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# A FIXED POINT THEOREM IN REFLEXIVE BANACH SPACES

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In this note we shall prove the following fixed-point theorem.

THEOREM. *Let $B$ be a reflexive Banach space, $K$ a nonempty bounded, closed and convex subset of $B$ and let $T : K \to K$ be a map such that*

$$\operatorname{diam}[T(D)] < \operatorname{diam}(D) \tag{1}$$

*holds for every closed and convex subset $D$ of $K$, containing more than one element and mapped into itself by $T$. Then $T$ has a fixed point in $K$.*

PROOF. Let $\mathcal{F}$ denote a family of all non-empty closed and convex subsets of $K$ which $T$ maps into itself. Then using the result of Smulian [7, p. 327] and Zorn's Lemma it follows that $\mathcal{F}$ has a minimal element, say $C$. Since $T(C) \subseteq C \in \mathcal{F}$ it follows that $\operatorname{Cl}[\operatorname{co} T(C)] \subseteq C$ and hence

$$T(\operatorname{Cl}[\operatorname{co} T(C)]) \subseteq T(C) \subseteq \operatorname{Cl}[\operatorname{co} T(C)].$$

This implies $\operatorname{Cl}[\operatorname{co} T(C)] \in \mathcal{F}$ and by the minimality of $C$ we have

$$\operatorname{Cl}[\operatorname{co} T(C)] = C. \tag{2}$$

As $\operatorname{diam}(\operatorname{co} S) = \operatorname{diam}(S)$ for every subset $S$ of $K$ [5, p. 17], (2) implies

$$\operatorname{diam}[T(C)] = \operatorname{diam}(C). \tag{3}$$

Now, using (1) we conclude that $C$ is a singleton, say $C = u$. Therefore, $u$ is a fixed-point of $T$, and the proof is complete.

We remark that maps considered in [2], [4] and [6] satisfy the condition (1), and therefore our theorem is a certain generalization of corresponding fixed-point theorems. We shall illustrate this on a theorem given in [6].

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THEOREM A [6]. *Let $K$ and $D$ be as in the previous theorem and $T : K \to K$ mapping satisfying the following conditions:*

$$\begin{aligned}\|Tx - Ty\| \leq \max\{&\|x - Tx\|,\ \|y - Ty\|,\ a\|x - Ty\| + b\|y - Tx\|,\\ &(\|x - y\| + \|x - Tx\| + \|y - Ty)/3\};\\ &x, y \in K,\ \ a \geq 0,\ \ b \geq 0,\ \ a + b < 1,\end{aligned} \tag{4}$$

$$\sup_{z \in D} \|z - Tz\| < r\ \operatorname{diam}(D), \quad 0 < r = r(D) < 1. \tag{5}$$

*Then $T$ has a unique fixed point in $K$.*

PROOF. If $\operatorname{diam}(D) > 0$, then by (4) and (5) for every $x, y \in D$ we have

$$\|Tx - Ty\| \leq \max\{r, (a + b),\ (1 + 2r)/3\} \cdot \operatorname{diam}(D) < \operatorname{diam}(D).$$

Therefore, $T$ satisfies (1), and by the previous theorem, $T$ has a fixed point in $K$. Since condition (4) implies that $T$ may have at most one fixed point, the proof is complete.

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