PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 36 (50), 1984, pp. 67–78

ASYMPTOTIC PROPERTIES OF CONVOLUTION PRODUCTS OF SEQUENCES

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Abstract. Suppose three sequences $\{a_n\}_{\mathbf{N}}$, $\{b_n\}_{\mathbf{N}}$ and $\{c_n\}_{\mathbf{N}}$ are related by the equation $c_n = \sum_{k=0}^{n} a_{n-k}b_k$. In this paper we examine the asymptotic behavior of c_n/a_n under various conditions on $\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$. If $\sum_{k=0}^{\infty} |b_k| < \infty$ we discuss conditions under which $c_n/a_n \to \sum_{k=0}^{n} b_k$ and give sharp rate of convergence results. From our results we obtain asymptotic expansions of the form

$$c_n = a_n \sum_{k=0}^{\infty} b_k + (a_n - a_{n-1}) \sum_{k=1}^{\infty} kb_k + O(|a_n - a_{n-1}|/n).$$

1. Introduction.

The convolution product of two sequences $\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$ of real numbers is the sequence $c = a * b$ defined by

$$c_n = (a * b)_n = \sum_{k=0}^{n} a_{n-k}b_k \quad (n \geq 0).$$

Given a sequence b one is often interested i relating the asymptotic behavior of the two sequence a and c. Results of this type have been examined by a number of authors. See for instance [1], [10], [14]. For applications in probability theory we refer to [7], [8], [11]. Also related is the paper of Pavlov [12] where the asymptotic behavior of the number of solutions to $x^k = a$ in the symmetric group of order n is considered.

In this paper we want to study the so-called regular variation of sequences.

DEFINITION. A sequence $\{r_n\}_{\mathbf{N}}$ of nonnegative real numbers varies regularly at infinity and with index $\rho \in \mathbf{R}$ if $r_n \sim n^\rho L(n)$ $(n \to \infty)$ where L is slowly varying (s.v.), i.e. $\lim_{t\to\infty} L(tx)/L(t) = 1$ for all $x > 0$. We will often use the notation $r = \{r_n\} \in RV_\rho$. □

AMS Subject Classification (1980): Primary 40 A 05, 40 A 25.

For s.v. functions we refer to [5] and [13]. For regularly varying sequences, see [2] and [15].

The next paragraph contains our main results. Proofs are given in Section 3.

2. Main results

Let $\{a_n\}_{\mathbf{N}}$, $\{b_n\}_{\mathbf{N}}$ and $\{c_n\}_{\mathbf{N}}$ be sequences of nonnegative real numbers related by

$$c_n = (a * b)_n = \sum_{k=0}^{n} a_{n-k} b_k \quad (n \geq 0).$$

The next result is an immediate consequence of Feller [9, p. 447]. In it we describe the asymptotic behavior of c_n in case $\sum_{k=0}^{\infty} b_k$ may be infinite.

LEMMA 2.1. *Let $(\sum_{k=0}^{n} b_k)_{\mathbf{N}} \in RV_\beta$ with $\beta \geq 0$. Then for any $\alpha \geq 0$ and any s.v. $L(x)$ the following two statements are equivalent: as $n \to \infty$*

$$\text{(i)} \quad \sum_{k=0}^{n} a_k \sim n^\alpha L(n), \qquad \text{(ii)} \quad \sum_{k=0}^{n} c_k \sim \frac{\Gamma(1+\alpha)\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)} n^\alpha L(n) \sum_{k=0}^{n} b_k.$$

Furthermore, if $\{a_n\}_N$ is nondecreasing and if $\alpha + \beta > 0$, then (i) *is equivalent to*

$$\text{(iii)} \quad c_n \sim (\alpha+\beta) \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} n^{\alpha-1} L(n) \sum_{k=0}^{n} b_k \qquad (n \to \infty). \ \square$$

If $\{a_n\}_{\mathbf{N}}$ is nonincreasing, the following analogue of lemma 2.1 (i) $\Leftrightarrow$ (iii) will be proved.

THEOREM 2.2. *$\{b_n\}_{\mathbf{N}} \in RV_\beta$ with $\beta > -1$ and let $\{a_n\}_{\mathbf{N}}$ be nonincreasing. For any α, $0 \leq \alpha \leq 1$ and any s.v. $L(x)$ the following two statements are equivalent: as $n \to \infty$*

$$\text{(i)} \quad \sum_{k=0}^{n} a_k \sim n^\alpha L(n) \qquad \text{(ii)} c_n \sim \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} n^\alpha L(n) b_n. \ \square$$

In case $\sum_{k=0}^{\infty} b_k < \infty$ (i.e. $\beta = 0$) and $\{a_n\}_{\mathbf{N}} \in RV_\alpha$, $\alpha > 0$, lemma 2.1 (iii) gives

$$\lim_{n \to \infty} c_n / a_n = \sum_{k=0}^{\infty} b_k.$$

The same result also holds in case $\{a_n\}_{\mathbf{N}} \in RV_\alpha$, $\alpha \leq 0$, if we assume $\sum_{k=0}^{\infty} k^{-\alpha+\varepsilon} b_k < \infty$ for some $\varepsilon > 0$.

LEMMA 2.3. *$\{a_n\}_{\mathbf{N}} \in RV_\alpha$, $\alpha \in \mathbf{R}$ and let $\{b_n\}_{\mathbf{N}}$ be a sequence of real numbers such that $\sum_{k=0}^\infty |b_k| < \infty$. In case $\alpha < 0$ assume that $\sum_{k=0}^\infty k^{-\alpha+\varepsilon}|b_k| < \infty$ for some $\varepsilon > 0$. Then*

$$\lim_{n\to\infty} c_n/a_n = \sum_{k=0}^\infty b_k. \ \square$$

The rate at which c_n/a_n converges to $\sum_{k=0}^\infty b_k$ heavily depends on the rate at which a_{n+1}/a_n converges to 1. More precisely we shall prove

THEOREM 2.4. *Suppose $\{a_n\}_{\mathbf{N}}$ is a sequence of positive real numbers such that for some $\beta \in \mathbf{R}$,*

$$\lim_{n\to\infty} n(a_{n-1}/a_n - 1) = \beta. \tag{2.1}$$

Let $\{b_n\}$ be a sequence of real numbers such that $\sum_{k=1}^\infty k|b_k| < \infty$ and in case $\beta \le 0$ such that $\sum_{k=1}^\infty k^{\beta+1+\varepsilon}|b_k| < \infty$ for some $\varepsilon > 0$. Then

$$\lim_{n\to\infty} n\left(\frac{c_n}{a_n} - \sum_{k=0}^\infty b_k\right) = \beta \sum_{k=1}^\infty k b_k. \ \square \tag{2.2}$$

This result should be compared with Theorem 2 of Bojanić and Lee [1]. There it is assumed that for $n \ge 1$, $|a_{n-1}/a_n - 1| \le \delta_{n-1}$, where $\{\delta_n\}_{\mathbf{N}}$ is some sequence of positive numbers such that as $n \to \infty$, $n\delta_n = O\,(1)$ and $n(\delta_{n-1}/\delta_n - 1) = O\,(1)$. Under these conditions and the condition $\sum_{k=1}^\infty k^\alpha |b_k| < \infty$ for all $\alpha > 0$, Bojanić and Lee prove that as $n \to \infty$,

$$\left|\frac{c_n}{a_n} - \sum_{k=0}^\infty b_n\right| = O(\delta_n).$$

Using (2.1) we get the exact asymptotic result (2.2). Note that (2.1) implies that $\{a_n\}_{\mathbf{N}} \in RV_{-\beta}$ and in case $\beta \neq 0$ (2.1) is equivalent to the regular variation of $\{|a_{n-1} - a_n|\}_{\mathbf{N}}$.

Again talking differences of the sequence $\{a_{n-1} - a_n\}_{\mathbf{N}}$ we can obtain rate of convergence results for (2.2) also. More generally, for $n \ge 0$ let $p_{0,n} = a_n$ and $r_{0,n} = b_n$. For $k \ge 0$ define sequences $p_k = \{p_{k,n}\}_{\mathbf{N}}$ and $r_k = \{r_{k,n}\}_{\mathbf{N}}$ by induction as follows

$$p_{k+1,n} = \begin{cases} p_{k,n-1} - p_{k,n} & \text{if } n \ge k+1 \\ 0 & \text{elsewhere} \end{cases}$$

$$r_{k+1,n} = \sum_{p=n+1}^\infty r_{k,p} \qquad n \ge 0.$$

An application of Theorem 2.4 then yields

COROLLARY 2.5. *Suppose for some $k \ge 1$ $\{p_{k,n}\}_{\mathbf{N}}$ is a sequence of nonnegative real numbers such that for some $\beta \in \mathbf{R}$,*

$$\lim_{n\to\infty} n\frac{p_{k,n-1} - p_{k,n}}{p_{k,n}} = \beta. \tag{2.3}$$

Let $\{b_n\}_{\mathbf{N}}$ *be a sequence of real numbers such that* $\sum_{n=1}^{\infty} n^{k+1}|b_n| < \infty$ *and in case* $\beta \geq 0$ *such that* $\sum_{n=1}^{\infty} n^{k+1+\beta+\varepsilon}|b_n| < \infty$ *for some* $\varepsilon > 0$. *Then*

$$\lim_{n\to\infty} n\left(\left(c_n - \sum_{m=0}^{k-1} p_{m,n}R_m(1)\right)/p_{k,n} - R_k(1)\right) = \beta R_{k+1}(1) \tag{2.4}$$

where $R_m(1) = sum_{n=0}^{\infty} r_{m,n} (m = 0, 1, \ldots, k+1)$. □

In the case $k = 1$, Corollary 2.5 gives

$$\lim_{n\to\infty} n\left(\left(c_n - a_n\sum_{k=0}^{\infty} b_k\right)/(a_{n-1} - a_n) - \sum_{k=1}^{\infty} kb_k\right) = \beta\sum_{k=2}^{\infty}\frac{k(k-1)}{2}b_k.$$

Of course a similar corollary can be stated in terms of the sequence d_k defined by $d_{0,n} = a_n$ and for $k \geq 0$,

$$d_{k+1,n} = \begin{cases} d_{k,n-1} - d_{k,n} & (n \geq k+1) \\ 0 & (n < k+1). \end{cases}$$

In this case the result is

$$\lim_{n\to\infty}\left((-1)^k\left(c_n - \sum_{m=0}^{k-1}(-1)^m d_{m,n}R_m(1)\right)/d_{k,n} - R_k(1)\right) = \beta R_{k+1}(1). \tag{2.5}$$

For another result of this type, see Theorem 2.8 below.

The main disadvantage of Theorem 2.4 and its corollaries is that it does not include sequences of the form $a_n = n^\beta e^{-n^\alpha} (0 < \alpha < 1, \beta \in \mathbf{R})$. In Theorem 2.6 below we shall formulate a general result that deals with sequences which satisfy one or more of the properties listed below. Let us assume that $a_n \neq 0$ for all $n \geq N$ and let us consider the following properties

(P1) $$\sup_{n\geq N} |a_{n-1}/a_n| = m < \infty$$

(P2) $$\sup_{n\geq N}\left(\sum_{k=0}^{n}|a_{n-k}|\,|a_k|\right)/|a_n| = M < \infty$$

(P3) $$\lim_{n\to\infty} a_{n-1}/a_n = 1$$

(P4) $$\lim_{n\to\infty}\left(\sum_{k=0}^{n} a_{n-k}a_k\right)/a_n = \lim_{n\to\infty}\frac{(a*a)}{a_n} = 2\sum_{k=0}^{\infty} a_k, \text{ with } \sum_{0}^{\infty}|a_k| < \infty.$$

In case of sequences of positive numbers, sequences that satisfy (P3) and (P4) have been studied already. We refer to the basic paper [3] and also to [4], [6], [8], where some applications in probability theory are given. Now we prove

THEOREM 2.6. *$\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$ be sequences of real numbers such that $a_n \neq 0$ for $n \geq N$, $\sum_{k=0}^{\infty} |a_k| < \infty$ and $\sum_{k=0}^{\infty} |b_k| < \infty$.*

(i) *If $\{a_n\}_{\mathbf{N}}$ satisfies* (P1), (P2) *and if* $\sup_{n \geq N} |b_n|/|a_n| < \infty$, *then also*

$$\sup_{n \geq N} |c_n|/|a_n| < \infty.$$

(ii) *If $\{a_n\}_{\mathbf{N}}$ satisfies* (P2), (P3) *and if* $\lim_{n\to\infty} b_n/a_n = 0$, *then*

$$\lim_{n\to\infty} c_n/a_n = \sum_{k=0}^{\infty} b_k.$$

(iii) *If $\{a_n\}_{\mathbf{N}}$ satisfies* (P2), (P3) *and* (P4), *and if* $\lim_{n\to\infty} b_n/a_n = \alpha$ *for some* $\alpha \in \mathbf{R}$, *then*

$$\lim_{n\to\infty} c_n/a_n = \sum_{k=0}^{\infty} b_k + \alpha \sum_{k=0}^{\infty} a_k. \ \square$$

In the special case of regularly varying sequences we obtain the following extension of Lemma 2.3.

COROLLARY 2.7. *Let $\{a_n\}_{\mathbf{N}}$ and $\{b_n\}_{\mathbf{N}}$ be sequences of nonnegative real numbers such that $\sum_{n=0}^{\infty} a_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. If for some $\beta \leq -1$, $\{a_n\}_{\mathbf{N}} \in RV_\beta$ and if $b_n/a_n \to \alpha (n \to \infty)$ for some $\alpha \geq 0$, then also $\{c_n\}_{\mathbf{N}} \in RV_\beta$ and $c_n/a_n \to \sum_{k=0}^{\infty} b_k + \alpha \sum_{k=0}^{\infty} a_k (n \to \infty)$.* $\square$

Corollary 2.7 follows at once from Theorem 2.6 (iii) since regularly varying sequences obviously satisfy (P3) and (P4). Further examples of sequences satisfying (P3) and (P4) are (see e.g. [3]):

(i) $a_n = n^\beta e^{-n^\alpha}$, $0 < \alpha < 1$, $\beta \in \mathbf{R}$

(ii) $a_n = \exp -n(\log n)^{-\beta}$, $\beta > 0$

(iii) sequences $\{a_n\}_{\mathbf{N}}$ such that $a_{n+1} \sim a_n$ $(n \to \infty)$ and

$$\sup_{1 \leq k \leq [n/2]} a_{n-k}/a_n \leq K < \infty.$$

Using Theorem 2.6 we can also genralise Theorem 2.4 and its corollaries. With p_k and r_k as before we have

COROLLARY 2.8. *Suppose for some $k \geq 1$ that $\{p_{k,n}\}_{\mathbf{N}}$ and $\{r_{k,n}\}_{\mathbf{N}}$ are such that $\sum_{n=0}^{\infty} |p_{k,n}| < \infty$ and $\sum_{n=0}^{\infty} |r_{k,n}| < \infty$.*

(i) *If $\{p_{k,n}\}_{\mathbf{N}}$ satisfies* (P1), (P2) *and if* $\sup_{n \geq N} |r_{k,n}/p_{k,n}| < \infty$, *then*

$$\sup_{n \geq N} \left| \left(c_n - \sum_{m=0}^{k-1} p_{m,n} R_m(1) \right) / p_{k,n} \right| < \infty.$$

(ii) *If $\{p_{k,n}\}_{\mathbf{N}}$ satisfies* (P2), (P3), (P4) *and if* $r_{k,n}/p_{k,n} \to \alpha$ $(n \to \infty)$, *then*

$$\lim_{n\to\infty} \left(c_n - \sum_{m=0}^{k-1} p_{m,n} R_m(1) \right) / p_{k,n} = R_k(1). \ \square$$

When $k = 1$, Corollary 2.8 (i) gives

$$\sup_{n \geq N} \left| \left(c_n - a_n \sum_{k=0}^{\infty} b_k \right) // (a_{n-1} - a_n) \right| < \infty.$$

Hence, if $\sup_{n \geq N} \delta_n^{-1} |(a_{n-1} - a_n)/a_n| < \infty$ we obtain Bojanić and Lee's result mentioned before. Corollary 2.8 (ii) in case $k = 1$, gives

$$\lim_{n \to \infty} \left(c_n - a_n \sum_{k=0}^{\infty} b_k \right) / (a_{n-1} - a_n) = \sum_{k=0}^{\infty} k b_k$$

which in case (2.1) holds, gives (2.2).

Before proving these results it is worthwhile noting that our theorems can be extended as follows. Let $\{a_n\}_{\mathbf{N}}$ satisfy the following properties

(P'1) $$\sup_{n \geq N} \left(\sum_{k=0}^{n} |a_{n-k}||a_k| \right) / |a_n| = M < \infty$$

(P'3) $$\lim_{n \to \infty} a_{n+1}/a_n = 1/r (> 0), \quad \sum_{n=0}^{\infty} |a_k| r^k < \infty$$

(P'4) $$\lim_{n \to \infty} (a * a)_n / a_n = 2 \sum_{k=0}^{\infty} a_k r^k =: 2A(r).$$

Upon transforming a sequence $\{d_n\}_{\mathbf{N}}$ to the sequence $\{d'_n\}_{\mathbf{N}}$ defined by $d'_n = r^n d_n$ we obtain the following result as an immediate consequence of Theorem 2.6.

THEOREM 2.9. *Let $\{b_n\}_{\mathbf{N}}$ and $\{a_n\}_{\mathbf{N}}$ be sequences of real numbers and suppose $\{a_n\}_{\mathbf{N}}$ satisfies* (P'1), (P'3), (P'4). *If $\lim_{n \to \infty} b_n/a_n = \alpha$ for some $\alpha \in \mathbf{R}$, then*

$$\lim_{n \to \infty} c_n/a_n = B(r) + \alpha A(r)$$

where $B(r) = \sum_0^{\infty} b_n r^n$ and $A(r) = \sum_0^{\infty} a_n r^n$. □

3. Proofs

3.1. Proof of Theorem 2.2. (ii) ⇒ (i) This follows immediately from lemma 2.1. (i) ⇒ (ii) The proof is divided into two parts.

Part 1: $\alpha = 0$. First observe that (i) implies

$$a_n \leq 2n^{-1} L(n) \tag{3.1}$$

for n large enough. For any fixed $\varepsilon, t, 0 < \varepsilon < t < 1$ write

$$c_n = \left(\sum_{k=0}^{[n\varepsilon]} + \sum_{k=[n\varepsilon]+1}^{[nt]} + \sum_{k=[nt]+1}^{[n]} \right) a_{n-k} b_k =: I_1 + I_2 + I_3.$$

First consider I_1. From the monotonocity of $\{a_n\}_{\mathbf{N}}$ and (3.1) we have

$$0 \le I_1 \le a_{n-[n\varepsilon]} \sum_{k=0}^{[n\varepsilon]} b_k \le 2(n-[n\varepsilon])^{-1} L(n-[n\varepsilon]) \sum_{k=0}^{[n\varepsilon]} b_k.$$

Since $\{b_n\}_{\mathbf{N}} \in RV_\beta$ and L is s.v. we obtain

$$0 \le \limsup_{n\to\infty} \frac{I_1}{L(n)b_n} \le \frac{2\varepsilon^{\beta+1}}{(1-\varepsilon)(1+\beta)}. \tag{3.2}$$

To handle I_2 and I_3 we use the fact that nb_n asymptotically equals a nondecreasing sequence $B(n$. This follows from the representation theorem for regularly varying sequences [2] and $1+\beta > 0$. Hence for any $\delta > 0$ and n large we have

$$(1-\delta)B(n) \le nb_n \le (1+\delta)B(n).$$

Using this and the monotonocity of $B(n)$ we obtain

$$(1+\delta)B([nt]+1)n^{-1} \sum_{k=0}^{n-[nt]-1} a_k \le I_3 \le (1+\delta)B(n)([nt]+1)^{-1} \sum_{k=0}^{n-[nt]-1} ak. \tag{3.3}$$

Hence

$$(1-\delta)t^{\beta+1} \le \lim_{n\to\infty} \begin{bmatrix}\sup\\ \inf\end{bmatrix} \frac{I_3}{L(n)b_n} \le (1+\delta)t^{-1}. \tag{3.4}$$

In a similar way for I_2 we have

$$0 \le I_2 \le (1+\delta)B([nt])([n\varepsilon]+1)^{-1} \sum_{k=[n\varepsilon]+1}^{[nt]} a_{n-k}.$$

Since $\sum_{k=[n\varepsilon]+1}^{[nt]} a_{n-k} = \sum_{k=0}^{n-[n\varepsilon]-1} - \sum_{k=0}^{n-[nt]-1} a_k = o(L(n))$ $(n \to \infty)$ we obtain

$$\lim_{n\to\infty} I_2/L(n)b_n = 0 \tag{3.5}$$

Now combine (3.2), (3.3) and (3.4). Let $\delta \downarrow 0$, $\varepsilon \downarrow 0$, $t \uparrow 1$ to obtain (ii).

Part 2: $0 < \alpha \le 1$. First note that (i) is equivalent to

$$a_n \sim \alpha n^{\alpha-1} L(n) \quad (n \to \infty) \tag{3.6}$$

As in part 1 divide c_n into the three parts I_1, I_2 and I_3. As in part 1 we obtain

$$0 \le \limsup_{n\to\infty} \frac{I_1}{na_nb_n} \le \frac{(1-\varepsilon)^{\alpha-1}\varepsilon^{\beta+1}}{\beta+1}. \tag{3.7}$$

Also (3.3) remains valid, and we have

$$(1-\delta)t^{\beta+1}\frac{(1-t)^\alpha}{\alpha} \le \lim_{n\to\infty} \begin{bmatrix}\sup\\ \inf\end{bmatrix} \frac{I_3}{na_nb_n} \le (1+\delta)\frac{(1-t)^\alpha}{\alpha t}. \tag{3.8}$$

As for I_2 we have ($[x]$ denotes the integral part of x)

$$I_2 = \int_{n-[nt]}^{n-[n\varepsilon]-1} a_{[s]} b_{n-[s]} ds = \int_{(n-[nt])/n}^{(n-[n\varepsilon]-1)/n} n a_{[ns]} b_{n-[ns]} ds.$$

Using the uniform convergence properties of regularly varying sequences and functions (see e.g. [2], [13]) we obtain

$$\lim_{n\to\infty} \frac{I_2}{n a_n b_n} = \int_{1-t}^{1-\varepsilon} s^{\alpha-1}(1-s)^{\beta} ds. \tag{3.9}$$

Now combine (3.7), (3.8) and (3.9). Let $\delta \downarrow 0$, $\varepsilon \downarrow 0$, $t \uparrow 1$ to obtain

$$\lim_{n\to\infty} \frac{c_n}{n a_n b_n} = \int_0^1 (1-s)^{\beta} s^{\alpha-1} ds.$$

Using (3.6), the result follows. □

3.2. Proof of Lemma 2.3. We only have to prove the result in case $\alpha \le 0$ since the other case follows from Lemma 2.1. Let us write

$$c_n = \sum_{k=0}^{[n/2]} a_{n-k} b_k + \sum_{k=[n/2]+1}^{n} a_{n-k} b_k =: I_1 + I_2.$$

Since $\{a_n\}_{\mathbf{N}} \in RV_\alpha$ we have

$$\sup_{0\le k\le [n/2]} a_{n-k}/a_n \le K < \infty.$$

Hence Lebesgue's theorem yields $\lim_{n\to\infty} I_1/a_n = \sum_{k=0}^{\infty} b_k$ since for fixed k, $a_{n-k} \sim a_n$ $(n \to \infty)$.

Next consider I_2; since $\{a_n\}_{\mathbf{N}} \in RV_\alpha$ we have [13, p. 20] for any $\varepsilon > 0$,

$$\sup_{k\le n}(k^{-\alpha+\varepsilon} a_k) \sim n^{-\alpha+\varepsilon} a_n \quad (n \to \infty).$$

Hence for k such that $[n/2]+1 \le k \le n-N$ and n, N large, we have

$$a_{n-k} \le 2N^{\alpha-\varepsilon} n^{-\alpha+\varepsilon} a_n.$$

Hence for all k such that $[n/2]+1 \le k \le n$ we have

$$a_{n-k} \le M n^{-\alpha+\varepsilon} a_n \tag{3.10}$$

for some M and n sufficiently large. But then

$$0 \le \frac{|I_2|}{a_n} \le M' \sum_{k=[n/2]+1}^{n} k^{-\alpha+\varepsilon} |b_k| \to 0 \ (n \to \infty).$$

This proves the lemma. □

3.3. Proof of Theorem 2.4. and Corollary 2.5.

3.3.1. Proof of Theorem 2.4. Let us write

$$\frac{c_n}{a_n}-\sum_{k=0}^{\infty} b_k=\sum_{k=1}^{[n/2]} b_k\left(\frac{a_{n-k}}{a_n}-1\right)+\sum_{k=[n/2]+1}^{n} b_k\frac{a_{n-k}}{a_n}-\sum_{k=[n/2]+1}^{\infty} b_k =: I_1+I_2+I_3.$$

Since $n|I_3|\le n\sum_{k=[n/2]}|b_k|\le 2\sum_{k=[n/2]}^{\infty}k|b_k|$ we have

$$\lim_{n\to\infty} nI_3=0. \tag{3.11}$$

Next consider I_2. In case $\beta<0$, from (2.1) we have $a_{n-1}\le a_n$ for $n\ge N$. Hence for $n-k\ge N$ we have $a_N\le a_{n-k}\le a_n$. It follows that

$$0\le n|I_2|\le\left(\left(\max_{0\le k\le N} a_k\right)/a_N+1\right)n\sum_{k=[n/2]}^{n}|b_k|\le M\sum_{k=[n/2]}^{\infty}k|b_k|$$

so that

$$\lim_{n\to\infty} nI_2=0. \tag{3.12}$$

In case $\beta\ge 0$, from (2.1) it follows that $\{a_n\}_{\mathbf{N}}\in RV_{-\beta}$. Using (3.10) we obtain

$$0\le n|I_2|\le Mn^{\beta+\varepsilon+1}\sum_{k=[n/2]}^{\infty}|b_k|\le M'\sum_{k=[n/2]}k^{\beta+\varepsilon+1}|b_k|$$

so that (3.12) is also valid here.

Finally for I_1 note that (2.1) implies that for some M and all $n\ge 1$,

$$|a_{n-1}/a_n-1|\le M/n.$$

From [1, lemma 2] it follows that for $1\le k\le[n/2]$ and n large enough,

$$n|a_{n-k}/a_n-1|\le nk/(n-k)M'\le kM''.$$

Hence in I_1 we can apply Lebesgue's theorem giving

$$\lim_{n\to\infty} nI_1=\sum_{k=1}^{\infty}\lim_{n\to\infty} n\frac{a_{n-k}-a_n}{a_n}b_k=\beta\sum_{k=1}^{\infty}kb_k. \tag{3.13}$$

Combining (3.11) – (3.13) gives the desired result. □

3.3.2. Proof of Corollary 2.5. For a sequence $\{d_n\}_{\mathbf{N}}$ denote by $D(z)=\sum_{n=0}^{\infty}d_nz^n$ its generating function. Then we have for $k=0,1,\ldots$

$$R_{k+1}(z)=(R_k(1)-R_k(z))/(1-z),\quad P_{k+1}(z)=(z-1)P_k(z)+p_{k,k}z^k.$$

By induction it follows that for $k \geq 1$,

$$C(z) - \sum_{m=0}^{k-1} P_m(z) R_m(1) + \sum_{m=0}^{k-1} p_{m,m} z^m R_{m+1}(z) = P_k(z) R_k(z). \tag{3.14}$$

Now let us consider $\sum_{n=1}^{\infty} n^{\delta+1} |r_{m,n}|$ for $m = 0, 1, \ldots, k$ and some $\delta \geq 0$. From the definition of $\{r_{m,n}\}_{\mathbf{N}}$ it follows that

$$\sum_{n=1}^{\infty} n^{\delta+1} |r_{m,n}| \leq \sum_{n=1}^{\infty} n^{\delta+1} \sum_{q=n+1}^{\infty} |r_{m-1,q}| = \sum_{q=2}^{\infty} \sum_{n=1}^{q-1} n^{\delta+1} |r_{m-1,q}|$$

so that $\sum_{n=1}^{\infty} n^{\delta+1} |r_{m,n}| \leq \sum_{n=1}^{\infty} n^{\delta+2} |r_{m-1,n}|$.

By induction it follows that

$$\sum_{n=1}^{\infty} n^{\delta+1} |r_{m,n}| \leq \sum_{n=1}^{\infty} n^{\delta+m+1} |r_{0,n}| = \sum_{n=1}^{\infty} n^{\delta+m+1} |b_n|.$$

Now $\sum_{n=1}^{\infty} n^{k+1} |b_n| < \infty$ and in case $\beta \geq 0$, $\sum_{n=1}^{\infty} n^{k+1+\beta+\varepsilon} |b_n| < \infty$ for some $\varepsilon > 0$. Hence for $m = 0, 1, \ldots, k$,

$$\sum_{n=1}^{\infty} n |r_{m,n}| < \infty \text{ and in case } \beta \geq 0, \ \sum_{n=1}^{\infty} n^{+\beta+\varepsilon+1} |r_{m,n}| < \infty. \tag{3.15}$$

Hence (2.4) will follow from (2.3), (3.14) and Theorem 2.4 if we can prove that for $m = 0, 1, \ldots, k$

$$\lim_{n \to \infty} n r_{m,n} / p_{k,n} = 0. \tag{3.16}$$

Since $\{p_{k,n}\}_{\mathbf{N}} \in RV_{-\beta}$, in case $\beta < 0$, (3.16) follows at once from (3.15) and $p_{k,n} \to \infty$ $(n \to \infty)$.

In case $\beta \geq 0$ we write for $\varepsilon > 0$,

$$\frac{n r_{m,n}}{p_{k,n}} = \frac{n^{\beta+\varepsilon+1} r_{m,n}}{n^{\beta+\varepsilon} p_{k,n}}.$$

Since $n^{\beta+\varepsilon} p_{k,n} \to \infty$ $(n \to \infty)$, again (3.16) follows from (3.15). This completes the proof of the corollary. □

3.4. Proof of the Theorem 2.6 and Corollary 2.8.

3.4.1. Proof of Theorem 2.6. (i). For $n \geq 2N$ we have

$$\begin{aligned} |c_n| &\leq \sum_{k=N}^{n-N-1} \left| \frac{b_{n-k}}{a_{n-k}} \right| |a_{n-k}| |a_k| + \sum_{k=n-N}^{n} |b_{n-k}| |a_k| \\ &\leq \sup_{k \geq N} \left| \frac{b_k}{a_k} \right| \sum_{k=0}^{n} |a_{n-k}| |a_k| + \sup_{n-N \leq k \leq n} |a_k| \cdot \sum_{0}^{\infty} |b_k|. \end{aligned}$$

For n and k such that $N \leq n-k$, using (P1), it follows that $|a_{n-k}/a_n| \leq m^k$. Using (P2) it follows that $\sup_{n\geq 2N} |c_n|/|a_n| < \infty$ and hence also that $\sup_{n\geq N} |c_n/a_n| < \infty$. □

3.4.2. Proof of Theorem 2.6 (ii). For some fixed R and $n \geq R \geq N$ we have

$$\left|\frac{c_n}{a_n} - \sum_{k=0}^{\infty} b_k\right| \leq \sum_{k=0}^{R} |b_k| \left|\frac{a_{n-k}}{a_n} - 1\right| + \sum_{k=R+1}^{n} \left|\frac{b_k}{a_k}\right| \frac{|a_{n-k}||a_k|}{|a_n|} + \sum_{k=R+1}^{\infty} |b_k|$$
$$\leq \sum_{k=0}^{R} |b_k| \left|\frac{a_{n-k}}{a_n} - 1\right| + \sup_{k\geq R} \left|\frac{b_k}{a_k}\right| \cdot M + \sum_{k=R+1}^{\infty} |b_k|.$$

For fixed R, using (P3) we have

$$0 \leq \limsup_{n\to\infty} \left|\frac{c_n}{a_n} - \sum_{k=0}^{\infty} b_k\right| \leq \sup_{k\geq R} \left|\frac{b_k}{a_k}\right| \cdot M + \sum_{k=R+1}^{\infty} |b_k|.$$

Now let $R \to \infty$ to obtain (ii). □

3.4.3. Proof of Theorem 2.6 (iii). Since $b = \alpha a + d$ for some sequence d such that $\lim_{n\to\infty} d_n/a_n = 0$ we have $c = a * b = \alpha a * a + a * d$. Using (P4) and § 3.4.2 we obtain

$$\lim_{n\to\infty} \frac{c_n}{a_n} = 2\alpha \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} d_k = \sum_{k=0}^{\infty} b_k + \alpha \sum_{k=0}^{\infty} a_k. \ \square$$

3.4.4. Proof of Corollary 2.8. As in the proof of Corollary 2.5 we have

$$C(z) - \sum_{m=0}^{k-1} P_m(z) R_m(1) + \sum_{m=0}^{k-1} p_{m,m} z^m R_{m+1}(z) = P_k(z) R_k(z). \tag{3.17}$$

Now, if $\sup_{n\geq N} |r_{k,n}/p_{k,n}| < \infty$, by induction for $m = 0, 1, 2, \ldots, k-1$, using (P1), we have

$$\sup_{n\geq N} \left|\frac{r_{m,n}}{p_{kn}}\right| \leq sup_{n\geq N} \left|\frac{r_{m+1,n-1}}{p_{k,n}}\right| + \sup_{n\geq N} \left|\frac{r_{m+1,n}}{p_{k,n}}\right| < \infty.$$

which using Theorem 2.6 (i) proves (i).

If $\lim_{n\to\infty} r_{k,n}/p_{k,n} = \alpha$, by induction for $m = 0, 1, 2, \ldots, k-1$ using (P3), we have

$$\lim_{n\to\infty} \frac{r_{m,n}}{p_{k,n}} = \lim_{n\to\infty} \frac{r_{m+1,n-1} - r_{m+1,n}}{p_{k,n}} = 0. \tag{3.18}$$

From (3.17), (3.18) and Theorem 2.6 it follows that

$$\lim_{n\to\infty} \left(c_n - \sum_{m=0}^{k-1} p_{m,n} R_m(1)\right) / p_{k,n} = -\alpha p_{k-1,k-1} + \alpha \sum_{m=0}^{\infty} p_{k,m} + R_k(1).$$

Since $\sum_{m=0}^{\infty} p_{k,m} = p_{k-1,k-1}$ the proof of (ii) is complete. □

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(Received 17 07 1983)