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THE $(\psi, \xi, \eta, \overline{g})$ STRUCTURE ON SUBSPACES OF THE SPACE WITH THE $\varphi(4,-2)$ STRUCTURE

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Abstract. Let a tensor field φ, $\varphi \neq 0$, $\varphi \neq 1$, of type (1,1) and of class C^∞ be given on M^n such that $\varphi^4 - \varphi^2 = 0$, and rank $\varphi = n-1$. The structure $\Phi = 2\varphi - 1$ is an almost product structure. Φ induces on hypersurface K a Sato structure. In this paper it is proved that the structure Sato ψ induced by Φ on K^* is equal to the $\overline{\varphi}$.($\overline{\varphi}$ is the restriction of the structure φ on K^*).

Introduction. In [**1**] Yano, Houh and Chen consider the structure called a $\varphi(4,-2)$ structure, defined by a tensor field φ of type (1,1) satisfying $\varphi^4 - \varphi^2 = 0$ and they study the existence of this structure.

In this paper we study a $\varphi(4,-2)$ structure of rank $r = n-1$ and the restriction of the structure φ on the hypersurface K. In **3.** we shall examine the relation between the almost product structure $\Phi = 2\varphi^2 - 1$ and $\varphi/_{K^*}$.

1. Preliminaries. Let $\mathcal{M}^n$ be an n-dimensional differentiable manifold of class C^∞, and let the $C^\infty(1,1)$ tensor fields f_1 and f_2 be given such that $f_1^2 = 1$, $f_1^2 = 0$. Then f_1 is an almost product structure, and f_2 is an almost tangent structure. Let a tensor field $\varphi, \varphi \neq 0$ and $\varphi \neq 1$, of type (1,1) and of class C^∞ be given on $\mathcal{M}^n$ such that $\varphi^4 - \varphi^2 = 0$ and rank $\varphi =$(rank $\varphi^2 + \dim \mathcal{M}^n)/2 = r$.

Let $\mathbf{l} = \varphi^2$, $\mathbf{m} = 1 - \varphi^2$, then $\varphi\mathbf{l} = \mathbf{l}\varphi = \varphi^3$, $\varphi\mathbf{m} = \mathbf{m}\varphi = \varphi - \varphi^3$, $\varphi^2\mathbf{l} = \mathbf{l}^2 = \mathbf{l}$, $\varphi^2\mathbf{m} = \mathbf{m}\varphi^2 = 0$.

Let $\Phi = \mathbf{l} - \mathbf{m} = 2\varphi^2 - 1$. Then it is clear that Φ defines on $\mathcal{M}^n$ an almost product structure if $\varphi^2 \neq 1$. Let L and M be the distributions corresponding to $\mathbf{l}$ and $\mathbf{m}$ respectively. We assume that $\varphi' = \varphi/L$ is not the identity operator of L. Then φ acts on L as an almost product structure operator and on M as an almost tangent structure operator. Moreover, $\dim M = 2(n-r)$ and $\dim L = 2r - n$. Such a structure φ is called a $\varphi(4,-2)$ structure of rank r.

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If the rank of φ is maximal, $r=n$, the $\varphi(4,-2)$-structure is an almost product structure and if the rank of φ is minimal, $2r=n$, the $\varphi(4,-2)$-structure is an almost tangent structure.

In [**1**] it has been proved that a necessary and sufficient condition for an n-dimensional manifold to admit a tensor field φ, $\varphi\neq 0$ and $\varphi\neq 1$ of type (1,1) defining a $\varphi(4,-2)$-structure is that the group of the tangent bundle of the manifold be reduced to the group $0(h)\times 0(2r-n-h)\times 0(n-r)\times 0(n-r)$ $h=\dim L_1$, L_1 being the subspace of L corresponding to the eigen value $+1$ of φ:

With respect to the adapted frame the tensors g_{ij} and φ^i_j have the components

$$g=\begin{bmatrix} E_h & 0 & 0 & 0\\ 0 & E_{2r-n-h} & 0 & 0\\ 0 & 0 & E_{n-r} & 0\\ 0 & 0 & 0 & E_{n-r}\end{bmatrix}\quad \varphi=\begin{bmatrix} E_h & 0 & 0 & 0\\ 0 & -E_{2r-n-h} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & E_{n-r} & 0\end{bmatrix}$$

I. Sato [**2**] introduced and studied almost paracontact Riemannian manifold V with the structure (ψ,ξ,η,g) that is, an n-dimensional differentiable manifold with a tensor field ψ of type (1,1), a positive definite Riemannian metric g, a vector field ξ and a 1-form η satisfying

(1) $\psi^2=I-\otimes\xi,\ \psi\xi=0,\ \eta\psi=0,\ \eta(\xi)=1,$

(2) $\eta(X)=g(\xi,X),\ g(\psi X,\psi Y)=g(X,Y)-\eta(X)\eta(Y),\ X,Y\in\mathcal{X}(V)$

where I is the identity and $\mathcal{X}(V)$ denotes the set of differentiable vector fields on V. Such a manifold is called an almost paracontact Riemannian manifold, and its structure an almost paracontact Riemannian structure. A structure which satisfies only condition (1) is called a Sato structure. The following theorem is proved in [**4**].

THEOREM 1.1. *The almost product structure Φ induces on a hypersurface the Sato stucture ψ in the following way*

$$\Phi B=B\psi\oplus(\eta\otimes N),\quad \Phi N=B\xi,$$

where B is the differential of the immersion i Kinto $\mathcal{M}^n$.

Proof. $\Phi B=B\psi\oplus(\eta\otimes N),\qquad \Phi^2BX=\Phi[B\psi\oplus(\eta\otimes N)]X,$

$$\begin{aligned}\Phi^2BX=&\Phi[B\psi X\oplus\eta(X)N],\qquad BX=\Phi B(\psi X)+\eta(X)\Phi(N)\\ BX=&[B\psi\oplus\eta\otimes N](\psi X)+\eta(X)\Phi(N),\\ BX=&[B\psi^2(X)+\eta\psi(X)N]+\eta(X)\Phi(N)\\ BX=&B(X)-\eta(X)B\xi+0+\eta(X)B\xi,\qquad BX=BX\end{aligned}$$

and

$$\begin{aligned}\Phi N=&B\xi,\qquad \Phi^2N=\Phi B\xi,\\ N=&(B\psi\oplus(\eta\otimes N))\xi,\qquad N=B\psi\xi+\eta(\xi)N,\qquad N=N.\end{aligned}$$

2. The structure $(\overline{\psi},\xi,\eta,\overline{g})$, on K. We shall assume that rank $\varphi = n-1$. Then M is a 2-dimensional manifold. Let K be a hypersurface in $\mathcal{M}^n$ orthogonal on vector

$$N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{in } \mathcal{M}^n.$$

Let $\overline{\varphi}, \overline{m}$ and $\overline{g}$ be restrictions of the structure φ and tensors m and g on K, and let

$$\xi = \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right\} n-1 \quad \eta = \underbrace{(0,\dots,0,1)}_{n-1}.$$

$\overline{\varphi}, \overline{m}$ and $\overline{g}$ have matrixes of the form

$$\overline{\varphi} = \begin{bmatrix} E_h & 0 & 0 \\ 0 & -E_{2r-n-h} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \overline{m} = \begin{bmatrix} 0_h & 0 & 0 \\ 0 & 0_{2r-n-h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \overline{g} = \begin{bmatrix} E_h & 0 & 0 \\ 0 & E_{2r-n-h} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

THEOREM 2.1. $\overline{\varphi}$ *is a Sato structure.*

Proof. Since $\overline{\varphi}^2 = 1 - \overline{m}$, multiplying the corresponding matrixes it is clear that $\overline{m} = \xi\eta$, $\overline{\varphi}^2 = I - \eta \otimes \xi$, $\overline{\varphi}\xi = 0$, $\overline{\varphi\eta} = 0$, $\xi(\eta) = 1$, and moreover:

THEOREM 2.2. $(\overline{\varphi},\ \xi,\ \eta, \overline{g})$ *is an almost paracontact Riemannian structure on K.*

Proof. It is clear that $\eta(X) = \overline{g}(\xi, X)$, $\overline{g}(\overline{\varphi}X, \overline{\varphi}Y) = \overline{g}(X,Y) - \eta(X)\eta(Y)$ which prves the theorem.

In Theorem 1.1. it is proved that an almost product structure induces on a hypersurface a Sato structure. From this and from Theorems 2.1 and 2.2. we obtain the following:

THEOREM 2.3. *The almost product structure $\Phi = 2\varphi^2 - 1$ induces on K a structure Sato moreover an almost paracontact Riemannian structure.*

3. Relation between ψ and the $(\overline{\varphi},\xi,\eta,\overline{g})$ structure. We shall examine what conditions must be satisfied so that the structure ψ induced by $\Phi = 2\varphi^2 - 1$ on K^* is equal to the structure $\overline{\varphi}$.

Let K^* be the subspace of K whose vectors have the form

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ 0_1 \\ \vdots \\ 0_{2r-n-h} \\ z_1 \end{bmatrix}$$

THEOREM 3.1. *The almost product structure* Φ *induces on* K^* *the Sato structure* $\overline{\varphi}$.

Proof. We shall prove the relations $\Phi B = B\overline{\varphi} \oplus (\eta \otimes N)$ and $\Phi N = B\xi$ on K^*. That $\Phi N = B\xi$ is clear using

$$\Phi = \begin{bmatrix} E_h & 0 & 0 & 0 \\ 0 & E_{2r-n-h} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

To prove the relation $\Phi B = B\overline{\varphi} \oplus (\eta \otimes N)$ on K^*, we shall prove $BX = \Phi B(\overline{\varphi}X) + \eta(X)\Phi(N)$ for the vectors $X \in K^*$.

Let $X \in K$, we obtain

$$BX = \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ y_1 \\ \vdots \\ y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix}, \quad \Phi B(\overline{\varphi}X) + \eta(X)\Phi(N) = \Phi \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ -y_1 \\ \vdots \\ -y_{2r-n-h} \\ 0 \\ 0 \end{bmatrix} + z_1 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_h \\ -y_1 \\ \vdots \\ -y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ y_1 \\ \vdots \\ y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ -y_1 \\ \vdots \\ -y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix}$$

when $y_1 = 0, \dots y_{2r-n-h} = 0$. From this it is easy to see that $\Phi B = B\overline{\varphi} \oplus (\eta \otimes N)$ only on the space K^*. This proves the Theorem.

Since $\overline{\varphi}$ and $\overline{g}$ satisfy the following on $K^* : \eta(X) = \overline{g}(\xi, X)$, $g(\overline{\varphi}X, \overline{\varphi}Y) = g(X,Y) - \eta(X)\eta(Y)$, we have

THEOREM 3.2. *The almost product structure* Φ *induces on* K^* *the almost paracontant Riemannian structure* $(\overline{\varphi}, \xi, \eta, \overline{g})$.

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