_1) regular and such that*

$$\left|\prod_E \omega_1\right| \leq \omega_3.$$

Applying Corrollary 8, Theorem 5 and Theorem 6 we can conclude.

COROLLARY 12

(i) *if there is a filter* D *over* ω_2 *like in* (i) *of Theorem* 11 *and* $2^{\omega_1} = \omega_2$ *(which is true in the model of* (3)) *then*

if T *admits the pair* (ω_2, ω) *then* T *admits* $(2^{\omega_2}, \omega_2)$.

(ii) *if there is a filter* E *over* ω_3, *like in* (ii) *of Theorem* 11 *and* $2^{\omega_2} = \omega_3$ *(which is true in the model of* (3)) *then*

if T *admits the pair* (ω_3, ω_1) *then* T *admits* $(2^{\omega_3}, \omega_3)$.

The proof of Theorem 11 *is general giving rise to similar filters over the other cardinals, which can be used similarly as above.*

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