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NON-ANTICIPATIVE INTEGRAL TRANSFORMATIONS OF STOCHASTIC PROCESSES

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Summary. Let X be a stochastic process, defined on the interval $[0;1]$, and Y its non-anticipative integral transformation defined by

$$Y(t)=\int_0^t g(t,u)X(u)du \tag{1}$$

In this paper we shall investigate conditions related to the family

$$G=\{g(t,u),\ t\in[0;1],\ u\leq t\} \tag{2}$$

under which the process Y : 1° generates the spaces $H(Y;t)$ equal to the corresponding spaces $H(X;t)$ of the process $\mathbf{X}$; 2∘ belongs to the same class as the process X; 3° is continuous, provided X is continuous.

Introduction. Let $X=\{X(t),\ t\in[0;1]\}$ be a stohastic process of the second order, that is $\|X(t)\|<\infty$ for each $t\in[0;1]$; the inner product and the norm are defined as in [**2**]. The Hilbert spaces generated by elements $X(s),\ s\leq t$, will be denoted by $H(X;t)$. In the whole paper, we assume that all considered processes satisfy the following conditions: (a) $H(X;0)=0$ and (b) $X(t)$ is continuous in the quadratic mean for each t. From (b) it follows immediately that the space $H(X)$ is separable [**1**].

At first, we consider conditions related to the family G, under which the operator $A:X\to Y$ defined by (1) on the curve determined by X, is linearly extendable to the whole $H(X)$. We assume that $\|X(t)\|<M,\ t\in[0;1]$ and that the operator A is bounded, that is that there exists a constant $K>0$ such that $\|A\|<K$. For instance, it is sufficient to assume that the condition $\int_0^t\int_0^t g(t,u)g(t,v)dudv<\infty$ is satisfied for each $t\in[0;1]$.

THEOREM 1. *If the operator A from* (1) *is linearly and bounded on elements from $\{X(t),\ t\in[0;1]\}$, then it can be linearly extended to $H(X)$.*

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Proof. The operator A is linear on the curve $X(t)$ iff

(3) $$A(\alpha X(t_1)+\beta X(t_2))=\alpha AX(t_1)+\beta AX(t_2);\ \alpha,\beta\in R;\ t_1,\ t_2\in[0;1].$$

We shall show that the operator A is linear on $H(X)$, i.e., that

(4) $$A(\alpha x+\beta y)=\alpha Ax+\beta Ay,$$

holds for all $x,y\in H(X)$ and $\alpha,\ \beta\in R$.

The space $H(X)$ contains:

1° finite linear combinations of elements from the curve determined by $X(t)$;

2° limits of sequences of elements of the form 1°.

Let x_n and y_m be elements of the form:

$$x_n=\sum_{i=1}^{n}\alpha_iX(t_i),\ y_m=\sum_{j=1}^{m}\beta_jX(t_j^*);\ \alpha_i,\ \beta_j\in R;\ t_i,\ t_j^*\in[0;1].$$

Assuming that

$$\overline{t_i}=\begin{cases} t_i, & i=\overline{1,n}\\ t_{i-n}^*, & i=\overline{n+1,n+m}\end{cases},$$

we have

$$A(\alpha x_n+\beta y_m)=A\sum_{i=1}^{n+m}\gamma_iX(\overline{t_i})$$

where

$$\gamma_i=\begin{cases}\alpha\alpha_i, & i=\overline{1,n}\\ \beta\beta_{i-n}, & i=\overline{n+1,\ n+m}\end{cases},$$

so that (4) follows from (3).

Let x and y be elements of the form 1°; we have

(5) $$A(x+y)=A\ \underset{n,m\to\infty}{\text{l.i.m.}}(x_n+y_m)=\underset{n,m\to\infty}{\text{l.i.m.}}A(x_n+y_m).$$

Relation (5) is equivalent to

(6) $$\underset{n\to\infty}{\text{l.i.m.}}Ax_n+\underset{m\to\infty}{\text{l.i.m.}}Ay_m+A\ \underset{n\to\infty}{\text{l.i.m.}}x_n+A\ \underset{m\to\infty}{\text{l.i.m.}}y_m=Ax+Ay$$

From (5) and (6)

$$A(x+y)=Ax+Ay.$$

Therefore, operator A is linear on $H(X)$.

LEMMA 1. *The process* $X(t)\int_0^t h(t,u)dZ(u)$, *where* Z *is a process with orthogonal increments such that* $\|dZ(u)\|^2=\sigma dv$, $\sigma>0$, *is continuous in the quadratic mean iff the functions* $h(t,u)$ *are continuous in the first argument for almost all values of the second argument.*

Proof. Let the functions $h(t,u)$ be continuous in the point $t_0 \in [0;1]$ for every u, i.e.,

$$\text{Leb}\{u : | h(t,u) - h(t_0,u) | \to 0,\ t \to t_0\} = 1. \tag{7}$$

We shall show that the process $X(t)$ is continuous in the quadratic mean in t_0 that is

$$\|X(t) - X(t_0)\| \to 0,\ t \to t_0. \tag{8}$$

For each $t, t_0 \in [0;1]$ we have

$$\begin{aligned} &\|X(t) - X(t_0)\| = \\ &= \left\| \int_0^{\min\{t,t_0\}} [h(t,u) - h(t_0,u)] dZ(u) + I_{\{v:v \geq t_0\}}(t) \int_{t_0}^{t} h(t,u) dZ(u) - \right. \\ &\left. - I_{\{v:v<t_0\}}(t) \int_t^{t_0} h(t_0,u) dZ(u) \right\| = \sigma \int_0^{\min\{t,t_0\}} | h(t,u) - h(t_0,u) |^2 du + \\ &+ I_{\{v:v \geq t_0\}}(t) \sigma \int_{t_0}^{t} | h(t,u) |^2 du - I_{\{v:v<t_0\}}(t) \sigma \int_t^{t_0} | h(t_0,u) |^2 du. \end{aligned} \tag{9}$$

Since we have (7), the last sum tends to zero as $t \to t_0$; thus $X(t)$ is contiuous in the quadratic mean in t_0.

Moreover, it is easily seen that conversely, in view of (9), relation (8) follows from relation (7).

1. Equality of spaces $H(X,t)$ and $H(Y;t)$ for each $t \in [0;1]$. We shall investigate conditions related to the function family G, given by (2), under which the spaces $H(Y;t)$, generated by Y, are identical to the corresponding spaces $H(X;t)$ of X for each $t \in [0;1]$.

Let $X(t) = \sum_{i=1}^{N} \int_0^t h_i(t,u) dZ_i(u)$, where N is an arbitrary natural number or infinity, be the Hida-Cramer representation of the stohastic process $\{X(t),\ t \in [0;1]\}$ of the second order. It is known that $H(X;t) = \sum_{i=1}^{N} \oplus H(Z_i;t)$ iff the family $\{h_i(t,\cdot),\ t \in [0;1],\ i = 1,\dots,N\}$ is complete with respect to $F_Z = (F_1,\dots,F_N)(F_n(t) = E \mid Z_n(t) \mid^2,\ 0 \leq t \leq 1,\ n = \overline{1,N})$. The transformation of X, defined by (1) can be written as

$$Y = \sum_{i=1}^{N} \int_0^t \left[\int_v^t g(t,u) h_i(u,v) du \right] dZ_i(v). \tag{10}$$

According to [**5**] equality $H(Y;t) = \sum\limits_{i=1}^{N} \oplus H(Z_i;t),\ t \in [0;1]$ holds iff the family $\left\{\int\limits_v^t g(t,u)h_i(u,v)du,\ t \in [0;1],\ i = 1,\dots,N\right\}$ is complete with respect to $F_Z = (F_1,\dots,F_N)$. Let $f(\cdot) = (f_1(\cdot),\dots,f_N(\cdot))$ be the function from $L_2(dF_Z)$ such that

$$\sum_{i=1}^{N} \int_0^t \left[\int_v^t g(t,u)h_i(u,v)du\right] f_i(v)dF_i(v) = 0 \text{ for each } t \le t_0. \tag{11}$$

It is equivalant to

$$\int_0^t g(t,u)\left[\sum_{i=1}^{N} \int_0^u h_i(u,v)f_i(v)dF_i(v)\right]du = 0.$$

If the family G, from (2), is complete with respect to the Lebesque measure then from (11) it follows that

$$\sum_{i=1}^{N} \int_0^t h_i(u,v)f_i(v)dF_i(v) = 0 \text{ almost everywhere on } [0;t_0]. \tag{12}$$

If the functions $h_i(u,v),\ i = 1,\dots,N$ are continuous in the first argument for almost all v, then, relation (12) holds everywhere on $[0;t_0]$. By assumption, the family $\{h_i(u,\cdot),\ u \in [0;1];\ i = 1,\dots,N\}$ is complete, so that (11) implies $f_i(v) = 0,\ i = 1,\dots,N$ almost everywhere with respect to $F_Z = (F_1,\dots,F_N)$.

But since the continuity of functions $h_i(u,v),\ i = 1,\dots,N$ in the first argument for almost all values of the second argument, is equivalent to the continuity of X in the quadratic mean, we proved:

THEOREM 2. *If X is a second order stohastic process with multiplicity N(N is an arbitrary natural number of infinity), the process Y is defined by* (1), *the family G from* (2) *is complete with respect to the Lebesque measure, and X is continuous in the quadratic mean, then the equality $H(X;t) = H(Y;t)$ holds for every $t \in [0;1]$.*

CONSEQUENCE 1. *If X is a Markov process, the process Y is defined by* (1), *and the family G from* (2) *is complete with respect to the Lebesgue measure, then the equality $H(X;t) = H(Y,t)$ holds for each $t \in [0;1]$.*

Remark: The completeness of the family G with respect to the Lebesgue measure is sufficient, but not necessary for the equality of the spaces $H(X;t)$ and $H(Y;t),\ t \in [0;1]$.

2. Some conditions under which X and Y belong to the same class of stochastic processes. Here we shall determine some conditions related to the family G, from (2), under which the given process X and the process Y, defined by (1), belong to the same class of processes.

If X is Markov process, then according to [**3, 4**]

(13) $$X(t) = h(t)Z(t), \ \ t \in [0;1], \ \ h(t) \neq 0 \ \ \text{almost everywhere}$$

is its Hida-Cramer representation. Z is a process with orthogonal increments, so that (13) becomes

$$X(t) = \int_0^t h(t) dZ(u); \ \ u \in [0;1].$$

The transformation (1) of X becomes

(14) $$Y(t) = \int_0^t g_*(t,v) dZ(v)$$

where

(15) $$g_*(t,v) = \int_v^t g(t,u)h(u)du.$$

Assume that family G, from (2), is complete with respect to the Lebesgue measure. Then, according to Consequence 1., equality $H(X;t) = H(Y;t)$ holds for each $t \in [0;1]$. If Y is a Markov process, then [**5**] for each $s,t \in [0;1]$, $s \leq t$, the projection of $Y(t)$ onto $H(Y;s)$ coincides with the projection of $Y(t)$ onto element $Y(s)$

(16) $$P_{H(Y;s)}Y(t) = a(t,s)Y(s), \ \ s \leq t,$$

where $a(t,s)$ [**5**] is a scalar function defined for $s,t \in [0;1]$, $s \leq t$ by

$$a(t,s) = r(t,s)/r(s,s), \ \ s \leq t$$

and $r(s,t)$ is the correlation function of the process Y. Function $a(\cdot,\cdot)$ satisfies conditions (see [**3, 5**])

$$a(t_3,t_1) = a(t_3,t_2) \cdot a(t_2,t_1) \ \ \text{for any} \ \ t_1 \leq t_2 \leq t_3; \ t_1,t_2,t_3 \in [0;1],$$

and

$$a(t_2,t_1) = h(t_2)/h(t_1) \ \ \text{for any} \ \ t_1 \leq t_2; \ t_1,t_2 \in [0;1].$$

From (16) it follows that

$$(Y(t_3) - a(t_3,t_2)Y(t_2), \ \ Y(t_1)) = 0 \ \ \text{for all} \ \ t_1 \leq t_2 \leq t_3.$$

The last relation is, according to (14), equivalent to

$$\int_0^{t_1} [g_*(t_3,v) - a(t_3,t_2)g_*(t_2,v)]g_*(t_1,v)dv = 0$$

or

$$\int_0^{t_1} [g_*(t_3,v) - h(t_3)/h(t_2) \cdot g_*(t_2,v)] g_*(t_1,v) dv = 0.$$

Since $g_*(t_1,v) = 0$ for $v > t_1$, the last equality implies

$$\int_0^{\infty} [g_*(t_3,v) - h(t_3)/h(t_2) \cdot g_*(t_2,v)] g_*(t_1,v) dv = 0.$$

and that means

$$g_*(t_1,\cdot) \perp g_*(t_3,\cdot) - h(t_3)/h(t_2) \cdot g_*(t_2,\cdot) \text{ in } L_2(dv). \tag{17}$$

By analogy with the definition of the process with orthogonal increments, we can define the function families with orthogonal increments if under "increment" of the function family $G_*(t,\cdot) = \{g_*(t,\cdot),\ t \in [0;1]\}$ on (t_1,t_2) one means the difference $G_*(t_2,\cdot) - h(t_2)/h(t_1) \cdot G_*(t_1,\cdot)$, and the inner product is defined in the usual way as in $L_2(dv)$. Then condition (17) means that the family G_* from (15) has the orthogonal increments. In that way we proved:

THEOREM 3. *Let X be a Markov process, and let family G from* (2) *be complete with respect to the Lebesgue measure, and family G_* from* (15) *has orthogonal increments. Then the process Y, defined by* (1), *is a Markov process.*

COROLLARY 1. *If X is a stohastic process with orthogonal increments, and the family $G_* = \{g_*(t,\cdot),\ t \in [0;1]\}$ (where $g_*(t,v) = \int_v^t g(t,u)du$) has orthogonal increments, then the process Y, defined by* (1) *has orthogonal increments too.*

It is easily seen that by "increment" of the function family $G_* = \{g_*(t,\cdot),\ t \in [0;1]\}$ on (t_1,t_2) one means the difference $G_*(t_2,\cdot) - G_*(t_1,\cdot)$.

3. Some sufficient conditions for continuity of the process Y. Here, we determine the conditions related to the family G, from (2), under which the process Y, defined by (1) is continuous, if X is continuous.

Let $X(t) = \sum_{i=1}^{N} \int_0^t h_i(t,u) dZ_i(u)$ be the Hida-Cramer representation of X.

Then, according to (10)

$$Y(t) = \sum_{i=1}^{N} \int_0^t \left[\int_v^t g(t,u) h_i(u,v) du \right] dZ_i(v).$$

By Lemma 1, the process Y is continuous iff functions $g^*(t,v) = \int_v^t g(t,u) h_i(u,v) du$, $i = 1, \dots, N$ are continuous on t, for almost all v,

i.e., if Leb $\{v :| g^*(t,v) - g^*(t_0,v) |\to 0,\ t\to t_0\} = 1$.
But, this is equivalent to

$$0 = \lim_{t\to t_0}\left(\int_v^{t_0}[g(t,u)-g(t_0,u)]h_i(u,v)du + \int_{t_0}^{t} g(t,u)h_i(u,v)du\right) =$$
$$= \int_v^{t_0}\lim_{t\to t_0}[g(t,u)-g(t_0,u)]h_i(u,v)du,$$

i.e., to the condition that all functions from family G are continuous in t for almost all u. Thus, we have:

THEOREM 4. *Let X be a continuous second order stohastic process, and let the functions from family G be continuous in the first argument for almost all values of the second argument. Then, the process Y, defined by* (1) *is continuous.*

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