PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 37 (51), 1985, pp. 29–32

ON SOME GRAPHIC POLYNOMIALS WHOSE ZEROS ARE REAL

Ivan Gutman

Abstract. Polynomials which are formed by linear combination of the characteristic polynomial of a graph G and the characteristic polynomials of the vertex-deleted subgraphs of G have real zeros. The same is true for the linear combination of the matching polynomial of G and the matching polynomials of the vertex-deleted subgraphs of G. Several statements about the location of the zeros of these polynomials are obtained.

1. Introduction. Let G be a graph having n vertices, $n \geq 2$. Let the vertices of G be labelled by $v_1, v_2, \dots, v_n$. The subgraph obtained from G by deletion of v_r will be denoted by G_r.

Two polynomials associated with a graph have been extesively studied in the mathematical literature, namely the characteristic [**1**] and the matching polynomial [**2**]. They will be denoted by $\varphi(G)$ and $\alpha(G)$, respectively.

Both $\varphi(G)$ and $\alpha(G)$ are polynomials of degree n in the variable x. Their zeros will be denoted by $x_i, 1, 2, \dots, n$ and y_i, $i = 1, 2, \dots, n$, respectively. It is known [**1, 2**] that all x_i's and y_i's are real and that, in addition, the following interlacing relations holds:

(1) $x_i \leq x_i^r \leq x_{i+1}$ for $i = 1, \dots, n-1$

(2) $y \leq y_i^r \leq y_{i+1}$ for $i = 1, \dots, n-1$,

where x_i^r and y_i^r are the zeros of $\varphi(G_r)$ and $\alpha(G_r)$, respectively.

It is also known that

(3) $d\varphi(G)/dx = \sum_{r=1}^{n} \varphi(G_r)$, (4) $d\alpha(G)/dx = \sum_{r=1}^{n} \alpha(G_r)$.

We shall examine several classes of graphic polynomials and determine certain properties of their zeros.

Let A be an ordered n-tuple $(A_1, A_2, \dots, A_n)$ of positive real numbers.

AMS Subject Classification (1980): Primary 05 C 55.

Let $B \subset \{1, 2, \ldots, n\}$. For given A, B and graph G we define following six polynomials:

$$\varphi^*(G) = \sum_{r \in B} A^r \varphi(G_r), \qquad \varphi^-(G) = \varphi(G) - \varphi^*(G)$$

$$\varphi^+(G) = \varphi(G) + \varphi^*(G),$$

$$\alpha^*(G) = \sum_{r \in B} A_r \alpha(G_r), \qquad \alpha^-(G) = \alpha(G) - \alpha^*(G),$$

$$\alpha^+(G) = \alpha(G) + \alpha^*(G).$$

Note that for $A_1 = A_2 = \cdots = A_n = 1$ and $B = \{1, 2, \ldots, n\}$, $\varphi^*(G)$ and $\alpha^*(G)$ are equal to the first derivatives of $\varphi(G)$ and $\alpha(G)$, respectively, eqs. (3) and (4)

The following theorem can be understood as the main result of the present work.

THEOREM 1. (a) *For all A, B and G, all the zeros of $\varphi^*(G)$, $\varphi^-(G)$, $\varphi^+(G)$, $\alpha^*(G)$, $\alpha^-(G)$ and $\alpha^+(G)$ are real.* (b) *If these zeros are denoted by x_i^*, x_i^- x_i^+ y_i^*, y_i^- and y_i^+, respectively, then for $i = 1, \ldots, n-1$, $x_i^+ \leq x_i \leq x_i^- \leq x_i^* \leq x_{i+1}^+ \leq x_{i+1} \leq x_{i+1}^-$.*

2. Preliminaries. All the polynomials considered in the present paper will be assumed to have real coefficients and a positive leading coefficient, ($\varphi(G)$, $\alpha(G)$ and the polynomials introduced in Definition 1 meet, of course, these requirements.) The variable in all the polynomials considered is denoted by x.

Let P and Q be two polynomials of degree m and n, respectively. Let their zeros be $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_n$, respectively.

We say that P *separates* Q in the following two cases;

(a) if $m = n - 1$ and $q_i \leq p_i \leq q_{i+1}$ for $i = 1, \ldots, n-1$, and

(b) if $m = n$ and $q_i \leq p_i \leq q_{i+1}$ for $i = 1, \ldots, n-1$ and $q_n \leq p_n$.

Then we shall write P sep Q.

The relation P sep Q implies, of course, that all the zeros of both P and Q are real. We shall need the following simple property of the separation relation.

LEMMA 1. *If a polynomial S exists, which separates the polynomials P, Q and R, then from P sep Q and Q sep R it follows that P sep R.*

Using the notation of Definition 2, we can formulate the inequalities (1) and (2) in the following manner.

LEMMA 2. *For all $r = 1, 2, \ldots, n$, $\varphi(G_r)$ sep $\varphi(G)$ and $\alpha(G_r)$ sep $\alpha(G)$.*

LEMMA 3. *Let P, Q and R be polynomials, such that P sep R and Q sep R and let P and Q have equal degrees m. Then for A_1 and A_2 being arbitrary positive constants.*

$$\min\{p_i, q_i\} \leq s_i \leq \max\{p_i, q_i\}, \tag{5}$$

where p_i, q_i *and* s_i *are the zeros of* P, Q *and* $S = A_1 P + A_2 Q$*, respectively,* $i = 1, 2, \ldots, m$.

Proof. Let T be the greatest common divisor of P and Q and let $P = T \cdot P_0$ and $Q = T \cdot Q_0$. Then also $S = T \cdot S_0$ with $S_0 = A_1 P_0 + A_2 Q_0$.

If $p_i = q_i$, then the inequalities (5) hold in a trivial manner. It is, therefore, sufficient to prove (5) for the zeros of P_0, Q_0 and S_0,

Let $p_{0,i}, q_{0,i}$ and $s_{0,i}$, $i = 1, \ldots, m_0$ be the zeros of P_0, Q_0 and S_0, respectively, labelled in non-decreasing order.

Two cases are to be distinguished: m_0, the degree of P_0, Q_0 and S_0, is either even or odd. Here we shall suppose that m_0 is even; the proof for the case when m_0 is odd is fully analogous.

If m_0 is even, then for $x < p_{0,1}$ (respectively for $x < q_{0,1}$), the polynomial P_0 (respectively Q_0) has positive values. Therefore S_0 is necessarily positive for $x < \min\{p_{0,1}, q_{0,1}\}$. Similarly, in the interval $[\max\{p_{0,1}, q_{0,1}\}, \min\{p_{0,2}, q_{0,2})]$ the polynomial S_0 must be negative, in the interval $[\max\{p_{0,2}, q_{0,2}\}, \min\{p_{0,3}, q_{0,3}\}]S_0$ must be positive etc. Consequently, $s_{0,i}$ lies in the internal $[\min\{p_{0,i}, q_{0,i}\}, \max\{p_{0,i}, q_{0,i}\}]$, $i = 1, \ldots, m_0$. The requirements P sep R and Q sep R quarantee that the above intervals will not overlap. □

3. Some separation relations. From Lemma 2 we see that the polynomials $\varphi(G_r)$ and $\alpha(G_r)$ meet the requirements of Lemma 3. Hence we have the following immediate consequence of Lemma 3.

LEMMA 4. *For all* A, B *and* G,

$$\min_{r \in B}\{x_i^r\} \le x_i^* \le \max_{r \in B}\{x_i^r\} \qquad \min_{r \in B} \le y_i^* \le \max_{r \in B}\{y_i^r\}, \quad i = 1, \ldots, n-1.$$

A special case of the result above is obtained by taking into account eqs. (3) and (4).

COROLLARY. *If* x_i' *and* y_i' *are the zeros of the first derivative of* $\varphi(G)$ *and* $\alpha(G)$*, respectively, then for* $i = 1, \ldots, n-1$,

$$\min_r\{x_i^r\} \le x_i' \le \max\{x_i^r\}, \qquad \min_r\{y_i^r\} \le y_i' \le \max_r\{y_i^r\}.$$

THEOREM 2. *For all* A, B *and* G; $\varphi(G)$ *sep* $\varphi(G)$ *and* $\alpha^*(G)$ *sep* $\alpha(G)$

Proof. By eqn. (1) (or by Lemma 2), $x_i \le \min\{x_i^r\}$ and $\max\{x_i^r\} \le x_{i+1}$. Then by Lemma 4, $x_i \le x_i^* \le x_{i+1}$, and the first part of Theorem 2 follows. The proof of the second part is analogous. □

THEOREM 3. *For all* A, B *and* G,

$$\varphi^*(G) \text{ sep } \varphi^-(G), \;\; \varphi^*(G) \text{ sep } \varphi^+(G), \;\; \alpha^*(G) \text{ sep } \alpha^-(G), \;\; \alpha^*(G) \text{ sep } \alpha^+(G).$$

Proof. We prove here only the first of the four statements given in the theorem, assuming besides that n is even. The proof in the case of odd n, as well as the proof of the additional three separation relations, follows in a completely analogous manner.

Let us further assume that the zeros of $\varphi(G)$ and $\varphi^*(G)$ are mutually distinct. (When this is not the case, then we have to find the greatest common divisor of $\varphi(G)$ and $\varphi^*(G)$ and to proceed similarly as in the proof of Lemma 3.) We already know that $\varphi^*(G)$ sep $\varphi(G)$. Since n is assumed to be even, for $x < x_1$ the polynomial $\varphi(G)$ has positive values, where as for $x < x_1^*$, $\varphi^*(G)$ is negative. Furthermore, $x_1 < x_i^*$. Therefore, $\varphi(G) - \varphi^*(G)$ will be positive for $x < x_1$. Similar arguments show that $\varphi(G) - \varphi^*(G)$ will be negative in the interval $[x_1^*, x_2]$, positive in the interval $[x_2^*, x_3]$ etc. Therefore, the zeros of $\varphi^-(G)$ lie in the intervals $[x_i, x_i^*]$, $i = 1, \dots, n-1$ and another zero lies in $[x_n, \infty)$. These intervals cannot overlap because of Theorem 2.

This proves that $\varphi^*(G)$ separates $\varphi(G)$ if n is even. □

THEOREM 4. *For all A, B and G,*

$$\varphi^-(G) \text{ sep } \varphi(G), \;\; \varphi(G) \text{ sep } \varphi^+(G), \;\; \alpha^-(G) \text{ sep } \alpha(G), \;\; \alpha(G) \text{ sep } \alpha^+(G).$$

Proof. In the proof of the previous theorem it was shown that the zeros x_i^- of $\varphi^-(G)$ lie in the interval $[x_i, x_i^*]$, i. e. $x_i \leq x_i^-$ for $i = 1, 2, \dots, n$ This, however, is just the first separation relation given in Theorem 4. Etc. □

THEOREM 5. *For all A, B and G, $\varphi^-(G)$ sep $\varphi^+(G)$ and $\alpha^-(G)$ sep $\alpha^+(G)$.*

Proof. Apply Lemma 1 to Theorem 3 and 4. Note that $\varphi^*(G)$ and $\alpha^*(G)$ play now the role of the polynomial S. □

By proving Theorems 2–5 we have, of course, also completed the proof of Theorem 1. It can be sean that Theorem 1 is, in fact, a consequence of the interlacing relations (1) and (2). It would be interesting to see if results similar to those given in Theorem 1 hold also for subgraphs obtained by deletion of more that one vertex from the graph.

REFERENCES

[1] D. Cvetković, M/ Doob, H. Sachs, *Spectra of Graphs—Theory and Applications*, Academic Press, New Yotk, 1980.

[2] C. D. Godsil, I. Gutman, *On the theory of the matching polynomial*, J. Graph Theory **5** (1981), 137–144.

Prirodno-matematički fakultet
34000 Kragujevac
Yugoslavia

(Received 06 02 1984)