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SEMIGROUPS IN WHICH SOME IDEAL IS A COMPLETELY SIMPLE SEMIGROUP

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0. In this paper we consider semigroups without zero containing an ideal which is a completely simple subsemigroup. The main result is the Theorem 1.1. which gives a structural description for such semigroups. In 2. we give a generalization of the Theorem in [**1**], as well a structural description of semigroups characterized by the Theorem 3.1. in [**3**].

1. At first let us construct a semigroup in which some ideal is a completely simple semigroup.

Let $m[I, G, J, P]$ be a Rees matrix semigroup (I, J are nonempty sets, G is a group, P is a matrix $J \times I$ with entries $p_{ji} \in G$).

Let T be a partial semigroup so that $(I \times G \times J) \cap T = \varnothing$.

Let

$$\xi : T \to \tau(I) \quad \text{and} \quad \eta : T \to \tau(J)$$

be mappings, where $\tau(I)$ and $\tau(J)$ are semigroups of all mappings of I in I, J in J, respectively, i.e. $\xi : p \mapsto \xi_p$, $\eta : p \mapsto \eta_p$, such that for all p, $q \in T$ is fulfilled:

(a) If $pq \in T$ then $\xi_{pq} = \xi_q \xi_p$

(b) If $pq \notin T$ then $\xi_q \xi_p$ is a constant mapping;

(c) If $pq \in T$ then $\eta_{pq} = \eta_p \eta_q$

(d) If $pq \notin T$ then $\eta_p \eta_q$ is a constant mapping.

Let

$$\varphi : T \times I \to G$$

be a mapping which satisfies

(e) If $pq \in T$ then $\varphi(pq, i) = \varphi(p, i\xi_i)\varphi(q, i)$

(f) $p_{j,i\xi_p} \varphi(p, i) p^{-1}_{j\eta_p, i}$ does not depend upon $i \in I$.

Let us denote the term in (f) by $\psi(p,j)$, i.e.

$$\psi(p,j) = p_{j,i\xi_p}\varphi(p,i)p^{-1}_{j\eta_p,i}. \tag{1.1}$$

Let us define the multiplication in $(I\times G\times J)\cup T$ by:

(i) $(i,x,j)\cdot(k,y,l) = (i,xp_{jk}y,l)$

(ii) $p\cdot(i,x,j) = (i\xi_p,\varphi(p,i)x,j)$

(iii) $(i,x,j)\cdot p = (i,x\psi(p,j),j\eta_p)$

where $p\subset T$, $(i,x,j),(k,y,l)\in I\times\times J$, and $\psi(p,j)$ is given by (1.1);

(iv) If $pq=r$ in T then $pq=r$ in $(I\times G\times J)\cup T$;

If $pq\notin T$ then

$$pq = (i\xi_q\xi_p,\ \varphi(p,\ i\xi_q)\varphi(q,\ i)p^{-1}_{j\eta_p\eta_q,i},\ j\eta_p\eta_q).$$

Let us denote $(I\times G\times J)\cup T$ with multiplication "$\cdot$" by $n[I,\ G,\ J,\ P;\ T,\ \varphi,\ \xi,\ \eta]$, i.e.

$$((I\times G\times J)\cup T,\cdot) = n(I,\ G,\ J,\ P;\ T,\ \varphi,\ \xi,\ \eta].$$

LEMMA 1.1. *$n[I,\ G,\ J,\ P;\ T,\ \varphi,\ \xi,\ \eta]$ is a semigroup.*

Proof. First, the multiplication is well defined. For the cases (i), (ii) and (iii), and (iv) if $pq\in T$, this follows immediately. Let $p,q\in T$ and $pq\notin T$. As in this case $i\xi_q\xi_q$ =const and $j\eta_p\eta_p$ =const, we have only to prove that the term

$$\varphi(p,\ i\xi_p)\varphi(q,i)\cdot p^{-1}_{j\eta_p\eta_p,i}$$

does not depend upon $i\in I$. Indeed, using (1.1) we have

$$\begin{gathered}\varphi(p,\ i\xi_q)\varphi(q,\ i)\cdot p^{-1}_{j\eta_p\eta_q,i} = \\ = p^{-1}_{j,i\xi_q\xi_p}\cdot p_{j,i\xi_q\xi_p}\varphi(p,\ i\xi_q)\cdot p^{-1}_{j,\eta_p,i\xi_q}\cdot p_{j\eta_p,i\xi_Q}(q,i)\cdot p^{-1}_{j,\eta_p\eta_q,i} = \\ = p^{-1}_{j,i\xi_q\xi_p}\psi(p,j)\psi(q,j\eta_p).\end{gathered}$$

Since the right side of this equality does not depend upon $i\in I$, it follows that the left side does not depend upon $i\in I$, too.

By direct verification we get that $n[I,G,J,P,T,\varphi,\xi,\eta]$ is a semigroup. For instance, if $p,q\in T$, $pq\notin T$ and $X=(k,x,l)$, then

$$\begin{gathered}pq\cdot X = (i\xi_q\xi_p,\varphi(p,i\xi_q)\varphi(q,i)\cdot p^{-1}_{j\eta_p\eta_q,i},j\eta_p\eta_q)\cdot(k,x,l) = \\ = (i\xi_q\xi_p,\varphi(p,i\xi_q)\varphi(q,i)\cdot p^{-1}_{j\eta_p\eta_q,i}\cdot p_{j\eta_p\eta_q,k}\cdot x,l)\end{gathered}$$

Since i is arbitrary, j is arbitrary, then for $i=k$, it follows

$$pq\cdot X = (k\,\xi_q\,\xi_p,\ \varphi(p,k\,\xi_q)\varphi(q,\ k)\cdot x,l).$$

On the other side

$$\begin{gathered}p\cdot qX = p\cdot q(k,\ x,\ l) = p\cdot(k\,\xi_q,\ \varphi(q,\ k)\cdot x,\ l) = \\ = (k\,xi_q\xi_p,\varphi(p,\ k\,\xi_q)\varphi(q,\ k)\cdot x,\ l).\end{gathered}$$

So we get

$$pq \cdot X = p \cdot qX.$$

Similarly for other cases, which proves the lemma.

LEMMA 1.2 *In a semigroup* $n = n[I, G, J, P; T, \varphi, \xi, \eta]$ *there is an ideal which is a completely simple subsemogroup (without zero).*

Proof. Subsemigroup $I \times G \times J$ ia an ideal in n, which follows immediately from the definition of the multiplication in n, (i)–(iv). From (i) we have that $I \times G \times J = m[I, G, J, P]$ is the Rees matrix semigroup over the group G, which is completely simple (without zero) [**4**].

THEOREM 1.1. *In a semigroup* S *there is an ideal which is a completely simple subsemigroup (without zero) if and if a semigroup* S *is isomorphic to a semigroup* $n[I, G, J, P; T, \xi, \xi, \eta]$.

Proof. Let K be an ideal in a semigroup S which is a completely simple semigroup without zero. Let $T = S \backslash K (\neq \varnothing)$. Then T is a partial semigroup. Since K can be represented by a Rees matrix semigroup, we have $K \cong m[I, G, J, P]$, so we get a isomrphism $S = K \cup T \cong m[I, G, J, P] \cup T$, and we identify K with $m[I, G, J, P]$. Then, in K, left ideals are

$$L_j = \{(i, g, j) : i \in I, \;\; g \in G\}, \;\text{ for }\; j \in J;$$

right ideals

$$R_i = \{(i, g, j) : g \in G, \;\; j \in J\}, \;\text{ for }\; i \in I$$

and bi-ideals

$$B_{ij} = \{(i, g, j) : g \in G\} = R_i \cap L_j.$$

For all left ideals L_j in K we have $KL_j = L_j$. Indeed, let $(i, g, j) \in L_j$ then

$$(i, g, j) = (i, g, l) \cdot (k, p_{lk}^{-1}, j) \in KL_j$$

so $L_j \subseteq KL_j$. By this we have $KL_j = L_j$.

Using this we have

$$SL_j = SKL_j \subseteq KL_j = L_j$$

whence we have that $L_j (j \in J)$ is a left ideal in S, too.

Similarly, every right ideal R_i in K, as like as bi-ideal B_{ij} in K, is also a right ideal in S, and also bi-ideal in S, respectively.

Let $p \in T$ and $(i, e, j) \in L_j$, where e is the identity of a of group G. Then

$$p \cdot (i, e, j) = (k, g, j) \in L_j$$

since L_j is a left ideal in S and

$$k = i\xi_{p,j}, g = \varphi(p, i, j)$$

so

$$p \cdot (i, e, j) = (i\xi_{p,j}, \varphi(p, i, j), j). \tag{1.2}$$

On the other side

$$\begin{aligned} p \cdot (i, e, j) &= p \cdot (i, e, l) \cdot (k, p_{lk}^{-1}, j) = \\ &= (i\xi_{p,l}, \varphi(p, i, l), l) \cdot (k, p_{lk}^{-1}, j) = \\ &= (i\xi_{p,l}, \varphi(p, i, l), j) \end{aligned} \tag{1.3}$$

From (1.2) and (1.3) it follows

$$\begin{aligned} i\xi_{p,j} &= i\xi_{p,l}, \\ \varphi(p, i, j) &= \varphi(p, i, l), \end{aligned}$$

whence $\xi_{p,j}$ depends only upon p and $\varphi(p, i, j)$ depends only upon p and i.

So we have

$$p \cdot (i, e, j) = (i\xi_p, \varphi(p, i), j), \tag{1.4}$$

where φ is mapping which maps $T \times I$ in G and ξ_p maps I in I i.e. $\varphi : T \times I \to G$ and $\xi_p : I \to I$.

Similarly, for $(i, e, j) \in R_i$, which is a right ideal in S, and $p \in T$ we get

$$(i, e, j) \cdot p = (i, \psi(p, j), j\eta_p), \tag{1.5}$$

where $\psi : T \times J \to G$ and $\eta_p : J \to J$.

Since

$$(i, e, j)p(i, e, j) = (i, e, j)(i\xi_p, \varphi(p, i), j) = (i, p_{j,i\xi_p}\varphi(p, i), j)$$

and

$$(i, e, j)p(i, e, j) = (i, \psi(p, j), \eta_p)(i, e, j) = (i, \psi(p, j)p_{j\eta_p,i}, j)$$

we have

$$\psi(p, j) = p_{j,i\xi_p}\varphi(p, i) \cdot p^{-1}_{j\eta_p,i} \tag{1.6}$$

whence it follows that the term on the right side in (1.6) does not depend upon $i \in I$.

Now, let $p \in T$ and $g \in G$. Then

$$p \cdot (i, g, j) = p \cdot (i, e, l)(k, p_{lk}^{-1} \cdot g, j) = (i\xi_p, \varphi(p, i) \cdot g, j).$$

Similarly we have

$$(i, g, j) \cdot p = (i, g \cdot \psi(p, j), j\eta_p).$$

Let $p, q \in T$ and $pq \in T$. Then

$$pq \cdot (i, e, j) = (i\xi_{pq}, \varphi(pq, i), j) = (i\xi_q\xi_p, \varphi(p, i\xi_q)\varphi(q, i), j)$$

whence

$$\begin{aligned} \varphi(p, q, i) &= \varphi(p, i\xi_q)\varphi(q, i) \\ \xi_{pq} &= \xi_q\xi_p. \end{aligned} \tag{1.7}$$

Similarly

$$(ie, j) \cdot pq = (i\, psi(pq, j), j\eta_{pq}) = (i, \psi(p, j)\psi(q, j\eta_p), j_p\eta_q)$$

whence

$$\begin{aligned} \psi(pq, j) &= \psi(p, j)\psi(q, j\eta_p) \\ \eta_{pq} &= \eta_p\eta_q. \end{aligned} \tag{1.8}$$

Let $p.q \in T$ and $pq \notin T$, i.e. $pq = (i, g, j)$. Then $pq = (i, g, j) = (i, g, j) \cdot (k, p_{jk}^{-1}, j) = p(k\xi_q, \varphi(q, k) \cdot p_{jk}^{-1}, j) = (k\xi_q\xi_p, \varphi(p, k\xi_q) \cdot \varphi(q, k)p_{jk}^{-1}, j)$, so we have

$$\begin{aligned} g &= \varphi(p, k\xi_q)\varphi(q, k)p_{jk}^{-1} \\ i &= k\xi_q\xi_p, \end{aligned} \tag{1.9}$$

where $k \in I$ is arbitrary, so $\xi_q\xi_p$ is a constant mapping for $[q \notin T$.

Similarly

$$\begin{aligned} pq = (i, g, j) &= (i, p_u^{-1}, l)(i, g, j) = (i, p_u^{-1}, l)pq = \\ &= (i, p_u^{-1}, \psi(p, l), l\eta_p)q = (i, p_u^{-1}, \psi(p, l)\psi(q, l\eta_p), l\eta_p\eta_q), \end{aligned}$$

whence

$$\begin{aligned} g &= p_u^{-1} \cdot \psi(p, l)\psi(q, l\eta_p) \\ j &= l\eta_p\eta_q, \end{aligned} \tag{1.10}$$

where $l \in J$ is arbitrary, so $\eta_p\eta_q$ is a constant mapping for $pq \notin T$.

Using (1.9) and (1.10) it follows that g does not depend upon k and l, but only upon p and q.

This proves that S is isomorphic with a semigroup n.

The converse follows from Lemma 1.1.

2. In this section we give two more theorems which are consequences of the Theorem 1.1. Previously we give some lemmas which characterize some properties of bi-ideals.

LEMMA 2.1. *Let B be a bi-ideal of a semigroup S. Then uBv is a bi-ideal in S for arbitrary $u, v \in S$.*

Proof. $uBvSuBv \subseteq uBSBv \subseteq uBv$.

LEMMA 2.2. *Let M be a minimal bi-ideal in S and B an arbitrary bi-ideal in S. Then $M = uBv$ for every u, $v \in M$.*

Proof. Using Lemma 2.1. uBy is a bi-ideal in S. Even more

$$uBv \subseteq MBM \subseteq MSM \subseteq M.$$

Since M is a minimal bi-ideal then $uBv = M$.

LEMMA 2.3. *Let M be a minimal bi-ideal in S. Then sMt is a minimal bi-ideal in S for each $s, t \in S$.*

Proof. According to Lemma 2.1. sMt is a bi-ideal in S. Let us prove that sMt is a minimal bi-ideal in S. Let the converse hold. Then there is a bi-ideal N such that $n \subseteq sMt$ and $N \neq sMt$, i.e. $N = \{sht : h \in H,\, H \subseteq M,\, H \neq M\}$. As N is a bi-ideal, then for all $x \in S$, h_1, $h_2 \in H$, we have

$$sh_1t \cdot x \cdot sh_2t \in N$$

whence $h_1\, txsh_2 \in H$, and so

$$h_1\, tSsh_2 \subseteq H.$$

Since h_1, $h_2 \in M$ and tS is a bi-ideal, M is a minimal bi-ideal, then according to Lemma 2.2. we have

$$M = h_1tSsh_2$$

whence $M \subseteq H$. Contradiction.

LEMMA 2.4.*Let M be a minimal bi-ideal in S. Then all minimal bi-ideals in S are of the form sMt, where s, $t \in S$.*

Proof. By using both Lemma 2.2 and Lemma 2.3.

LEMMA 2.5. *The union of all minimal bi-ideals in S is an ideal in S.*

Proof. Let M be any minimal bi-ideal in S. Then according to Lemma 2.4.

$$D = \cup\{sMt : s,\, t \in S\} = SMS$$

is the union of all minimal bi-ideals in S. Then

$$DS = SMSS \subseteq SMS = D,$$
$$SD = SSMS \subseteq SMS = D.$$

Accordingly, D is an ideal in S.

LEMMA 2.6. *Let B be a bi-ideal in S. Then B is a minimal bi-ideal in S if and only if B is a group.*

Proof. Let B be a minimal bi-ideal in S. For any $b_1, b_2 \in B$ we have $B = b_1Bb_2$ (Lemma 2.2.). Let $a \in B$, then there is $x \in B$ such that $a = b_1xb_2$, and thus

$$a = b_1y, \quad a = zb_2$$

where $y = xb_2 \in B$, $z = b_1x \in B$. According, those equations are solvable upon y and z for each a, b_1, $b_2 \in B$. Thus, B is a group.

Conversely let B be a group and, at the same time let it not be a minimal bi-ideal. Then there is such a bi-ideal in S that $M \subseteq B$ and $M \neq B$. Let $x \in B\backslash M$. Then for any $u, v \in M \subseteq B$, there is $y \in B$ such that $x = yuv$, because B is a group ($y = u^{-1}xv^{-1}$). Whence $x \in MBM \subseteq MSM \subseteq M$. A contradiction.

THEOREM 2.1. *A semigroup S has at least left ideal L such that L is a union of groups which are right ideals in L if and only if S is isomorphic to a semigroup $n[I, G, J, P; T, \varphi, \xi, \eta]$.*

Proof. Let L be a left ideal in S i.e. $SL \subseteq L$ and $L = \bigcup_{\alpha\in A} G_\alpha$, where $G_\alpha(\alpha \in A)$ are groups, such that $G_\alpha L \subseteq G_\alpha$, for all $\alpha \in A$. Then G_α are bi-ideals in S. In fact

$$G_\alpha SG_\alpha \subseteq G_\alpha SL \subseteq G_\alpha L \subseteq G_\alpha, \text{ for all } \alpha \in A.$$

As G_α are groups, then G_α are minimal bi-ideals in S (Lemma 2.6.). Let K be a union of all minimal bi-ideals in S, i.e.

$$K = \bigcup_{\alpha\in\Lambda} H_\alpha$$

Then K is an ideal in S (Lemma 2.5.). According to Lemma 2.6. and The Theorem in [**1**], K is a completely simple semigroup (without zero). Thus S is semigroup having an ideal K which is a completely simple subsemigroups (without zero) and by using Theorem 1.1 a semigroup S is isomorphic with a semigroup $n[I, G, J, P; T, \varphi, \xi, \eta]$.

The converse follows by a straight verification.

COROLLARY [**2**] *A semigroup S has at least one left ideal which is a group if and only if S is isomorphic to a semigroup $n[I, G, J, P; T, \varphi, \xi, \eta]$, where I is a singleton.*

Proof. Let $L = G$ be a left ideal in S which is a group. As $GG = G$ we have that G is a right ideal in L, and thus the reuirements of the Theorem 2.1 are funlfiled.

Since

$$L_j = \{(i, g, j) : g \in G,\, i \in I\} = \cup\{G_{ij} : i \in I\},$$

for monominial set $I = \{i_0\}$ we have

$$L_j \simeq L = G_{i_0j} = G_j.$$

Since $T \times \{i_0\} \simeq T$, then the mapping $\varphi : T \to G$ is a homomorphism, because for any $p \in T$, $\xi_p\{i_0\} \to \{i_0\}$, so ξ_p is any identity mapping.

Since each Rees matrix semigroup $S = m[I,\, G,\, J,\, P]$ is isomorphic with some Rees matrix semigroup $S \simeq m[I,\, G,\, J,\, Q]$ where Q is some normalized sendwich matrix (i.e. such a matrix which has at least one row and at least one column with all the entries equal to the identity of o group G) then in this case, as $I = \{i_0\}$, we have

$$Q = \begin{pmatrix} e \\ e \\ \vdots \\ e \end{pmatrix}$$

where e is the identity of a group G [**4**]. Then $p_{ji} = e$, for each $j \in J$. Then

$$\varphi(p) = \psi(p)$$

and the multiplication is the same as in the Theorem in [**2**].

The converse is proved by a verification.

THEOREM 2.3. *A semigroup S contains at least one minimal left ideal and at least one minimal right ideal if and only if S is isomorphic to a semigroup* $n[I,\, G,\, J,\, P;\, T,\, \varphi,\, \xi,\, \eta]$.

Proof. It follows from Theorem 1.1 and Theorem 3.1 in [**3**].

REFERENCES

[1] Čupona G., *On completely simple semigroups*, Glasnik Mat. fiz. i astr. Tom 18. No 3 (1963), 159–164.

[2] Čupona G., *Semogroups in which some left ideal is a group*, God. Zbor. PMF A, T. **14** (1963), 15–17.

[3] Clifford A. H., *Semigroups containing minimal ideals*, Amer. J. Math., **70** (1948), 521–526.

[4] Petrich M., *Structure of regular semigroups*, Cahiers Math. **11**, Montpellier (1977)