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ON THE STABILITY OF THE FUNCTIONAL QUADRATIC ON A-ORTHOGONAL VECTORS

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Abstract. Let X be a complex Hilbert space ($\dim X$ is at least three) and A a bounded selfadjoint operator on X ($\dim AX$ is neither 1 nor 2). In this paper we study a continuous functional h on X which is approximately quadratic on A-orthogonal vectors (i.e. (α_1) is satisfied provided that $(Ax, y) = 0$). We find that there exists a unique continuous functional h_1 (given by (α_2)) which is quadratic on A-orthogonal vectors (i.e. (α_3) holds) and which is near h (i.e. (α_4) holds).

In [1] the "approximately linear" mapping $f : E_1 \to E_2$ was considered, where E_1, E_2 are Banach spaces and a linear mapping $T : E_1 \to E_2$ which is near the mapping f was found. In [2] we consider the mapping $\varphi : X \to X$, where X is a complex Hilbert space which is "approximately additive" on A-orthogonal vectors, and we found the mapping $\varphi_1 : X \to X$ which is additive on A-orthogonal vectors and near the mapping φ. In this paper we consider the analogous problem for a functional which is "approximately square" on A-orthogonal vectors. For this functional the following theorem will be proved:

THEOREM. *Let X be a complex Hilbert space $\dim X \geq 3$, $A : X \to X$ a bounded selfadjoint operator, $\dim AX \neq 1, 2$, $\theta \geq 0$ and $p \in [0, 2)$ real numbers and h a continuous functional defined on X. If*

$$|h(x+y) + h(x-y) - 2h(x) - 2h(y)| \leq \theta[|(A_x, x)|^{p/2}] \qquad (\alpha_1)$$

whenever $(Ax, y) = 0$, then by

$$h_1(x) = \lim_{n\to\infty} 2^{-2n} h(2^n x), \qquad (\alpha_2)$$

a continuous functional is defined on X such that

$$h_1(x+y) + h_1(x-y) = 2h_1(x) + 2h_1(y) \qquad (\alpha_3)$$

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* Izrada ovog rada potpomognuta je od strane Republičkog fonda za naučni rad SRBiH.

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whenever $(Ax, y) = 0$. *Furthermore there exist a real number* $\varepsilon > 0$ *such that*

$$|h(x) - h_1(x)| \leq |(Ax, x)|^{p/2} \cdot \varepsilon. \qquad (\alpha_4)$$

PROOF. It is obvious that $h(0) = 0$.

1° Let $(Ax, x) = 0$ for some x in $X (x \neq 0)$. Then it follows that

$$|h(x + x) + h(x - x) - 2h(x) - 2h(x)| \leq 2\theta|(Ax, x)|^{p/2} = 0, \quad |h(2x) - 4h(x)| = 0$$

respectively

$$h(2x)/2^2 = h(x) \qquad (1)$$

Therefore, for each natural number n, taking into consideration that (1) holds $h(2^n x)/2^{2n} = h(x)$ and it is obvious that $\lim_{n\to\infty} h(2^n x)/2^{2n} = h(x)$.

2° Let $(Ax, x) \neq 0$ for some $x \in X$. Let us show first that there exists a $y \in X$ such that $(Ay, y) \neq 0$ and $(Ax, y) = 0$. Let $Y = \{y \mid y \in X,\ (y, Ax) = 0\}$ and let $(Ay, y) = 0$, for each $y \in Y$. From this for $y_1, y_2 \in Y$ we have $(A(y_1 + y_2),\ y_1 + y_2) = 0$, that is $(Ay_1, y_2) + (Ay_2, y_1) = 0$. If we replace iy_2 with y_2, we get $-i(Ay_1, y_2) + i(Ay_2, y_1) = 0$, and with the above equality we have $(Ay_2, y_1) = 0$ for every $y_1, y_2 \in Y$. Since for $y \in Y$ it follows that $Ay \perp Y$, $Ay = \alpha(y)Ax$, for each $y \in Y$. Since for each z in X can be written in the form $z = \beta x + y (x \notin Y,\ y \in Y)$, we have $Az = \beta Ax + Ay = (\beta + \alpha(y))Ax$, which contradicts the hypothesis that $\dim A(X) \neq 1$. Therefore there exists a vector $y \in X$, such that $y \perp Ax$ and $(Ay, y) \neq 0$. Without loss of generality we can assume that $(Ay, y) = \pm(Ax, x)$.

a) Let $(Ax, x) \neq 0$, $(Ax, y) = 0$, and furthermore let $(Ay, y) = (Ax, x)$. From this it follows that $(A(x + y), x - y) = 0$. Then we have that

$$\begin{aligned} &|h(x + y + x - y) + h(x + y - x + y) - 2h(x + y) - 2h(x - y)| \\ &\quad \leq \theta[|(A(x + y), x + y)|^{p/2} + |(A(x - y), x - y)|^{p/2}] \end{aligned}$$

or $|h(2x) + h(2y) - 2h(x + y) - 2h(x - y)| \leq \theta 2^{p/2+1} \cdot |(Ax, x)|^{p/2}$. Moreover

$$\begin{aligned} |[h(2x) - 4h(x)] + [h(2y) - 4h(y)]| &= |[h(2x) + h(2y) - 2h(x + y) - 2h(x - y)] \\ &+ [2h(x + y) + 2h(x - y) - 4h(x) - 4h(y)]| \leq |h(2x) + h(2y) - 2h(x + y) \\ &- 2h(x - y)| + 2|h(x + y) + h(x - y) - 2h(x) - 2h(y)| \\ &\leq \theta 2^{p/2+1} \cdot |(Ax, x)|^{p/2} + 2^2\theta|(Ax, x)|^{p/2} = 2\theta|(Ax, x)|^{p/2}(2 + 2^{p/2}). \end{aligned}$$

From that dividing by 2^2 we obtain

$$|[h(2x)/2^2 - h(x)] + [h(2y)/2^2 - h(y)]| \leq \theta|(Ax, x)|^{p/2}(1 + 2^{p/2-1}). \qquad (2)$$

We check that the following inequality

$$\begin{aligned} &|[h(2^n x)/2^{2n} - h(x)] + [h(2^n y)/2^{2n} - h(y)]| \\ &\quad \leq \theta|(Ax, x)|^{p/2} \cdot (1 + 2^{p/2-1}) \cdot \sum_{k=0}^{n} 2^{k(p-2)} \qquad (*) \\ &\quad \leq |(Ax, x)|^{p/2} \cdot (1 + 2^{p/2-1})2^2/(2^2 - 2^p) \end{aligned}$$

holds for each natural number n. It is obvious that (*) holds for $n = 0$. Let (*) hold for $n = 0, 1, \dots, k-1$. We prove that (*) holds for $n = k$.

$$\begin{aligned}
&|[h(2^k x)/2^{2k} - h(x)] - h(x)] \\
&\quad + [h(2^k y)/2^{2k} - h(y)]| = |[h(2^{k-1} \cdot 2x)/2^{2(k-1)+2} - h(2x)/2^2] \\
&\quad + [h(2^{k-1} \cdot 2y)/2^{2(k-1)+2} - h(2y)/2^2] + [h(2x)/2^2 - h(x)] + [h(2y)/2^2 - h(y)]| \\
&\quad \le 1/2^2 \cdot |[h(2^{k-1} \cdot 2x)/2^{2(k-1)} - h(2x)] + [h(2^{k-1} \cdot 2y)/2^{2(k-1)} - h(2y)]| \\
&\quad + |[h(2x)/2^2 - h(x)] + [h(2y)/2^2 - h(y)]| \le 1/2^2 \cdot \theta|(A(2x), 2x)|^{p/2}(1 + 2^{p/2-1}) \\
&\quad \cdot 2^2/(2^2 - 2^p) + \theta|(Ax,x)|^{p/2}(1 + 2^{p/2-1}) = \theta|(Ax,x)|^{p/2}(1 + 2^{p-2} \cdot 2^2/(2^2 - 2^p)) \\
&\quad = \theta|(Ax,x)|^{p/2} \cdot (1 + 2^{p/2-1})2^2/(2^2 - 2^p).
\end{aligned}$$

Thus (*) holds for $n = k$; hence it holds also for each natural n. We prove that the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ converges for the $x, y \in X$ above. For $m > n > 0$ we have

$$\begin{aligned}
&|[h(2^m x)/2^{2m} + h(2^m y)/2^{2m}] - [h(2^n x)/2^{2n} + h(2^n y)/2^{2n}]| = 1/2^{2n} \\
&\quad \cdot |[h(2^{m-n} \cdot 2^n x)/2^{2(m-n)} - h(2^n x)] + [h(2^{m-n} \cdot 2^n y)/2^{2(m-n)} - h(2^n y)]| \\
&\quad \le 1/2^{2n} \cdot \theta|(A(2^n x), 2^n x)|^{p/2}(1 + 2^{p/2-1})2^2/(2^2 - 2^p) = \theta|(Ax,x)|^{p/2}(1+ \\
&\quad 2^{p/2-1}) \cdot 2^2/(2^2 - 2^p) \cdot 2^{n(p-2)}.
\end{aligned}$$

Therefore, the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is a Cauchy sequence; so it converges for the $x, y \in X$ above.

Let us show first that there exists a $z \in X$ such that $(Ax, z) = 0$, $(Ay, z) = 0$ and $(Ay, y) = (Az, z) = \pm(Ax, x)$. Let P be a projection of X onto Y parallel to Ax. Then $Ay = \alpha(y)Ax + PAy$, and therefore $(PAy, y) = (Ay, y)$. (The last equality holds if y is replaced by any other $u \in Y$.) Let $Z = \{z \mid z \text{ in } Y, (PAy, z) = 0\}$. If $(PAz, z) = 0$, for all $z \in Z$, then $(Az, z) = 0$, for all $z \in Z$. Hence, $(Az, z_1) = 0$, for all $z, z_1 \in Z$. This means that $Az \perp Z$, so that $PAz = \alpha(z) \cdot PAy$, for all $z \in Z$. It is easy to check that x is not in Y and y is not in Z and that x and y are linearly independent. So every $y' \in Y$ can be written in the form $y' = \alpha x + \beta y + z$, and moreover

$$\begin{aligned}
PAy' &= \alpha PAx + \beta PAy + PAz \\
&= \beta PAy + PAz = \beta PAy + \alpha(z)PAy = (\beta + \alpha(z))PAy,
\end{aligned}$$

for all $y' \in Y$.

Let $u \in X$. Then

$$Au = \alpha(u)Ax + PAu \tag{δ}$$

Since $u = \alpha_1 x + \beta_1 y + z$, this amounts to $PAu = \alpha_1 PAx + \beta_1 PAy + PAz = \beta_1 PAy + PAz$. Taking into consideration what was already proved, i.e. that $PAz = \beta(z) \cdot PAy$, we have $PAu = \beta(u) \cdot PAy$. Thus the relation (δ) becomes $Au = \alpha(u) \cdot Ax + \beta(u) \cdot PAy$, for all $u \in X$ which contradicts the hypothesis

$\dim A(X) \neq 1, 2$. Therefore, there exists a $z' \in Z$ such that $(PAz', z') \neq 0$. But $z \in Z$ can be chosen such that $(PAz, z) = (PAy, y)$, i.e. $(Az, z) = (Ay, y)$. Besides $(Ay, z) = 0$ and $(Ax, z) = 0$. If $(Ax, z) = 0$, $(Ay, z) = 0$ and $(Ay, y) = (Az, z) = (Ax, x)$, then from what was proved earlier it follows that the sequences

$$\{h(2^n x)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty} \tag{Δ_1}$$

$$\{h(2^n y)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty} \tag{Δ_2}$$

converge. If we subtract (Δ_2) from (Δ_1), we conclude that the sequence: $\{h(2^n x)/2^{2n} - h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, which together with the fact the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, implies that the sequence $\{h(2^n x)/2^{2n}\}_{n=0}^{\infty}$ is convergent for the $x \in X$ above.

b) Let $(Ax, x) \neq 0$ and $(Ax, y) = 0$, and furthermore let $(Ay, y) = -(Ax, x)$. From this it follows that $(A(x \pm y).\ x \pm y) = 0$. Then using 1° we have $h[2^n(x \pm y)]/2^{2n} = h(x \pm y)$ for each natural number n; hence, the sequences $\{h[2^n(x \pm y)]/2^{2n} = h(x \pm y)\}_{n=0}^{\infty}$ are convergent for the $x, y \in X$ above.

Since for every $x, y \in X$, for which $(Ax, y) = 0$ and $(Ay, y) = -(Ax, x)$ and all $n = 0, 1, 2, \ldots$ we have

$$\begin{aligned}|h[2^n(x+y)]/2^{2n} + h[2^n(x-y)]/2^{2n} - 2h(2^n x)/2^{2n} - 2h(2^n y)/2^{2n}|\\ \leq 2\theta 2^{n(p-2)}|(Ax, x)|^{p/2}\end{aligned}$$

we conclude that the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ converges for the $x,\ y \in X$ above. Similarly we conclude that the sequence $\{h(2^n x)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ is convergent, for each $x, z \in X$, for which $(Ax, z) = 0$ and $(Az, z) = -(Ax, x)$ (Before we showed that such a z exists). Also, using a), for that z it holds that $(Ay, z) = 0$ and $(Ay, y) = (Az, z)$, and we concluded that the sequence $\{h(2^n y)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ is convergent. From the fact that the sequences $\{h(2^n x)/2^n + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ and $\{h(2^n y)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ are convergent, we conclude that the sequence $\{h(2^n x)/2^n - h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, which together with the fact that the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, implies that the sequence $\{h(2^n x)/2^{2n}\}_{n=0}^{\infty}$ is convergent for the $x \in X$ above. From that we conclude that for any $x \in X$, $\lim_{n\to\infty} h(2^n x)/2^{2n}$ exists. Let $h_1(x) = \lim_{n\to\infty} h(2^n x)/2^{2n}$, for all $x \in X$. Let us prove that for the functional h_1 we have: $h_1(x+y) + h_1(x-y) = 2h_1(x) + 2h_1(y)$, for all $x, y \in X$ for which is $(Ax, y) = 0$. For every $x, y \in X$ for which $(Ax, y) = 0$ and all $n = 0, 1, 2, \ldots$ it holds that

$$\begin{aligned}|h[2^n(x+y)]/2^{2n} + h[2^n(x-y)]/2^{2n} - 2h(2^n x)/2^{2n} - 2h(2^n y)/2^{2n}|\\ \leq 2^{n(p-2)}\theta[|(Ax, x)|^{p/2} + |(Ay, y)|^{p/2}]\end{aligned}$$

Letting $n \to \infty$ we have $h_1(x+y) + h_1(x-y) = 2h_1(x) + 2h_1(y)$. Let x, y, z be such that $(Ax, y) = 0$, $(Ax, z) = 0$, $(Ay, z) = 0$ and $\pm(Ax, x) = (Ay, y) = (Az, z)$. Then for all $n = 0, 1, 2, \ldots$ using the relation $(^*)$ we have

$|[\lim_{n\to\infty} h(2^n x)/2^{2n} - h(x)] + [\lim_{n\to\infty} h(2^n y)/2^{2n} - h(y)]| \le |(Ax,x)|^{p/2} \cdot \varepsilon_1$ where $\varepsilon_1 = \theta(1+2^{p/2-1}) \cdot 2^2/(2^2-2^p)$ or

$$\begin{aligned} |[h_1(x) - h(x)] + [h_1(y) - h(y)]| &\le \varepsilon_1 |(Ax,x)|^{p/2} \\ |[h_1(x) + h_1(y)] - [h(x) + h(y)]| &\le \varepsilon_1 |(Ax,x)|^{p/2}. \end{aligned} \qquad (\delta_1)$$

By analogy we can prove that

$$|[h_1(x) + h_1(z)] - [h(x) + h(z)]| \le \varepsilon_1 |(Ax,x)|^{p/2} \qquad (\delta_2)$$

$$|[h_1(y) + h_1(z)] - [h(y) + h(z)]| \le \varepsilon_1 |(Ax,x)|^{p/2} \qquad (\delta_3)$$

Taking into consideration the relations (δ_1), (δ_2) and (δ_3) it is easy to check that: $|h(x) - h_1(x)| \le \varepsilon|(Ax,x)A^{p/2}$, for all x in X where $\varepsilon_1 = 3/2 \cdot \theta(1 + 2^{p/2-1} \cdot 2^2/(2^2 - 2^p)$. Before proving a continuity functional, we introduce the following: for $x, y \in X$ for which $(Ax,y) = 0$ and $(Ay,y) = \pm(Ax,x)$ let $\hat{h}_n^{\pm}\{x,y\} = h(2^n x)/2^{2n} \pm h(2^n y(/2^{2n}$. For $m > n > 0$ we have

$$\begin{aligned} &|\hat{h}_m^{+}\{x,y\} - \hat{h}_n^{+}\{x,y\}| = |h(2^m x)/2^{2m} + h(2^m y)/2^{2m} - h(2^n x)/2^{2n} - h(2^n y)/2^{2n}| \\ &= |[h(2^m x)/2^{2m} - h(2^n x)/2^{2n}] + [h(2^m y)/2^{2m} - h(2^n y)/2^{2n}]| = 1/2^{2n} \\ &\cdot |[h(2^{m-n} \cdot 2^n x)/2^{2(m-n)} - h(2^n x)] + [h(2^{m-n} \cdot 2^n y)/2^{2(m-n)} - h(2^n y)]| \\ &\le \theta/2^{2n} \cdot |(A(2^n x), 2^n x)|^{p/2}(1 + 2^{p/2-1}) \cdot 2^2/(2^2 - 2^p) \\ &= 2^2/(2^2 - 2^p) \cdot \theta |(Ax,x)|^{p/2} \cdot (1 + 2^{p/2-1}) \cdot 2^{n(p-2)}. \end{aligned}$$

It means that

$$|\hat{h}_m^{+}\{x,y\} - \hat{h}_n^{+}\{x,y\}| \le 2^2\theta/(2^2 - 2^p) \cdot |(Ax,x)|^{p/2}(1 + 2^{p/2-1})2^{n(p-2)} \qquad (^{**})$$

A similar relation holds for the function $\hat{h}_n^{-}$. The domain functionals $\hat{h}_n^{pm}$ are set as follows:

$$\mathcal{D}^{\pm} \equiv \{\{x,y\} \mid (Ax,y) = 0, \ \ (Ay,y) = \pm(Ax,x), x, y \in X\} \subset X \times X.$$

The functionals $\hat{h}_n^{\pm}$ are continuous on $\mathcal{D}^{\pm}$, and from $(^{**})$ it follows that $\hat{h}_n^{\pm}$ uniformly converges on $\mathcal{D}^{\pm} \cap S$, where S is any sphere in $X \times X$, and the functionals

$$\hat{h}^{\pm}\{x,y\} = \lim_{n\to\infty} [h(2^n x)/2^{2n} \pm h(2^n y)/2^{2n}] \qquad (3)$$

are continuous on $\mathcal{D}^{\pm}$.

We now prove the continuity of the functional h_1. Let the sequence $\{x_n\}_{n=1}^{\infty}$ (in X) converge to x_0 (in X).

A) Suppose that $(Ax_0, x_0) \ne 0$.

1° Let $y_0 \in X$ ($y_0 \ne 0$) such that $(Ax_0, y_0) = 0$ and $(Ay_0, y_0) = (Ax_0, x_0)$. Then $\{x_0, y_0\} \in \mathcal{D}^{+}$. As before, we can find a z_0, such that $(Ax_0, z) = 0$, $(Ay_0, z_0) = 0$ and $(Ay_0, y_0) = (Az_0, z_0) = (Ax_0, x_0)$. Therefore $\{x_0, z_0\} \in \mathcal{D}^{+}$, and $\{y_0, z_0\} \in$

$\mathcal{D}^+$. Since $(Ax_n, x_n) \to (Ax_0, x_0) \neq 0$ begins from any number n, $(Ax_n, x_n) \neq 0$, and we can begin from that n, we make a sequence

$$\tilde{y}_n = y_0 - x_n/(Ax_n, x_n) \cdot (Ay_0, x_n) \quad (n = 1, 2, \dots) \tag{4}$$

Obviously, $\tilde{y}_n \to y_0$ $(n \to \infty)$ and $(Ax_n, \tilde{y}_n) = 0$ $(n = 1, 2, \dots)$. From that we obtain

$$(A\tilde{y}_n, \tilde{y}_n) = (A\tilde{y}_n, y_0) = (\tilde{y}_n, Ay_0) = (y_0, Ay_0) - (Ay_0, x_n)/(Ax_n, x_n) \cdot (x_n, Ay_0) \tag{5}$$

For sufficiently large n, the difference on the right-hand side of this equality shall be different from zero, and we can begin from this n and make the sequence

$$y_n = \sqrt{(Ax_n, x_n)/(A\tilde{y}_n, \tilde{y}_n} \cdot \tilde{y}_n \qquad (n = 1, 2, \dots) \tag{α}$$

which has the following characteristics:

$$(Ax_n, y_n) = 0, \ (Ax_n, x_n) = (Ay_n, y) \text{ and } y_n \to y_0 \quad (n \to \infty).$$

Therefore, the pair $\{x_n, y_n\}$ are in $\mathcal{D}^+$. Since $(Ay_0, y_0) \neq 0$ and $(Ay_n, y_n) \to (Ay_0, y_0) \neq 0$ $(n \to \infty)$, we can make the sequence

$$\tilde{z}_n = z_0 - y_n/(Ay_n, y_n) \cdot (Az_0, y_n) - x_n/(Ax_n, x_n) \cdot (Az_0, x_n) \quad (n = 1, 2, \dots). \tag{6}$$

Hence $\tilde{z}_n \to z_0$ $(n \to \infty)$ and

$$\begin{aligned}(Ay_n, \tilde{z}_n) &= (Ay_n, z_0 - y_n/(Ay_n, y_n) \cdot (Az_0, y_n) - x_n/(Ax_n, x_n) \cdot (Az_0, x_n)) \\ &= (Ay_n, z_0) - (Ay_n, z_0)(Ay_n, y_n)/(Ay_n, y_n) - (Ax_n, z_0)/(Ax_n, x_n) \cdot (Ay_n, x_n) \\ &= (Ay_n, z_0) - (Ay_n, z_0) = 0.\end{aligned}$$

Therefore $(Ay_n, \tilde{z}_n) = 0$ $(n = 1, 2, \dots)$. Besides that, we have

$$\begin{aligned}(A\tilde{z}_n, \tilde{z}_n) = (A\tilde{z}_n, z_0) = (z_0, Az_0) - (Az_0, y_n)/(Ay_n, y_n) \cdot (y_n, Az_0) \\ - (x_n, Az_0)/(Ax_n), x_n) \cdot (Az_0, x_n).\end{aligned} \tag{7}$$

For a sufficiently large n the difference on the right hand side of this equality shall be different from zero, and we can begin from this n, and make the sequence

$$z_n = \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)} \cdot \tilde{z}_n \quad (n = 1, 2, \dots) \tag{β}$$

which has the following characteristics: $(Ay_n, z_n) = 0$, $(Ay_n, y_n) = (Az_n, z_n)$ and $z_n \to z_0$ $(n \to \infty)$. Therefore, the pairs $\{y_n, z_n\}$ are in $\mathcal{D}^+$. We now show that the pairs $\{x_n, z_n\}$ are in $\mathcal{D}^+$:

$$\begin{aligned}(Ax_n, z_n) &= \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)} \cdot (Ax_n, \tilde{z}_n) = \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)} \cdot (Ax_n, z_0 \\ &\quad - y_n/(Ay_n, y_n) \cdot (Az_0, y_n) - x_n/(Ax_n, x_n) \cdot (Az_0, x_n)) \\ &= \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)}[(Ax_n, z_0) - (Ay_n, z_0)(Ax_n, y_n)/(Ay_n, y_n) \\ &\quad - (Ax_n, z_0)(Ax_n, x_n)/(Ax_n, x_n)] = 0.\end{aligned} \tag{***}$$

Therefore $(Ax_n, z_n) = 0$ and $(Ax_n, x_n) = (Ay_n, y_n) = (Az_n, z_n)$. Hence the pairs $\{x_n, z_n\}$ are in $\mathcal{D}^+$ $(n = 1, 2, \ldots)$ too.

From these considerations we obtain $x_n \to x_0$, $y_n \to y_0$, z_n, z_0, $(n \to \infty)$.

Since the functional $h^+\{x, y\} = \lim_{n\to\infty}[h(2^n x)/2^{2n} + h(2^n y)/2^{2n}] = h_1(x) + h(y)$ is continuous on $\mathcal{D}^+$, we have

$$\begin{aligned} h^+\{x_n, y_n\} &\to h^+\{x_0, y_0\} = h_1(x_0) + h_1(y_0);\ h^+\{y_n, z_n\} \to h^+\{y_0, z_0\} \\ &= h_1(y_0) + h_1(z_0) h^+\{x_n, z_n\} \to h^+\{x_0, z_0\} = h_1(x_0) + h_1(z_0). \end{aligned}$$

Subtract the second equation from the first to obtain $h^+\{x_0, y_0\} - h^+\{y_0, z_0\} = h_1(x_0) - h_1(z_0)$. By adding this equation to the third we obtain $h^+\{x_0, y_0\} - h^+\{y_0, z_0\} + h^+\{x_0, z\} = 2h_1(x_0)$. Therefore from $x_n \to x_0$ $(n \to \infty)$ it follows that

$$\begin{aligned} 2h_1(x_n) = h^+\{x_n, y_n\} - h^+\{y_n, z_n\} + h^+\{x_n, z_n\} &\to h^+\{x_0, y_0\} - h^+\{y_0, z_0\} \\ + h^+\{x_0, z_0\} = 2h_1(x_0) \end{aligned}$$

i.e.

$$h_1(x_n) \to h_1(x_0).$$

2° We proceed analogously and for the case $(Ax_0, y_0) = 0$ and $(Ay_0, y_0) = -(Ax_0, x_0)$ from $x_n \to x_0$ it follows that $h_1(x_n) \to h_1(x_0)$ $(n \to \infty)$.

B) Suppose that $(Ax_0, x_0) = 0$.

1° Let $Ax_n = 0$ $(n = 1, 2, \ldots)$. Then we put $y_n = 0$ $(n = 1, 2, \ldots)$. Then $(Ax_n, y_n) = 0$, $(Ax_n, x_n) = (Ay_n, y_n) = 0$ $(n = 1, 2, \ldots)$. Therefore, the pairs $\{x_n, y_n\}$ are in $\mathcal{D}^\pm$ $(n = 1, 2, \ldots)$, and we obtain $h^+\{x_n, y_n\} = h_1(x_n) + h_1(y_n)$, $h^-\{x_n, y_n\} = h_1(x_n) - h_1(y_n)$ or $2h_1(x_n) = h^+\{x_n, y_n\} + h^-\{x_n, y_n\}$. Since the functionals h^+ and h^- are continuous, we have

$$2h_1(x_n) = h^+\{x_n, y_n\} + h^-\{x_n, y_n\} \to h^+\{x_0, y_0\} + h^-\{x_0, y_0\} = 2h_1(x_0).$$

Therefore from $x_n \to x_0$ it follows that $h_1(x_n) \to h_1(x_0)$ $(n \to \infty)$.

2° Suppose now that $Ax_n \neq 0$ $(n = 1, 2, \ldots)$ and let $h_1(x) \not\to h_1(x_0)$ $(n \to \infty)$ i.e. let there be an $\varepsilon_0 > 0$ and let for the subsequence $\{x_{n_k}\}_{k=1}^\infty$ of the sequence $\{x_n\}_{n=1}^\infty$ the following hold

$$|h_1(x_{n_k}) - h_1(x_0)| > \varepsilon_0 \tag{8}$$

Put $x_{n_k} = \bar{x}_k$ $(k = 1, 2, \ldots)$; then $|h_1(\bar{x}_k) - h_1(x_0)| > \varepsilon_0$. We construct the sequence $\{z_k\}_{k=1}^\infty = \{A\bar{x}_k/\|A\bar{x}_k\|\}_{k=1}^\infty$ $(\|z_k\| = 1,\ k = 1, 2, \ldots)$. Since the sequence $\{z_k\}_{k=1}^\infty$ is bounded we can separate from it the subsequence $\{z_{k_p}\}_{p=1}^\infty$ which converges weakly to any z_0. Let $y_0 \neq 0$, $y_0 \perp z_0$ and $(Ay_0, y_0) \neq 0$. Then $(z_{k_p}, y_0) = (A\bar{x}_{k_p}/\|A\bar{x}_{k_p}\|, y_0) \to 0$ $(p \to \infty)$. We construct the sequence $\{y_p\}_{p=1}^\infty$, $y_p = y_0 - (A\bar{x}_{k_p}, y_0)/\|A\bar{x}_{k_p}\|^2 \cdot A\bar{x}_{k_p}$ $(p = 1, 2, \ldots)$. Obviously $y_p \to y_0$ and $(A\bar{x}_{k_p}, y_0) \to (Ax_0, y_0) = 0 (p \to \infty)$. We shall show that $(A(x_0 \pm y_0), x_0 \pm y_0) \neq 0$.

From $(A(\bar{x}_{k_p} \pm y_p), \bar{x}_{k_p} \pm y_p) = (A\bar{x}_{k_p}, \bar{x}_{k_p}) \pm (A\bar{x}_{k_p}, y_p) \pm (Ay_p, y_p)$ letting p tend to infinity we get

$$(A(\bar{x}_{k_p} \pm y_p), \bar{x}_{k_p} \pm y_p) \to (A(x_0 \pm y_0), x_0 \pm y_0) = (Ay_0, y_0) \neq 0.$$

Therefore we have $(Ax_0 \pm y_0), x_0 \pm y_0) = (Ay_0, y_0) \neq 0$. Then using A), the functional h_1 is continuous at the points $x_0 \pm y_0$ and y_0.

Since $(A\bar{x}_{k_p}, y_p) = 0$ $(p = 1, 2, \dots)$ and taking into consideration that h_1 is a quadratic functional on A-orthogonal vectors, we have $h_1(\bar{x}_{k_p} + y_p) + h_1(\bar{x}_{k_p} - y_p) = 2h_1(\bar{x}_{k_p}) + 2h_1(y_p)$ or $2h_1(\bar{x}_{k_p}) = h_1(\bar{x}_{k_p} + y_p) + h_1(\bar{x}_{k_p} - y_p) - 2h_1(y_p)$. If $p \to \infty$, we get $2h_1(\bar{x}_{k_p}) \to h_1(x_0 + y_0) + h_1(x_0 - y_0) - 2h_1(y_0) = 2h_1(x_0)$ which contradict (8). From 1° and 2° it now follows that in case $(Ax_0, x_0) = 0$, $x_n \to x_0$, $h_1(x_n) \to h_1(x_0)$ $(n \to \infty)$. This together with A) shows that h_1 is a continuous functional. Let us prove that the functional h_1 is unique. For that proof we shall use the fact that the quadratic functional is "square homogeneous" i.e. $h_1(ax) = |a|^2 h_1(x)$, for all $x \in X$ and a is a complex number. Suppose that there is a continuous functional $h_2 \neq h_1$ for which $h_2(x + y) + h_2(x - y) = 2h_2(x) + 2h_2(y)$, for all $x, y \in X$ for which $(Ax, y) = 0$ and $|h(x) - h_2(x)| \leq \tilde{\varepsilon} \cdot |(Ax, x)|^{p/2}$, for all $x \in X$. Then $|h_1(x) - h_2(x)| \leq |h_1(x) - h(x)| + |h(x) - h_2(x)| \leq (\varepsilon + \tilde{\varepsilon})|(Ax, x)|^{p/2}$, for all $x \in X$, or

$$\begin{aligned} |h_1(x) - h_2(x)| &= |1/n^2 \cdot h_1(nx) - 1/n^2 \cdot h_2(nx)| = 1/n^2 \cdot |h_1(nx) - h_2(nx)| \\ &\leq 1/n^2 \cdot (\varepsilon + \tilde{\varepsilon}) n^p |(Ax, x)|^{p/2} = (\varepsilon + \tilde{\varepsilon}) n^{p-2} |(Ax, x)|^{p/2}. \end{aligned}$$

From this $\lim_{n \to \infty} |h_1(x) - h_2(x)| = 0$, for all $x \in X$. This proves the theorem.

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