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# EDGE-COLORING OF A FAMILY OF REGULAR GRAPHS

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**Abstract.** Let $G(m)$ denote the composition graph $G[mK_1]$. An obvious necessary condition for $G(m)$ to be 1-factorable is that $G$ is regular and $mp$ is even, where $p$ is the number of vertices of $G$. It is conjectured that this is also a sufficient condition. For regular $G$ it is proved that $G(m)$ is 1-factorable if at least one of the following conditions is satisfied: (a) $G$ is 1-factorable, (b) $G$ is of even degree and $m$ is even, (c) $m$ is divisible by 4, (d) $G$ has a 1-factor and $m$ is even, (e) $G$ is cubic and $m$ is even. The results are used to solve some other problems.

**1. Introduction.** The necessary background and terminology of graph theory can be found in [**1, 5**], in particular in the paper by Fiorini and Wilson [**4**].

A factor $F$ of $G$ is a subgraph of $G$ with the vertex set $V(F) = V(G)$, and a *factization* $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ of $G$ is a family of factors $F_1, F2, \ldots, F_k$ whose edge sets $E(F_1), E(F_2), \ldots, E(F_k)$ partition the edge set $E(G)$. A $d$-regular factor of $G$ is said to be a *d-factor.* If there exists a factorization of $G$ into $d$-factors then it is called a *d-factorization* and $G$ is said to be *d factorable.*

Vizing's well-known theorem [**12**] which states that every graph $G$ with maximum degree $\Delta(G)$ can be edge-colored with at most $\Delta(G) + 1$ colors, gives rise to the famous *Classification problem.* Obviously there is no edge-coloring of $G$ using less than $\Delta(G)$ colors. A graph $G$ is said to be of *class* 1 if it can be edge-colored with $\Delta(G)$ colors, otherwise it is of *class* 2. A regular graph is of class 1 if and only if it is 1-factorable. The classification problem is extremely difficult even for regular graphs.

Various partial results in this field were obtained in several directions by different authors [**3, 6, 8, 9**]. A good survey is given by Fiorini and Wilson [**4**]. Recently Kotzig [**7**] investigated 1-factorization of Cartesian products of regular graphs. The aim of this paper is to generalize the results of Laskar and Hare [**8**], and Parker [**9**], and to prepare ground for 1-factorization of other products of regular graphs [**11**].

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By $G[H]$ we denote the composition of graphs, also known as the lexicographic product. In the particular case when $H$ is the null graph $mK_1$ on $m$ vertices, we give the composition $G[mK_1]$ a special symbol: $G(m)$. This notation almost agrees with that of Bouchet [**2**] except that he uses $m$ as a subscript. In our notation the regular multipartite graph $K_{n(m)}$ can be written as $K_n(m)$.

If $G$ has $p$ vertices and $q$ edges, then $G(m)$ has $mp$ vertices and $m^2q$ edges. Each vertex of degree $d$ in $G$ gives rise to $m$ vertices of degree $md$ in $G(m)$. This means that $G(m)$ is $k$-regular, if and only if $k = dm$ and $G$ is $d$-regular. An obvious necessary condition for $G(m)$ to be 1-factorable is that $G$ is regular and $mp$ is even, where $p$ is the number of vertices of $G$. The authors have no counter-example to the converse statement. This paper is devoted to the study of the following conjecture.

1.1. CONJECTURE. *Let $G$ be a graph on $p$ vertices and $m > 1$. Then $G(m)$ is 1-factorable if and only if $G$ is regular and $pm$ is even.*

**2. Main results.** The Conjecture 1.1 is supported by the following theorems:

2.1. THEOREM. *If $G$ is 1-factorable, then $G(m)$ is 1-factorable.*

2.2. THEOREM. *If $G$ is regular of even degree, then $G(2m)$ is 1-factorable.*

2.3. THEOREM. *For any regular graph $G$, the graph $G(4m)$ is 1-factorable.*

2.4. THEOREM. *If $G$ is regular and has a 1-factor, then $G(2m)$ is 1-factorable.*

2.5. THEOREM. *If $G$ is a cubic graph, then $G(2m)$ is 1-factorable.*

The proofs of these theorems are given in Sections 3 and 4. Here are three simple corollaries:

2.6. COROLLARY. *if $G$ is a regular bipartite graph, then $G(m)$ if 1-factorable for arbitrary $m$.*

*Proof.* Apply König's well-known theorem [**5**, Theorem 9.2] and our Theorem 2.1!

2.7. COROLLARY. (Laskar and Hare, [**8**]) *The complete $n$-partite graph $Kn(m)$ each of whose parts has exactly $m$ vertices is 1-factorable if and only if $mn$ is even.*

*Proof.* If $mn$ is odd then $K_n(m)$ obviously has no 1-factor. For even $n$ the graph $K_n$ is 1-factorable, [**5**, Theorem 8.1], hence Theorem 2.1 applies. If, however, $n$ is odd and $m$ is even, Theorem 2.2 applies.

2.8. COROLLARY. (Parker, [**9**]) *The generalized cycle $C_n(m)$ is 1-factorable if and only if $mn$ is even.*

*Proof.* Substitute $C$ for $K$ in the proof of the Corollary 2.7 and omit the reference.

Some other partial results are given in the subsequent sections.

**3. Proofs.** In this section we introduce some lemmas necessary to prove all theorems stated in Section 2 except Theorem 2.5 which is proved in the next section.

3.1. LEMMA. *Let* $F_1 \oplus F_2 \oplus \cdots \oplus F_k$ *be a factorization of* $G$. *Then* $F_1(m) \oplus F_2(m) \oplus \cdots \oplus F_k(m)$ *is a factorization of* $G(m)$.

*Proof.* Trivial!

*Proof* of Theorem 2.1. If $G$ is 1-factorable then let $G = F_1 \oplus F_2 \oplus \cdots \oplus F_d$ be one of its 1-factorizations. $F_i(m)$ is a $m$-regular bipartite graph, for $i = 1, 2, \ldots, d$. By König's theorem [**5**, Theorem 9.2] it is 1-factorable. By Lemma 3.1 the graph $G(m)$ can be factored into factors $F_i(m)$, which turned out to be 1-factorable, therefore $G(m)$ itself is 1-factorable.

The easy proofs of the following two lemmas are omitted.

3.2. LEMMA. *If* $G_1, G_2, \ldots, G_k$ *are components of graph* $G$:

$$G = G_1 \cup G_2 \cup \cdots \cup G_k$$

*then*

$$G(m) = G_1(m) \cup G_2(m) \cup \cdots \cup G_k(m)$$

3.3. LEMMA. $G(km) = G(k)(m)$.

The key to Theorem 2.2 is the following lemma:

3.4. LEMMA. $C_n(2)$ *is 1-factorable.*

*Proof.* Each vertex of graph $C_n(2)$ is labeled by an ordered pair $(u, k)$, $u \in \{0, 1, \ldots, n-1\}$ and $k \in \{0, 1\}$. It is assumed that two consecutive vertices on a cycle are labeled by consecutive integers modulo $n$. Obviously all the edges joining $(i, 0)$ to $((i+1) \mod n, 1)$, for $i = 0, 1, \ldots, n-1$ constitute a 1-factor of $C_n(2)$. Removing this 1-factor we obtain a $n$-gonal prism. Each prism is edge-3-colorable: choose an arbitrary 3-coloring for both base cycles and use the missing color at each vertex for its lateral edge. This proves the lemma.

*Proof* of Theorem 2.2. Since by Lemma 3.3 we have $G(2m) = G(2)(m)$, by Theorem 2.1 its is enough to prove that $G(2)$ is 1-factorable. As $G$ is regular of even degree, by Petersen's theorem [**10**, p. 200] it is 2-factorable. Using Lemmas 3.1 and 3.2 it is sufficient to prove that $C_n(2)$ is 1-factorable. This is established by Lemma 3.4.

*Proof* of Theorem 2.3. By Lemma 3.3 we have $G(4m) = G(2)(2m)$. Since $G(2)$ is regular of even degree, Theorem 2.2 applies.

*Proof* of Theorem 2.4. If the degree of $G$ is even, this theorem is just a special case of Theorem 2.2. Otherwise remove the 1-factor and apply the same theorem.

**4. Cubic graphs.** In this section Theorem 2.5 is proved by induction on the number of bridges. The induction basis is established using Theorem 2.4 and

Petersen's theorem concerning the existence of 1-factors in cubic graphs [**10**, p. 218]. Lemma 4.3 is crucial in the proof of Theorem 2.5. It guarantees the existence of a special edge-coloring of $G(2)$, for an arbitrary cubic (multi) graph $G$.

For a graph $G$, an edge-coloring of $G(2)$ is called simple if it satisfies the following condition: for every edge $e$ in $G$, either $e(2)$ is colored with only two colors or the edge $e$ lies on a unique cycle $C$ such that both $C(2)$ and $e(2)$ are edge-4-colored. In connection with this definition the following corollary to Lemma 3.4 is of interest:

4.1. COROLLARY. *For every cycle $C$ there exists an edge-4-coloring of $C(2)$ such that for every edge $e$ of $C$ four colors are used to color the edges of $e(2)$.*

*Proof.* The construction of the proof of Lemma 3.4 gives the desired edge-4-coloring of $C(2)$.

This corollary and the following lemma are used in the proof of Lemma 4.3.

4.2. LEMMA. *For every path $P$ there exists an edge-4-coloring of $P(2)$ such that for every edge $e$ of $P$ two colors are used to color the edges of $e(2)$.*

*Proof.* Trivial.

4.3. LEMMA. *For every cubic graph $G$ there exists a simple edge-6-coloring of $G(2)$.*

*Proof.* Without loss of generality assume that $C$ is connected. The lemma is proved using induction on the number of bridges in $G$. By Petersen's theorem [**1O**, p. 218] every cubic (multi)graph $G$ with at most one bridge is a sum of a 1-factor and a 2-factor: $G = F_1 \oplus F_2$. By Theorem 2.1 the factor $F_1(2)$ of $G(2)$ is edge-2-colorable and by Corollary 4.1 the factor $F_2(2)$ is edge-4-colorable in such a way that these colorings give rise to a simple edge-6-coloring of $G(2)$. This takes care of the induction basis.

Let $G$ be an arbitrary cubic graph with $k$ bridges ($k \geq 2$). By the induction hypothesis, for all cubic graphs with less than $k$ bridges there exists a simple edge-6-coloring. Two possibilities arise.

*Case* 1. In $G$ there are 3 bridges incident with the same vertex $v$, as depicted in Figure 1a. We construct the three cubic graphs $H_1$, $H_2$ and $H_3$ of Figue 1b. In each of these three graphs the number of bridges is at least two less than in $G$. By the induction hypothesis we can get simple edge-6-colorings of $H_1(2)$, $H_2(2)$ and $H_3(2)$. The bridges $e_1$, $e_2$, and $e_3$ do not belong to any cycle, therefore $e_1(2)$, $e_2(2)$, and $e_3(2)$ are each colored with only two colors. By appropriate permutation of the colors we can construct such colorings that $e_1(2)$ is colored with colors 1 and 2, $e_2(2)$ with colors 3 and 4, and $e_3(2)$ with colors 5 and 6. Combining the three parts colored in this way, we get a simple edge-6-coloring of $G(2)$.

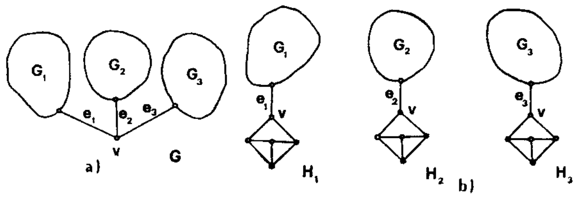


Fig 1

*Case* 2. No two bridges have a common vertex. Then we have a situation shown in Figure 2a, where $u$ and $v$ are distinct vertices and $G_1$ and $G_3$ are blocks. The same arguments as in Case 1 apply for the two graphs of Figure 2b. Graph $H_1(2)$ [or $H_3(2)$] admits and edge-6-coloring and the two colors used for $e_1(2)$ [or $e_3(2)$] can be chosen in advance. Graph $L$ depicted in Figure 2c is obtained from $G_2$ by adding a new edge $e$, incident with $u$ and $v$. By the induction hypothesis we can get a simple edge-6-coloring of $L(2)$. Cutting the edge $e$ in $L$ we obtain $M$, as depicted in Figure 2d. The simple edge-6-coloring of $L(2)$ induces an edge-6-coloring of $M(2)$ in which $f_1(2)$ and $f_2(2)$ are colored the same way as $e(2)$. Again we have two cases:

*Case* 2.1. If $e(2)$ is edge-2-colored in $L(2)$, the induced coloring of $M(2)$ is simple.

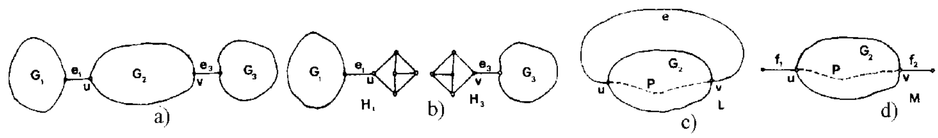


Fig 2.

*Case* 2.2. Otherwise there is a unique cycle $C$ in $L$ such that $e$ lies on $C$ and $C(2)$ is edge-4-colored in $L(2)$. Let $P = C - e$. Obviously $P$ is a path. Starting with the induced edge-6-coloring of $M(2)$ we replace the coloring of $P(2)$ with the coloring of Lemma 4.2 and we use the two unused colors at $u$ to color $f_1(2)$ and the two unused colors at $v$ to color $f_2(2)$. This new coloring of $M(2)$ can be readily seen to be a correct simple edge-6-coloring.

Combining the simple edge-6-coloring of $M(2)$ [which was obtained in both sub-cases] with the suitably adjusted simple edge-6-colorings of $H_1(2)$ and $H_3(2)$ gives us the required simple edge-6-coloring of $G(2)$. The lemma is proved.

*Proof* of Theorem 2.5. By Lemma 3.3 and Theorem 2.1 it suffices to prove that $G(2)$ is 1-factorable. This follows from Lemma 4.3.

Lemma 4.3. guarantees that for any cubic graph $G$ there exists a simple edge-6-coloring of $G(2)$. This in turn implies that for any bridge $e$ of $G$ the edges of $e(2)$ are colored with only two colors. The following proposition generalizes this observation to arbitrary edge-6-colorings of $G(2)$.

4.5. PROPOSITION. *Let $e$ be a bridge of a cubic graph $G$. For an arbitrary edge-6-coloring of $G(2)$, the edges of $e(2)$ are colored with exactly two colors.*

*Proof.* As $e(2)$ is isomorphic to the complete bipartite graph $K_{2,2}$ at least two colors are needed to color its edges. It is a bit harder to see that no more than two colors are used in an edge-6-coloring. Let $u$ and $v$ be the end-points of the bridge $e$, i.e. $e = uv$. The edges of $e(2)$ are $e_{i,j} = (u,i)(v,j)$, $i,j \in \{0,1\}$. Take any edge-6-coloring of $G(2)$ and any edge $e_{i,j}$ of $e(2)$. Let $c$ be the color of $e_{i,j}$. If we show that $e_{1-i,1-j}$ is also colored by $c$, we will establish the lemma. Each color in the given edge-coloring determines a 1-factor in $G(2)$. In particular the color $c$ determines a 1-factor in $G(2)$ and also in the two-vertex-deleted graph $H = G(2) - (u,i) - (v,j)$. Since $e_{1-i,1-j}$ constitutes a bridge joining two odd components in $H$, it lies on every 1-factor in $H$, in particular in the 1-factor determined by color $c$.

**5. Concluding remarks.** In order to apply our theorems to more general compositions of graph we need the following simple proposition:

5.1. PROPOSITION. *If $H$ is a 1-factorable graph on $m$ vertices and $G(m)$ is 1-factorable, then the composition $G[H]$ is 1-factorable t 1-factorable too.*

*Proof.* Since $G[H]$ can be factored into $G(m)$ and $pH$, where $p$ is the number of vertices of $G$, the Proposition follows readily.

This proposition, combined with the other results of this paper, gives an interesting corollary.

5.2. COROLLARY. *If $H$ is 1-factorable, on $m$ vertices then $G[H]$ is 1-factorable if at least one of the following is true:*

(a) *$G$ is regular of even degree,*

(b) *$m$ is divisible by 4,*

(c) *$G$ is regular and has a 1-factor,*

(d) *$G$ is cubic.*

A generalization of Proposition 5.1 and of Corollary 5.2 is obtained using different methods in a forthcoming paper [**11**]. However the methods do not give a generalization of the other results of this paper.

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*Note added in proof.* M. Truszczyński found a counterexample to our Conjecture 1.1.

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