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A NOTE ON MINKOWSKI FUNCTIONALS OF A TOPOLOGICAL VECTOR SPACE

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Abstract. We consider under what locally convex spaces the lower semi-continuous semi-norms coincide with the semi-norms which are continuous relative to the strong topology Furthermore in a topological vector space we study relations between the lower semi-continuous (F)-semi-norms and the (F)-semi-norms which are continuous relative to the strong topology in the sense of topological vector spaces.

1. Introduction. Let $E(t)$ be a Hausdorff locally convex space and E' be its dual. It is well known that the locally convex topology defined by the system of lower semi-continuous semi-norms relative to t is identical with the strong topology $\beta(E, E')$. Then the semi-norms which are lower semi-continuous relative to t are always $\beta(E, E')$-continuous. In this note, we consider that under what conditions the converse mentioned above is always true. Further in a Hausdorff topological vector space $E(t)$ we replace "semi-norm" with "(F)-semi-norm" and study relations between the lower semi-continuous (F)-semi-norms relative to t and the (F)-semi-norms which are continuous relative to the strong topology in the sense of topological vector spaces. Throughout this note, we consider the spaces which are Hausdorff locally convex spaces or Hausdorff topological vector spaces over the real or complex field K. We mostly use notations and definitions of Adasch, Ernst and Keim [1] and Köthe [3]. In section 2 we consider the case of locally convex spaces and consider the case of topological vector spaces in section 3.

2. In locally convex spaces. We shall need the following results.

(1) Let P be the set of real numbers. Then a real valued function $f(x)$ on a topological space R is lower semi-continuous if and only if for each $r \in P$ the subset $\{x \mid f(x) \leq r\}$ is closed (Köthe [3]).

(2) Let $E(t)$ be a topological vector space and $f(x)$ be a linear functional. Then $f(x)$ is continuous if and only if $f^{-1}(0) = \{x \mid f(x) = 0\}$ is closed in $E(t)$ (Robertson and Robertson [4]).

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Now using these results, we obtain the following

THEOREM 1. *Let $E(t)$ be a locally convex space and E' be its dual. Then the semi-norms which are lower semi-continuous relative to t coincide with the $\beta(E,E')$-continuous semi-norms if and only if the Mackey topology $\tau(E,E')$ is identical with the strong topology $\beta(E,E')$.*

Proof. If $\tau(E,E')=\beta(E,E')$, then $U=\{x \mid p(x)\leq 1\}$ is $\tau(E,E')$-closed for an arbitrary $\beta(E,E')$-continuous semi-norm $p(x)$ on E. From a property of locally convex topologies on E which are compatible with the dual pair (E,E'), U is t-closed. Thus $p(x)$ is lower semi-continuous on $E(t)$ from (1). Conversely to show that $\tau(E,E')=\beta(E,E')$, if we denote the dual space of $E(\beta(E,E'))$ by E'_β, it suffices to show that $E'_\beta=E'$. If $u(x)$ is an arbitrary $\beta(E,E')$-continuous linear functional, then the semi-norm $p(x)=|u(x)|$ for all $x\in E$ is a $\beta(E,E')$-continuous semi-norm. By the assumption, $p(x)$ is lower semi-continuous on $E(t)$. Hence $\bigcap_{0<r}\{x \mid p(x)\leq r\}$ is t-closed. Since $\bigcap_{0<r}\{x \mid p(x)\leq r\}=\{x \mid p(x)=0\}$, the subset $\{x \mid u(x)=0\}$ is t-closed. So $u(x)$ is t-continuous from (2). This completes the proof.

Example 1. Let ψ be a finite sequence space. Then in the dual pair (l^1,ψ), we consider the space $l^1(\sigma(l^1,\psi))$. In $l^1(\sigma(l^1,\psi))$, it has its dual ψ and $\beta(l^1,\psi)$ is identical with the l^1-norm topology.

Thus l^∞ is the dual of $l^1(\beta(l^1,\psi))$. In l^∞, we denote by e the sequence such that $e=(a_n)$, $a_n=1$ for all $n\in N$. Then if we consider the semi-norm $p(x)=|\langle e,x\rangle|=|\sum_{i=1}^{\infty}x_i|$ for all $x\in l^1$, then $p(x)$ is $\beta(l^1,\psi)$-continuous but not lower semi-continuous relative to $\sigma(l^1,\psi)$ since e is an element of l^∞ but not of ψ.

3. In topological vector spaces. Following Adasch, Ernst and Keim [1], we begin with explanations of some definitions.

Definition 1. Let E be a vector space over K. A sequence (U_n) of subsets U_n of E is called a string if (i) every U_n is balanced and absorbing, (ii) (U_n) is summative, that means $U_{n+1}+U_{n+1}\subset U_n$ for all $n\in N$. The subset U_n is called the *n-th knot* of (U_n). Further let $E(t)$ be a topological vector space. A string (U_n) in $E(t)$ is called *topological* (*closed*) if every U_n is a neighbourhood of 0 (a closed subset).

Remark. In a topological vector space, we can take a set of strings whose knots form a base of neighbourhoods of 0.

Definition 2. An *(F)-semi-norm* on a vector space E over K is a real valued function $p: E\to R$ (R is the real field) satisfying that for all $x,y\in E$ (i) $p(x)\geq 0$, (ii) $p(x+y)\leq p(x)+p(y)$, (iii) $p(\lambda x)\leq p(x)$ for all $\lambda\in K$ with $|\lambda|\leq 1$ and (iv) if (λ_n), $\lambda_n\in K$ converges to 0, then $(p(\lambda_n x))$ converges to 0.

Definition 3. Let (U_n) be a string on a vector space E. Then we call $q(x)$ the *associated (F)-semi-norm* of (U_n) if the (F)-semi-norm is given by the following process. First we prepare the rational dyadic numbers $\delta>0$ of the following form;

$\delta = n + \sum_{k=1}^{\infty} \varepsilon_k \cdot 1/2^k$ ($n = 0$ or $n \in N$), where $\varepsilon_k = 1$ for at most finitely many $k \in N$ and $\varepsilon_k = 0$ otherwise. For each δ, we define a subset W_δ of E by $W_\delta = \sum_1^n U_1 + \sum_{k=1}^{\infty} \varepsilon_k U_{k+1}$. If we put $q(x) = \inf\{\delta \mid x \in W_\delta\}$ for all $x \in E$, then $q(x)$ is an (F)-semi-norm on E.

Definition 4. An (F)-semi-norm $p(x)$ on a vector space E has $(*)$ property if $\{x \mid p(x) \leq 1/2^{n-1}\} = \{x \mid q(x) \leq 1/2^{n-1}\}$ $n \in N$, where $q(x)$ is the associated (F)-semi-norm of the string (U_n), $U_n = \{x \mid p(x) \leq 1/2^{n-1}$, $n \in N$.

Using these definitions, we obtain the following

PROPOSITION. *Let $E(t)$ be a topological vector space and (U_n) be a closed string in $E(t)$. Then there exists an (F)-semi-norm $p(x)$ with $(*)$ property which is lower semi-continuous and satisfies $U_n \subset \{x \mid p(x) \leq 1/2^{n-1}\} \subset U_{n-1}$ for $n \geq 2$.*

Proof. Using a similar method to Definition 3, for (U_n) if we define a closed subset V_δ of E by $V_\delta = \overline{\sum_1^n U_1 + \sum_{k=1}^{\infty} \varepsilon_k U_{k+1}}$ for each rational dyadic number δ and if we put $p(x) = \inf\{\delta \mid x \in V_\delta\}$, then clearly $p(x)$ is an (F)-semi-norm. Further $p(x)$ is lower semi-continuous, since $\{x \mid p(x) \leq r\} = \bigcap_{r \leq \delta} V_\delta$ for each nonnegative number r.

Now we shall show that $p(x)$ has $(*)$-property. Let $q(x)$ be the associated (F)-semi-norm of the string (W_n), where each $W_n = \{x \mid p(x) \leq 1/2^{n-1}\}$ $n \in N$. For each $n \in N$, if we set $W_n' = \{x \mid q(x) \leq 1/2^{n-1}\}$, then clearly W_n is contained in W_n'. Conversely for an arbitrary $x \in W_n'$, we obtain that

$$x \in W_n + W_m = \bigcap_{1/2^{n-1} < \delta} V_\delta + \bigcap_{1/2^{m-1} < \delta} V_\delta \subset \overline{U_n + U_k} + \overline{U_m + U_k} \quad \text{for all } k, m \in N$$

from the definition of the associated (F)-semi-norm. On the other hand, as each

$$W_n = \bigcap_{i \in N} (V_{1/2^{n-1} + 1/2^{i-1}}) = \bigcap_{i \in N} \overline{U_n + U_i},$$

if we put $k = i + 3$ and $m = i + 3$, $i \in N$ in the above formula, then we have $x \in \overline{U_n + U_{i+3}} + \overline{U_{i+3} + U_{i+3}} \subset \overline{U_n + U_i}$ for all $i \in N$ and hence x is an element of W_n. Finally from $W_n = \bigcap_{i \in N} \overline{U_n + U_i}$, we obtain that $U_n \subset W_n \subset \overline{U_n + U_n} \subset U_{n-1}$ for $n \geq 2$.

In a topological vector space $E(t)$, we can define a linear topology if we take knots of all closed strings as a base of neighbourhoods of 0. We call this linear topology the *strong topology* of $E(t)$ and denote it by t^b. A topological vector space $E(t)$ is called *barrelled in* L if t is identical with t^b [1]. If $E(t)$ is a locally convex space and E is the dual of $E(t)$, then clearly t^b is finer than $\beta(E, E')$. As in the case of locally convex spaces, barrelledness in L is expressed by the lower semicontinuous (F)-semi-norms from Proposition.

COROLLARY. *A topological vector space $E(t)$ is barrelled in L if and only if t is identical with the linear topology which is generated by the system of the t-lower semi-cohtinuous (F)-semi-norms with $(*)$ property.*

In a topological vector space $E(t)$, it seems to be a gap between the t^b-continuous (F)-semi-norms and the t-lower semi-continuous ones since we cannot apply duality methods and properties of semi-norms. As in Theorem 1, veen though we set a locally convex space $E(t)$ with its dual E' such that $E(t^b)$ is barrelled in L, if E' is a proper subspace of E'_b, where E'_b is the dual of $E(t^b)$, then the t^b-continuous (F)-semi-norms do not coincide with the t-lower semi-continuous ones. For instance, $\psi(\delta(\psi, psi))$ or $l^1(\delta(l^1, \psi))$ in Example 1 applies. Hence we shall consider a negative partial answer for the rest of this note. Before giving a theorem, we need the following two lemmas.

LEMMA 1. *Let $E(t)$ be a metrizable topological vector space and (U_n) be a closed string whose knots form a base of neighbourhoods of* 0. *Then the associated (F)-norm $p(x)$ of (U_n) has $(*)$ property. (We call $p(x)$ (F)-norm if $p(x)$ is an (F)-semi-norm such that $p(x) = 0$ implies $x = 0$.)*

Proof. Clearly each U_n is contained in $V_n = \{x \mid p(x) \leq 1/2^{n-1}\}$ from Definition 3. Conversely for an arbitrary $x \in V_n$, if $p(x) < 1/2^{n-1}$, then x is an element of $W_{1/2^{n-1}} = U_n$. If $p(x) = 1/2^{n-1}$, then we obtain that $x \in W_{1/2^{n-1}} + 1/2^{m-1} = U_n + U_m$ for all $m \in N$, hence $x \in \bigcap_{m \in N} U_n + U_m = \overline{U_n} = U_n$ from the assumption. Thus it holds that $U_n = V_n$ for all $n \in N$ and clearly $p(x)$ is an (F)-norm.

In a topological vector space $E(t)$, for an arbitrary subset A of E we denote by $B(A)$ the balanced hull of A. Clearly the subset $B(A) = \{y \mid y = \lambda x$ for all $x \in A$ and all $\lambda \in K$ with $|\lambda| < 1\}$. From this we obtain the following

LEMMA 2. *Let $E(t)$ be a topological vector space and C be a compact subset in $E(t)$. Then the balanced hull $B(C)$ of C is compact.*

Proof. Since, in $E(t)$, the bilinear map F from the product space $K \times E(t)$ onto $E(t)$ defined by $(\lambda, x) \to \lambda x$ for each $\lambda \in K$ and each $x \in E$ is continuous, if we consider the image of the compact subset $\{\lambda \mid |\lambda| \leq 1\} \times C$, we obtain the conclusion.

Using these lemmas, we prove the following

THEOREM 2. *Let $E(t)$ be a topological vector space whose strong topology t^b is a metrizable linear topology and strictly finer than t. Further, it is assumed that in $E(t)$ there exists a t^b-bounded sequence (x_n) which does not converge relative to t^b but converges to* 0 *relative to t. Then among the (F)-semi norms with $(*)$ property on E, the t^b-continuous (F)-semi-norms do not coincide with the t-lower semi-continuous (F)-semi-norms.*

Proof. In $E(t)$, let $p(x)$ be a t-lower semi-continuous (F)-norm which generates t^b and let (x_n) be a sequence which satisfies the assumptions and $p(x_n) \geq \varepsilon > 0$ for all $n \in N$ and some positive number ε. These restrictions do not lose generality from Proposition and the property of the sequence (x_n). Now we shall construct a t^b-closed string (U_n) whose knots form a base of t^b-neighbourhoods of 0 and which is not t-closed. In the first place, we set the first knot $U_1 = \{x \mid p(x) \leq d\}$ with

$\varepsilon > d > 0$. From t^b-boundedness of (x_n), there is a positive number $\alpha(1)$ such that $p(\alpha(1)x_n) \leq d/2^3$ for all $n \in N$. Further, let B_1, be the subset $\{x \mid p(x) \leq d/2^3\}$ and C_1 be the subset $\{x \mid p(x) \leq d/2^4\}$, then we take a $y(1) \in B_1 - C_1$ such that $y(1) \notin [x_n]$ for all $n \in N$, where each $[x_n]$ is the one dimensional subspace generated by x_n. (It is possible if we take a sufficiently small d.) If we set the second knot

$$U_2 = C_1 \cup B((\alpha(1)x_n + y(1))_n),$$

then U_2 has the following properties: (i) as U_2 is contained in B_1+B_1, $p(x) \leq d/2^2$ for all $x \in U_2$, (ii) U_2 is absorbing and balanced, and (iii) since C_1 is t-closed and the seqnence $(\alpha(1)x_n + y(1))$ converges to $y(1)$ relative to t, for sufficiently large n each element $\alpha(1)x_n + y(1)$ of U_2, does not belong to C_1. For the rest knots, we can determine $\alpha(i)$, B_i, C_i and $y(i)$, $i \geq 2$ and set U_{i+1} in the similar manner. Thus we can obtain the string (U_n) which generates t^b. So it is sufficient to show that each U_i is t^b-closed but not t-closed.

Since for each U_{i+1} the sequence $(\alpha(i)x_n)$ converges to 0 relative to t, from Lemma 2 the subset $C_i \cup B((\alpha(i)x_n + y(i))_n) \cup B((y(i)))$ is t-closed. Hence we show that for each $\gamma y(i) \in B((y(i)))$ and $\gamma y(i) \notin C_i$ there exists a t^b-neighbourhood S of 0 with $(\gamma y(i)+S) \cap U_i = \Phi$. If for any t^b-neighbourhood O of 0 $(\gamma y(i)+O) \cap U_i \neq \Phi$, since C_i is t-closed and each balanced hull $B((\alpha(i)x_n+y(i)))$ of $\alpha(i)x_n+y(i)$, $n \in N$ is t^b-compact, then there exists a subsequence $(\varepsilon_j(\alpha(i)x_{nj}+y(i)))$ with $0 < |\varepsilon_j| \leq 1$, $j \in N$ which converges to $\gamma y(i)$ relative to t^b and hence its scalar sequence (ε_j) converges to γ. Then as it holds that $p(\gamma(\alpha(i)x_{nj} + y(i)) - \varepsilon_j(\alpha(i)x_{nj} + y(i))) \leq p(\gamma - \varepsilon_j)\alpha(i)x_{nj}) + p((\gamma - \varepsilon_j)y(i))$ for all $j \in N$, the sequence $(\gamma(\alpha(i)x_{nj} + y(i)))$ also converges to $\gamma y(i)$ relative to t^b from t^b-boundedness of (x_n) but this fact contradicts the assumption that $p(x_n) \geq \varepsilon > 0$ for all $n \in N$. Thus each U_{i+1} is t^b-closed. Finally if we set the sequence $(\alpha(i)x_n + y(i))$ which is contained in U_{i+1}, then it converges to $y(i)$ relative to t which does not belong to U_{i+1}.

Consequently as each U_{i+1}, $i \in N$ is not t-closed, the associated (F)-norm $g(x)$ of the string (U_i) has $(*)$-property from Lemma 1 and is not t-lower semi-continuous but t^b-continuous.

Example 2. We set the sequence space $l^{1/2} = \{x = (x_n) | \sum_{n=1}^{\infty} |x_n|^{1/2} < \infty\}$ and on this space consider the following two linear topologies: one is the linear topology $t_{1/2}$ generated by the natural paranorm $\|\cdot\|_{1/2}$ with $\|x\|_{1/2} = \sum_{n=1}^{\infty} |x_n|^{1/2}$ for all $x \in l^{1/2}$ and the other is the topology of simple convergence t_s. $l^{1/2}(t_{1/2})$ is complete metrizable and has a base of neighbourhoods of 0 which consists of $U_\varepsilon = \{x \mid \|x\|_{1/2} \leq \varepsilon\}$, $\varepsilon > 0$. So $l^{1/2}(t_{1/2})$ is barrelled in L. On the other hand, since t_s is strictly coarser than $t_{1/2}$ and every U_ε is t_s-closed, we have $t_s^b = t_{1/2}$ from the above fact. If we consider the sequence (x_n) such that $x_1 = (1, 0, 0, \ldots)$, $x_2 = 0, 1, 0, \ldots)$, $\ldots$, then $l^{1/2}(t_s)$ has a sequence satisfying the conditions of Theorem 2. Hence the $t_{1/2}$-continuous (F)-semi-norms with $(*)$-property do not coincide with the t_s-lower semi-continuous ones.

Example 3. Let t_1 be the l^1-norm topology on $l^{1/2}$, then $l^{1/2}(t_1)$ is barrelled but not barrelled in L (Khaleelulla [2]). As t_1 is strictly coarser than $t_{1/2}$ but strictly finer than t_s, $t_1^b = t_{1/2}$ from the same argument as in Example 2. As in the case of Example 2 if we consider the sequence (x_n) such that $x_1 = (1, 0, 0, \dots)$, $x_2 = (1/2^2, 1/2^2, 0, 0, \dots)$, $\dots$, $x_n = (1/n^2, 1/n^2, \dots, 1/n^2, 0, 0, \dots)$, $\dots$, then in $l^{1/2}(t_1)$ the $t_{1/2}$-continuous (F)-semi-norms with $(*)$-property do not coincide with the t_1-lower semi-continuous ones.

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