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THE GENERAL LINEAR EQUATION ON VECTOR SPACES

Jovan Kečkić

Abstract. General solution of linear equation of the form (1) and (3) are obtained by means of the generalized inverse functions. The obtained theorems are applied to equations on near-rings, linear functionals, matrix, differential and functional equations.

1. General theorems

Let X and Y be nonempty sets and let $f : X \to Y$ be a surjection. The existence of a function $g : Y \to X$ such that $(\forall y \in Y) f(g(y)) = y$ is a well-known equivalent of the Axiom of Choice, due to Bernays [**1**] (see also [**2**]). By a slight modification of the argument, we prove the following

THEOREM 1. *Suppose that X and Y are nonempty sets, and let $a \in X$, $b \in Y$. If $f : X \to Y$ and $f(a) = b$, then there exists a function $g : Y \to X$ such that:*

(i) $fgf = f$, *i.e.* $(\forall x \in X) f(g(f(x))) = f(x)$;

(ii) $g(b) = a$.

Proof. If $f(X)$ is a singleton, then $f(X) = \{b\}$ and the function $g : Y \to X$ defined by $(\forall y \in Y) g(y) = a$ satisfies (i) and (ii).

If $f(X)$ contains more than one element, let $X_y = \{x \mid f(x) = y\}$, where $y \in f(X)$. Then $X_y \neq \varnothing$, $f(X)\backslash\{b\} \neq \varnothing$, and according to the Axiom of Choice there exists a function $G : f(X)\backslash\{b\} \to X\backslash X_b$ such that $G(y) \in X_y$. The function $g : Y \to X$ defined by

$$g(y) = \begin{cases} G(y), & y \in f(X)\backslash\{b\} \\ a, & y = b \\ H(y), & y \in Y\backslash f(X) \end{cases}$$

where $H : Y\backslash f(X) \to X$ is an arbitrary function (for example, $H(y) = a$, for every $y \in Y\backslash f(X)$) satisfies the conditions (i) and (ii).

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Indeed, (ii) is trivial. To prove (i) notice that for arbitrary $x \in X$ we have

$$g(f(x)) = \begin{cases} G(f(x)), & x \in X \backslash X_b (\Leftrightarrow f(x) \neq b) \\ a, & x \in X_b \quad (\Leftrightarrow f(x) = b) \end{cases}$$

and so

$$f(g(f(x))) = \begin{cases} f(x), & f(x) \neq b \\ a, & f(x) = b \end{cases} = f(x)$$

which completes the proof.

We now apply Theorem 1 to the general linear equation on groups. Namely, suppose that $(G_1, *)$ and $(G_2, \circ)$ are groups whose neutral elements are denoted by e_1 and e_2, respectively. If $f : G_1 \to G_2$ is a homomorphism, then $f(e_1) = e_2$, and hence, according to Theorem 1 there exists a function $g : G_2 \to G_1$ such that $fgf = f$ and $g(e_2) = e_1$.

THEOREM 2. *Consider the equation in x:*

(1) $$f(x) = e_2$$

The general solution of the equation (1) *is given by:*

(2) $$x = t * \overline{g(f(t))},$$

where $t \in G_1$ is arbitrary and $\overline{u}$ denotes the inverse of $u (\in G_1$ or $G_2)$.

Proof. The proof of this statement is straight forward. Namely, since f is a homomorphism, from (2) follows

$$f(x) = f(t * \overline{g(f(t))} = f(t) \circ f\overline{(g(f(t)))} = f(t) \circ \overline{f(g(f(t)))} = f(t) \circ \overline{f(t)} = e_2,$$

which means that (2) is a solution of (1). Conversely, suppose that x_0 is a solution of (1), i.e. that $f(x_0) = e_a$. Then, putting $t = x_0$ into (2) we get

$$x = x_0 * \overline{g(f(x_0))} = x_0 * \overline{g(e_2)} = x_0 * \overline{e_1} = x_0 * e_1 = x_0.$$

In other words, the solution x_0 of (1) is obtained from (2) by putting $t = x_0$, which means that (2) is the general solution of (1).

Consider now the nonhomogeneous equation in x:

(3) $$f(x) = c$$

where $c \in G_2$ is given. The equation (3) has a solution if and only if

(4) $$f(g(c)) = c$$

In that case the general solution of (3) is given by

(5) $$x = t * \overline{g(f(t))} * g(c)$$

where $t \in G_1$ is arbitrary. Indeed, if (3) has a solution, then from (3) follows $g(f(x)) = g(c)$, and again $f(g(f(x))) = f(g(c))$. But $fgf = f$ which together with (3) and the last equality implies (4). Conversely, if (4) holds, then $g(c)$ is clearly a solution of (3). The fact that (5) is the general solution of (3) is easily verified.

Remark. If g is the inverse function of f then (1) and (3) have unique solutions, namely: e_1 and $g(c)$, respectively.

Problem. According to Theorem 1, for a homomorphism $f : G_1 \to G_2$ there exists a function $g : G_2 \to G_1$ such that $fgf = f$ and $g(e_2) = e_1$. What additional conditions, if any, are needed to ensure that g is also a homomorphism?

2. The case of vector spaces

If V_1 and V_2 are vector spaces over a scalar fields S and if $f \in \text{Hom}(V_1, V_2)$, i. e. $f : V_1 \to V_2$ is a homomorphism, then there exists a function $g : V_2 \to V_1$ such that $fgf = f$ and $g(0) = 0$, and we obtain corresponding conclusions about the equations $f(x) = 0$ and $f(x) = c$.

However, in this case it is possible to obtain the form of the general solution of those equations. Namely, we have

THEOREM 3. *If $f \in \text{Hom}(V_1, V_2)$, the general solution of the equation $f(x) = 0$ has the form $x = h(t)$, where $h \in \text{Hom}(V_1, V_1)$ and $t \in V_1$ is arbitrary.*

Proof. We first prove that there exists a homomorphism $g : f(V_1) \to V_1$ such that $fgf = f$. Indeed, since $f(V_1)$ is a vector space, it has a basis $B = \{b_1, b_2, \dots\}$. Moreover, $b_i \in f(V_1)$ and so the set $X_i = \{x \mid f(x) = b_i\}$ is not empty. Hence, according to the Axiom of Choice, there exists a function $g : B \to V_1$ such that $g(b_i) = g_i \in X_i$. For arbitrary $y = \sum_{k=1}^{n(y)} \alpha_k b_k \in f(V_1)$ define $g(y) = \sum_{k=1}^{n(y)} \alpha_k g_k$. The function $g : f(V_1) \to V_1$ defined in this way is clearly a homomorphism and it is easily verified that for all $x \in V_1$ we have $f(g(f(x))) = f(x)$. Hence, the general solution of $f(x) = 0$ is $x = t - g(f(t)) = (i - gf)(t)$, where $i : V_1 \to V_1$ is the identity mapping and $t \in V_1$ is arbitrary. Since $h = i - gf \in \text{Hom}(V_1, V_1)$, the theorem is proved.

Therefore, the general solution of the linear equation $f(x) = 0$ is a linear function of an arbitrary element t. However, in order to obtain the solution explicitly. it is necessary to construct the function g.

3. Applications

We now investigate some cases in which the function g can be determined.

3.1. Linear equations on near-rings. Suppose that $(P, +, \cdot)$ is a near-ring (i. e. the group $(P, +)$ need not be commutative). The function $f : P \to P$ defined by $f(x) = axb$, where $a, b \in P$ are fixed, is a homomorphism.

If a, b are regular elements of P, i. e. if there exist $\overline{a}, \overline{b} \in P$, such that $a\overline{a}a = a$, $b\overline{b}b = b$, then the function $g : P \to P$ defined by $g(x) = \overline{a}x\overline{b}$ is such that $fgf = f$. Hence, the general solution of the equation $axb = 0$ is: $x = t - \overline{a}atb\overline{b}$. The nonhomogeneous equation $axb = c$ has a solution if and only if $a\overline{a}c\overline{b}b = c$; in that case, the general solution is $x = t - \overline{a}atb\overline{b} + \overline{a}c\overline{b}$. For instance, the general solution of $axb = ab$ is: $x = t - \overline{a}atb\overline{b} + \overline{a}ab\overline{b}$, where $t \in P$ is arbitrary.

More general equations, together with applications to matrix equations are considered in [**3**].

3.2. Linear functionals. Let V be a vector space overr the field S, let $f : V \to S$ be a linear functional on V and consider the equation in x:

$$f(x) = 0 \tag{6}$$

We suppose that there exists $x_0 \in V$ such that $f(x_0) \neq 0$; otherwise (6) holds for all $x \in V$.

For the function $g : S \to V$ defined by $g(s) = sx_0/f(x_0)$ it is easily verified that $fgf = f$. Hence, the general solution of (6) is $x = t - x_0 f(t)/f(x_0)$, where $t \in V$ is arbitrary. Moreover, the general solution of the nonhomogeneous equation $f(x) = c$ is: $x = t + (c - f(t))x_0/f(x_0)$, where $t \in V$ is arbitrary.

Various applications of this result, particularly to integral equations, are given in [**4**].

3.3. The function f satisfies a polynomial equation. Let $f : V \to V$, where V ia a vector space over a field S and suppose that the function f satisfies an equation of the form:

$$\lambda_n f^n + \lambda_{n-1} f^{n-1} + \cdots + \lambda_1 f + \lambda_0 i = 0, \tag{7}$$

where $\lambda_0, \ldots, \lambda_n \in S$, $i : V \to V$ is the identity mapping and f^k is the k-th iterate of f. We have the following conclusions:

(i) If $\lambda_0 \neq 0$, then the function g defined by

$$g = -\lambda_0^{-1}(\lambda_n f^{n-1} + \lambda_{n-1} f^{n-2} + \cdots + \lambda_1 i)$$

is the inverse of f.

(ii) If $\lambda_0 = 0$, $\lambda_1 \neq 0$, then the function g defined by

$$g = -\lambda_1^{-1}(\lambda_n f^{n-2} + \lambda_{n-1} f^{n-3} + \cdots + \lambda_2 i)$$

is such that $fgf = f$.

Hence, in those cases it is possible to write down the general solutions of the equations $f(x) = 0$ and $f(x) = c$.

Remark. If $\lambda_0 = \lambda_1 = 0$, then $x = \lambda_n f^{n-1}(t) + \cdots + \lambda_2 f(t)$, where $t \in V$ is arbitrary, is clearly a solution of the equation $f(x) = 0$, but examples can be constructed to show that this solution need not be general.

In particular, if f can be written in the form

$$f(x) = \sum_{\nu=1}^{m} \sigma_{1\nu} A_\nu(x) \qquad (\sigma_{1\nu} \in S), \tag{8}$$

where the linear functions $A_1, \dots, A_m : V \to V$ form a semigroup, then

$$f^k(x) = \sum_{\nu=1}^{m} \sigma_{k\nu} A_\nu(x) \qquad (k = 1, \dots, m), \tag{9}$$

and eliminating the $A_\nu(x)$'s between (8), (9) and $i(x) = x$, we arrive at an equation of the form (7).

This method was applied in [**5**] to the linear matrix equation

$$A_1 X B_1 + \cdots + A_m X B_m = 0.$$

3.4. Differential equations. This example shows how the existing theory of linear differential equations can be interpreted within the framework of the general method given here. Namely, it can be shown [**6**] that the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is equavilent to the equation

$$y - \frac{W(y,\psi)}{W(\varphi,\psi)}\varphi - \frac{W(\varphi,y)}{W(\varphi,\psi)}\psi = 0,$$

where φ and ψ are linearly independent solutions of (10) and $W(u,\nu) = u'\nu - u\nu'$. However, for the function f defined by

$$f(y) = y - \frac{W(y,\psi)}{W(\varphi,\psi)}\varphi - \frac{W(\varphi,y)}{W(\varphi,\psi)}\psi$$

we have $f^2 = f$, and hence the general solution of $f(y) = 0$, i. e. the general solution of (10) is

$$y = t - f(t) \quad (t \text{ arbitrary twice differentiable function) i.e.}$$

$$y = \frac{W(t,\psi)}{W(\varphi,\psi)}\varphi + \frac{W(\varphi,t)}{W(\varphi,\psi)}\psi.$$

Since it can be shown that the expressions $W(t,\psi)/W(\varphi,\psi)$ and $W(\varphi,t)/W(\varphi,\psi)$ do not depend on x (provided that p has a primitive function), the last expression takes the familiar form: $y = C_1\varphi + C_2\psi$, where C_1 and C_2 are arbitrary constants.

This method of approach to linear differential equations has certain advantages over the standard method. They are discussed in [**6**].

3.5. Equations on algebras. Suppose that V is a commutative algebra, and cosider the equation in $x \in V$:

$$a_{11}A_1x + \cdots + a_{1n}A_nx = 0, \tag{11}$$

where $a_{11}, \dots, a_{1n} \in V$, $A_1, \dots, A_n : V \to V$ are linear functions with the properties:

(i) $G = \{A_1, \dots, A_n\}$ is a group of order n;

(ii) $A_i(\nu A_j x) = A_i(\nu)A_i(A_j x)$ for all $x, \nu \in V$ and $i, j = 1, \dots, n$, Then, if we put

$$f(x) = \sum_{\nu=1}^{n} a_{1\nu}A_\nu x, \tag{12}$$

it again follows that

$$f^k(x) = \sum_{\nu=1}^{n} a_{k\nu}A_\nu x \qquad (k = 1, \dots, n), \tag{13}$$

and again eliminating the $A_\nu x$'s between (12), (13) and $i(x) = x (i \in G)$, we arrive at an equation of the form

$$a_n f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 i = 0.$$

Though the coefficients $a_0, \dots, a_n$ belong to V, it can be shown, by a technique similar to Prešić's [**7**] that $f(a_k f^k) = a_k f^{k+1}$, and so the function g can be formed analogously as in 3.3. The fact that $f(a_k f^k) = a_k f^{k+1}$ corresponds to the condition "compatible with the group G" which appears in [**7**].

Remark. The equation (11) can be treated in the same way as Prečić [**7**] solved its special case, the equation for $\varphi : E \to K$

$$a_1(x)\varphi(g_1x) + \cdots + a_n(x)\varphi(g_nx) = 0,$$

where $g_1, \dots, g_n : E \to E$ form a group of order n. In this case E is a nonempty set, K is field and $a_1, \dots, a_n : E \to K$.

Example. As an example, we solve the following functional equation

$$a(x)\varphi(x) + b(x)\varphi(-x) = 0, \tag{14}$$

where $a, b : R \to R$ are given, and $\varphi : R \to R$ is the unknown function. Let $f : R^R \to R^R$ be defined by

$$f(\varphi(x)) = a(x)\varphi(x) + b(x)\varphi(-x). \tag{15}$$

Then

$$\begin{aligned} f^2(\varphi(x)) =&(a(x)^2 + b(x)b(-x))\varphi(x)) \\ &+(a(x)b(x) + a(-x)b(x))\varphi(-x), \end{aligned} \tag{16}$$

and elimination of $\varphi(x)$ and $\varphi(-x)$ between (15), (16) and $i(\varphi x) = \varphi(x)$ leads to the equation

$$f^2 - (a(x) + a(-x))f + (a(x)a(-x) - b(x)b(-x))i = 0. \tag{17}$$

If $a(x)a(-x) \neq b(x)b(-x)$, f has its inverse f^{-1} and $\varphi(x) \equiv 0$ is the only solution of (14). Suppose that $a(x)a(-x) = b(x)b(-x)$ and that $a(x) + a(-x) \neq 0$. Then (17) reduces to

$$f^2 - (a(x) + a(-x))f = 0,$$

and the function $g : R^R \to R^R$ defined by

$$g = (a(x) + a(-x))^{-1} i$$

is such that $fgf = f$, which is easily verified. Hence, the general solution of (14) is

$$\varphi(x) = t(x) - \frac{a(x)t(x) + b(x)t(-x)}{a(x) + a(-x)}$$

i. e.

$$\varphi(x) = \frac{a(-x)t(x) - b(x)t(-x)}{a(x) + a(-x)} \quad (t : R \to R \text{ is arbitrary}).$$

* * *

The research which lead to **[3]**—**[6]** and finally to this paper, was initiated mainly by **[7]** and **[8]**.

REFERENCES

[1] B. Bernays, *A system of axiomatic set theory II*, J. Symb. Logic **6** (1941), 1–17.

[2] H. Rubin, J. Rubin, *Equivalents of the Axiom of Choice*, Amsterdam, 1963.

[3] J. D. Kečkić, *Some linear equations on near-rings*, Math. Balkanica **9** (1979).

[4] J. D. Kečkić, *Reproductivity of some equations of analysis II*, Publ. Inst. Math. (Beograd) (N.S.) **33**(**47**) (1983), 109-118.

[5] J. D. Kečkić, *Explicit solutions of some linear matrix equations*, Publ. Inst. Math. (Beograd), **35**(**49**) (1984), 78–82.

[6] J. D. Kečkić, *Reproductivity of some equations of analysis*, Publ. Inst. Math. (Beograd) **31**(**45**) (1982), 73–81.

[7] S. Prešić, *Méthode de résolution d'une classe d'équations fonctionnelles linéaires*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. N° **115**—N° **121** (1963), 21–28.

[8] S. B. Prešić, *Une classe d'équations matricielles et l'équation fonctionnelle $f^2 = f$*, Publ. Inst. Math. (Beograd) **8**(**22**) (1968), 143–148.

Tikveška 2
11000 Beograd

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