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THE NOTIONS OF w-NET AND Y-COMPACT SPACE VIEWED UNDER INFINITESIMAL MICROSCOPE

Rade Živaljević

Abstract. Nonstandard analysis of A. Robinson is used to give a nonstandard description of the notion of w-net introduced by H.J. Wu in [**6**]. This concept leads to the notion of Y-compact space so that $[0,1]$-compact spaces are compact in the usual sence while R-compact spaces are E. Hewit's realcompact spaces.

Among typical applications of A.Robinson's infinitesimal (nonstandard) analysis are constructions of various kinds of completions of topological spaces, algebras, Banach spaces etc. If X is the object under consideration, one usually starts with a subset $A \subset^* X$ of the nonstandard picture of the object X, and defines the completion of X as the set $c(X) := A/\sim$ where $\sim$ is a suitable equivalence relation on the set A. Examples of this kind can be found in almost any book or paper concerning this subject, like A. Robinson [**5**], W.A.J. Luxemburg and K.D. Stroyan [**4**], M. Davis [**1**], W. Henson [**3**] or J.C. Dyre [**2**], and they are both standard (e. g. Stone-Čech compactification) and nonstandard (nonstandard hulls of Banach space).

In [**6**] Hueytzen J. Wu introduced the concept of w-net and used it, among other applications, to state a general form of the Tychonoff Compactness Theorem. Also, this notion leads to the definition of w-complete spaces and w-completions, which seems to be a useful unification and generalization of compactifications and the real-compactification of a Tychonoff space X.

The aim of this note is to give nonstandard characterizations of concepts mentioned above and to use them to discuss some properties of a naturally defined class of Y-compact spaces, for a given Hausdorff space Y. All definitions and concepts of nonstandard analysis used in this paper are standard and can be found in any text on the subject. For a one-page account of the main definitions and principles, it is recommended to the interested reader to see "Non-standard analysis for pedestrians" in [**2**]. The nonstandard model is assumed to be polysaturated.

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1. w-nets versus w-points

Definition 1. (Wu [**6**]) Let $\mathcal{A} = \{f_i\}_{i\in I}$ be a family of continuous functions on a topological space X. A net $\langle x_m | m \in D\rangle$, where $\langle D, \leq\rangle$ is a directed set, is called a w-net induced by $\mathcal{A}$ if $f_i\langle x_m | m \in D\rangle := \langle f_i x_m | m \in D\rangle$ converges for each $i \in I$.

Definition 2. Let $\mathcal{A}$ be as before. A point $x \in^* X$ is called w-point if for each $i \in I^* f_i(x)$ is near-standard in $*X_i$. The set of all w-points is denoted by $W(^*X)$.

PROPOSITION 1. *A net $\langle X_m | m \in D\rangle$ in space X is a w-net induced by the family $\mathcal{A}$ if and only if x_m is a w-point for all $m \in \inf(^*D)$ and $(\forall i \in I)\,(\forall m, n \in \inf(^*D))^* f_i x_m \approx^* f_i x_n$ where $\inf(^*D := \{m \in^* D | (\forall n \in D)\ n \leq m\}$*

Proof. The following two equivalences are well-known.

(a) $\langle y_m | m \in D\rangle$ converges iff $(\forall m, n \in \inf(^*D))(y_m, y_n \in ns(^*X)$ and $y_m \approx y_n)$.

(b) $\langle y_m | m \in D\rangle$ has a cluster point in $E = cl(E) \subset X$ iff $\exists m \in inf(*D))$ $(y_m \in ns(^*E))$.

As usual $m(x)$ is the monad of $x, ns(^*X) := \{y \in^* X | (\exists x \in X) y \in m(x)\}$ and $y_1 \approx y_2$ means that both y_1, y_2 are in the same monad.

The proposition easily follows from (a).

PROPOSITION 2. (Wu[**6**]) *Let $\mathcal{A} = \{f_i\}_{i\in I}$ be a family of continuous functions f_i on X into Hausdorff spaces X_i such that the topology on X is the initial (weak) topology induced by $\mathcal{A}$. Let E be a subset of X . The following are equivalent:*

(1) *Every w-net in E induced by $\mathcal{A}$ has a cluster point in the closure cl(E) of E.*

(2) *Every w-net in cl(E) induced by $\mathcal{A}$ converges in cl(E).*

Proof (nonstandard). (2)→(1) is trivial, so let us prove (1)→(2). By Proposition 1 and the equivalence (a) of its proof, (2) is equivalent with

(3) every w-point $x \in^* (clE)$ is near-standard $in^* X$.

Assume (1). Let $x \in^* (clE)$ be a w-point. Define $\mathcal{D} = \{O | O \subset X$ open and $x \in^* O\}, \langle \mathcal{D}, \supset\rangle$ is a directed set. Obviously $O \cap E \neq \varnothing$ for every $O \in \mathcal{D}$; so let $\langle y_O | O \in \mathcal{D}\rangle$ be a net in E such that $y_O \in O \cap E$ for every $O \in \mathcal{D}$. This net is a w-net. Indeed, let for a given $f_i \in \mathcal{A}$, $z_i = st^* f_i(x) \in X_i$. Then for every open $V \ni z_i$, $f_i^{-1} V$ is an open set such that $x \in (^*f_i)^{-1}(^*V) =^* (f_i^{-1} V)$. Hence, $f^{-1} V \in \mathcal{D}$, which proves that $\langle f y_O | O \in \mathcal{D}\rangle$ converges to z_i. By (1) $\langle y_O | O \in \mathcal{D}\rangle$ has a cluster point in clE, hence $y_m \in ns(clE)$ for some $m \in \inf(^*D)$. Let $y \in st\, y_m$ (we do not assume that X is Hausdorff). From the definition of $\mathcal{D}$ it follows that $(\forall f \in \mathcal{A})^* f(x) \approx^* f(y_m) \approx f(y)$. But, since the topology on X is induced by $\mathcal{A}$, we see that $x \in m(y)$, which proves that x is near-standard.

The proposition above yields the following version of the Tychonoff Compactness Theorem proved by Wu in [**6**].

THEOREM 1. *Let $\mathcal{A}$ be a family of continuous functions on a topological space X. Then X is compact iff*

(1) *$f(X)$ is contained in a compact subset C_f for each $f \in \mathcal{A}$ and*

(2) *every w-net induced by $\mathcal{A}$ has a cluster point in X.*

Proof. (nonstandard). Let $x \in^* X$. Condition (1) implies that ${}^*f(x) \in ns({}^*C_f)$ for every $f \in \mathcal{A}$, hence x is w-point. Condition (2) is by the proof of Proposition 2 equivalent with

(3) *every w-point $x \in^* X$ is near-standard.*

So, $x \in ns({}^*X)$. By the well-known nonstandard characterization of compactness X is compact.

2. w-completing of topological spaces

All definitions in this paragraph are nonstandard versions of definitions taken from the paper of Wu quoted above. Let $W({}^*X)$ be the set of all w-points, i.e., $W({}^*X) = \{x \in^* X | (\forall f_i \in \mathcal{A})^* f_i(x) \in ns({}^*X_i)\}$. For $x, y \in W({}^*X)$, let $x \sim y$ iff $(\forall f \in \mathcal{A})^* f(x) \approx^* f(y)$. Then, $\langle Y, e \rangle$, where $Y = W({}^*X)/\approx$ and $e: X \to Y$ is the mapping induced by $i: X \to^* X$ (note that it is not necessarily one-to-one) is called the w-transformation of X induced by $\mathcal{A} = \{f_i\}_{i \in I}$. It is clear that every function f_i can be extended to Y by $\overline{f_i}([x]) = st^* f_i(x)$ for $x \in W({}^*X)$ and $[x] \in Y$. It will be assumed that Y is equipped with the initial topology induced by the family $\overline{\mathcal{A}} = \{\overline{f_i}\}_{i \in I}$ which makes $e: X \to Y$ continuous and $e(X)$ dense in Y. Space Y is a "completion" of X in the following sence.

PROPOSITION 3. (Wu [**6**]) *Every w-net in Y induced by $\overline{\mathcal{A}}$ converges in Y.*

Proof (nonstandard). Let $\langle [y_n] | n \in D \rangle$ be an w-net in Y where $y_n \in W({}^*X)$. By saturation we can assume that there exists an internal function $y: {}^*D \to^* X$ such that $(\forall n \in D) y_n = y(n)$. Let $z_i \in Y_i$ be defined by $\lim\limits_{n \in D} st^* f_i(y_n) = z_i$. Let $\mathcal{H}_{i,O} = \{m \in^* D | f_i y(m) \in^* O\}$ and $\mathcal{G}_n = \{m \in^* D | n \leq m\}$ for $i \in I$, O is an open neighborhood of z_i and $n \in D$. Since $\lim \langle st^* f_i y_n) | n \in D \rangle = z_i$ the family

$$\mathcal{U} = \{\mathcal{H}_{i,O} | i \in I \text{ and } O \in \mathcal{B}(z_i)\} \cup \{\mathcal{G}_n | n \in D\}$$

has the finite intersection property. By saturation $\cap \mathcal{U} \neq \varnothing$. So let $m \in \cap \mathcal{U}$. It is easy to see that $y(m)$ must be a w-point in $*X$ and that $\langle [y_n] | n \in D \rangle$ converges to $y(m)$.

Definition 3. If $\langle Y, e \rangle$ is the w-transformation of the space X such that X is a Hausdorff space equipped with the initial topology induced by $\mathcal{A}$, then $e: X \to Y$ ia an embedding and the w-transformation is called w-completion of the space X.

The proof of the last proposition in this paragraph is left to the reader.

PROPOSITION 4. *The w-completion $\langle Y, e \rangle$ of the Hausdorff space X with respect to the family $\mathcal{A} = \{f_i\}_{i \in I}$ has the following universal property. For any pair*

$\langle Z, r\rangle$ and family $\mathcal{B} = \{g_i\}_{i\in I}$ such that Z is a Hausdorff space with the initial topology induced by $\mathcal{B}$, if $r : X \to Z$ is an embedding onto a dense subspace of Z and $(\forall i \in I)\underline{g_i} \circ r = f_i$, then there exists a unique mapping $j : Z \to Y$ such that $(\forall i \in I) g_i = \overline{f_i} \circ j$.

Definition 4. Let X and Y be two Hausdorff spaces such that the topology on X is induced by the family $\mathcal{A} = \mathrm{Map}(X, Y)$ of all continuous functions from X to Y(X will be referred to as a Y-Tychonoff space). If X coincides with its w-completion w.r.t. family $\mathcal{A}$, it is called a Y-compact space. Another characterization is the following. A Y-Tychonoff space is Y-compact if $(\forall x \in^* X)[(\forall f \in \mathrm{Map}(X,Y))^* f(x) \in ns(Y)] \to x \in ns(^*X)$.

Let us note that R-Tychonoff spaces are Tychonoff spaces in the usual sence, while R-compactness coincides with realcompactness. So it is not surprising that Y-compact spaces share some properties of realcompact spaces.

PROPOSITION 5. *Let us assume that the product $\Pi\langle X_i | i \in I\rangle$ of a family of Y-Tychonoff spaces is again a Y-Tychonoff space. Then if X_i is Y-compact for every $i \in I$, $\Pi\langle X_i | i \in I\rangle$ is also a Y-compact space.*

Proof. Let $f \in^* (\Pi\langle X_i | i \in I\rangle)$ be a w-point w.r.t. family $\mathrm{Map}(\Pi\langle X_i | \in I\rangle, Y)$. This implies that $(\forall i \in I)(f(i)$ is a w-point in $X_i)$ so, by assumption, let $g \in \Pi\langle Y_i | i \in I\rangle$ has the property $(\forall i \in I) st\, f(i) = g(i)$. But, the monad of $^*g$ is

$$m(^*g) = \{f \in^* (\Pi\langle X_i | i \in I\rangle) | (\forall i \in I) g(i) \approx f(i)\},$$

hence f is near-standard.

COROLLARY. *Products of realcompact spaces are realcompact.*

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Matematički institut
Knez-Mihailova 35
11000 Beograd
Jugoslavija

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