PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 40 (54), 1986, pp.117–141

MIXED NORM SPACES OF ANALYTIC AND HARMONIC FUNCTIONS, I

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Abstract. For an increasing absolutely continuous function $\varphi : (0,1) \to (0,+\infty)$ we define the spaces $H(p,q,\varphi)$, $p > 0$, and $h(p,q,\varphi)$, $p \geq 1$, (of analytic and harmonic functions f on the unit disc, respectively) by the requirement that the function $r \to \varphi(1-r)M_p(r,f)$, $0 < r < 1$, belongs to $Lq(\varphi'(1-r)dr/\varphi(1-r))$. These spaces are generalizations of those considered by Shields and Williams [17, 18] and Mateljević and Pavlović [13]. If $\varphi(2t) \leq C\varphi(t)$ we construct certain equivalent norms and use them to find the duals of $H(p,q,\varphi)$ and $h(p,q,\varphi)$. In particular, we have an improvement of the main result of [18]. Our main tools are a theorem of Hardy and Littlewood on Cesaro means of power series and a new integrability theorem for power series with positive coefficients.

0. Introduction

Throughout the paper $h(U)$ is the class of all complex-valued harmonic functions on the open unit disc U, φ is an increasing absolutely continuous function on the interval $(0,1]$ with $\varphi(0+) = 0$, and $0 < q \leq \infty$. Let X be a quasi-normed space contained in $h(U)$ such that for every $f \in s(X)$ the function $r \to \|f_r\|_X$, $0 < r < 1$, is measurable, where

$$s(X) = \{f \in h(U) : f_r \in X \quad \text{for all } r \in (0,1)\},$$

while f_ξ, $|\xi| \leq 1$, is defined by $f_\xi(z) = f(\xi z)$, $|z| < 1/|\xi|$. We define

$$\|f\|_{X(q,\varphi)} = \left\{ \int_0^1 [\varphi(1-r)\|f_r\|_X]^q dm_\varphi(r) \right\}^{1/q}, \quad f \in s(X), \tag{0.1}$$

where

$$dm_\varphi(r)/dr = \varphi'(1-r)/\varphi(1-r), \quad 0 < r < 1.$$

We consider the spaces

$$X(q,\varphi) := \{f \in s(X) : \|f\|_{X(q,\varphi)} < \infty\}.$$

AMS Subject Classification (1980): Primary 46E14, 30H05.

We shall mainly be concerned with the case when X is the usual Hardy space H^p, $0 < p \leq \infty$. In this case we have $s(X) = H(U)$ and $\|f_r\|_X = M_p(r, f)$, where $H(U)$ stands for the subspace of $h(U)$ consisting of analytic functions, and

$$M_p(r,f) = \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p d\theta/2\pi \right\}^{1/p}, \quad f \in h(U).$$

The space $H^p(q,\varphi) =: H(p,q,\varphi)$, is defined in [13] by using the measure $\mathrm{dr}/(1-r)$ instead of dm_φ, but the present definition is more convenient for our purposes. Note that if $\varphi(t) = t^\alpha$ for some $\alpha > 0$, then $dm_\varphi(r) = \alpha \mathrm{dr}/(1-r)$.

The spaces $H(p,q,\alpha)$, obtained by taking $\varphi(t) = t^\alpha$, have been considered by many authors. The spaces $H(p,q,\alpha)$ are called Bergman spaces, although they were introduced by Džarbashian [3, 4]. However, many important theorems concerning $H(p,q,\alpha)$ are contalned in Hardy's and Litteewood works. See [5, 7] for information and references.

One of interesting problems is to describe the dual (and the predual if it exists) of $H(p,q,\varphi)$. The first result in this direction is due to Zakharyuta and Yudovich [19]. Using some bounded projections from $L^\varphi(U)$ they found the dual of $H(p,p,\alpha)$ with $1 < p < \infty$. In a similar way Shields and Williams [17] solved the duality problem for the space $H(p,p,\varphi)$, $1 \leq p \leq \infty$, where φ is a "normal" function. Some other methods have been used by Duren, Romberg and Shields [6], Flett [7] and Shapiro [16]. However, these methods do not work in the case $p < 1$, $q > 1$ even if $\varphi(t) = t^\alpha$.

Recently Shields and Williams [18] considered the spaces $h^\infty(\infty,\varphi)$, where

$$h^p = \{f \in h(U) : \|f\|_p < \infty\}, \quad 0 < p \leq \infty,$$
$$\|f\|_p = \sup\{M_p(r,f) : 0 < r < 1\}, \quad f \in h(U).$$

They found the predual of $h^\infty(\infty,\varphi)$ for such functions φ as $\varphi(t) = 1/\log(e/t)$, $0 < t \leq 1$. In this case $h^\infty(\infty,\varphi)$ is isomorphic to the dual of $h^1(1,\varphi)$, but this is not true if "h" is replaced by "H".

In this paper we present a new approach to the duality problem for $H(p,q,\varphi)$ which can be used whenever $0 < p$, $q < \infty$ and

$$\sup \varphi(2t)/\varphi(t) < \infty \quad (0 < t < 1/2). \tag{0.2}$$

If this condition is satisfied we say that φ is *quasi-normal.* If, in addition,

$$\sup \varphi(at)/\varphi(t) < 1 \quad (0 < t < 1) \tag{0.3}$$

for some $a > 0$ then φ is said to be *normal.*

Our method is based on an extension of the following result of Mateljević and Pavlović [13].

Theorem A. *Let $\Delta_n(z) = \sum_{j \in J_n} z^j$, where $J_0 = \{0,1\}$ and $J_n = (j : 2^n \leq j \leq 2^{n+1} - 1\}$ for $n \geq 1$. If $1 < p < \infty$ and φ is normal then*

$$\left\{ \int_0^1 [\varphi(1-r) M_p(r,f)]^q \mathrm{dr}/(1-r) \right\}^q \sim \left\{ \sum_{n=0}^\infty [\varphi(2^{-n}) \|\Delta_n * f\|_p]^q \right\}^{1/q}, f \in H(U).$$

This theorem does not hold if $p \le 1$ or if φ is not normal, and we shall use some polynomials more complicated than Δ_n. Namely, for any lacunary sequence $w = \{\lambda_n\}_0^\infty$ of positive integers and any integer $N > 0$ we shall construct a sequence $w_n = w_{n,N,\Lambda}$, $n \ge 0$, of harmonic polynomials satisyfing the following conditions:

$$f = \sum_{n=0}^{\infty} w_n * f \quad \text{for all } f \in h(U), \tag{0.4}$$

with the series uniformly converging on compact subsets of U;

$$\hat{w}_n(j) = 0 \quad \text{if } |j| \notin [\lambda_{n-1}, \lambda_n) \quad (\lambda_{-1} := 0); \tag{0.5}$$

$$\|w_n * f\|_X \le C\|f\|_X, \quad f \in X,\ n \ge 0 \tag{0.6}$$

where $X = H^p$, $p > 1/(N+1)$, and C is a positive real constant not depending on f, n.

The construction of the sequence $\{w_n\}$ will be given in Section 2. A theorem of Hardy and Littlewood [9] concerning Cesaro means of power series plays the central role in the proof of (0.6).

Our main result asserts that if $\varphi(1/\lambda_n)$ behaves like 2^{-n} then

$$\|f\|_{X(q,\varphi)} \sim \left\{ \sum_{n=0}^{\infty} [\varphi(1/\lambda_n)\|w_n * f\|_X]^q \right\}^{1/q}, \quad f \in s(X), \tag{0.7}$$

where $X = H^p$, $p > 1/(N+1)$, or $X = h^p$, $p \ge 1$. In fact, the conditions (0.6) and (0.7) are satisfied if X is an arbitrary Banach "A-space" (to be defined in Section 1). The proof of the main result is given in Sections 3, 4 and is based on a new integrability theorem for lacunary power series with positive coefficients.

Using the above properties of $\{w_n\}$ one can reduce some problems for $X(q,\varphi)$ to the analogous ones for X. In Section 5 we show how to calculate the dual of $X(q,\varphi)$ if the dual of X is known. In particular, we have a solution to the duality problem for $H(p,q,\varphi)$, $p > 0$, and for $h(p,q,\varphi) := h^p(q,\varphi)$, $p \ge 1$, where φ is an arbitrary quasi-normal function. For example, it follows from our duality theorem and the Fefferman theorem that the dual of $H(1,q,\varphi)$ is isomorphic to the space $BMOA\ (q',\varphi)$, where $BMOA$ is the space of analytic functions of bounded mean oscilation. The dual of $h(p,q,\varphi)$, $p \ge 1$, is isomorphic to $h(p',q',\varphi)$, and this generalizes Theorem 4 [18]. See also Section 6.

If φ is a normal function then the dual of $H(p,q,\varphi)$ is simpler than in the general case (Section 7). If $p \ge 1$ it is isomorphic to $H(\infty,q',\varphi)$. Furthermore, in this case the spaces $H(p,q,\varphi)$ and $H(1,q,\psi)$, where, $\psi(t) = \varphi(t)t^{1/p-1}$ have the same dual.

In Part II we shall consider the coefficient multipliers from $X(q,\varphi)$ to $Y(q,\psi)$. For example, we shall prove that if $\psi(t) = \chi(\varphi(t))$, where χ is normal, then $H(p,q,\varphi)$ and $H(p,q,\psi)$ are isomorphic via a multiplier transform; this generalizes the well-known theorem of Hardy and Littlewood on fractional integration and differentiation in $H(q,p,\alpha)$. We shall also present a solution to a problem of Shields and Williams [18, Problem B].

1. A-Spaces

Let $h(U_R)$ denote the class of all harmonic functions on the disc $U_R = \{z : |z| < R\}$, $R > 0$. For a non-empty set E of integers let

$$h_E(U_R) = \{f \in h(U_R) : \operatorname{supp}(\hat{f}) \subset E\},$$

where $\hat{f}$ is uniquely determined by

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}, \quad z = re^{i\theta} \in U_R.$$

A quasi-normed space X is said to be an A-space if there exists a set E such that the following conditions hold.

I. $X \subset h_E(U)$, the inclusion being continuous.

II. $h_E(U_R) \subset X$ for all $R > 1$. More precisely, the restriction map $f \to f|_U$ is a continuous operator from $h_E(U_R)$ to X.

III. If $f \in X$ then $f_\xi \in X$ and

$$\|f_\xi\| \leq \|f\| \ \text{ for } \ |\xi| \leq 1.$$

We suppose $h_E(U_R)$ is endowed with the topology of uniform convergence on compact subsets of U_R. Recall that a quasi-norm $\|.\|$ on a linear space X is characterized by the following properties: 1. $f = 0$ if $\|f\| = 0$; 2. $\|\lambda f\| = |\lambda| \cdot \|f\|$; 3. $\|f + g\| \leq C\|f\| + C\|g\|$, where $C < \infty$ is independent of $f, g \in X$. Note also that if X satisfies Conditions I, II, then

$$s(X) = h_E(U).$$

It is hard to verify that H^p and h^p, $0 < p \leq \infty$, are A-spaces. Many classical sequence spaces may be regarded as being A-spaces. For example, the space l^p, $0 < p \leq \infty$, may be identified with the class of those $f \in H(U)$ for which

$$|f|_p := \left\{ \sum_{n=0}^{\infty} |\hat{f}(n)^p \right\}^{1/p} \leq \infty.$$

Further examples are Hardy-Orlicz spaces [11] and Orlicz sequence spaces.

Let $\mathcal{P}$ be the set of all harmonic polynomials (i.e. of those $f \in h(U)$ for which the support of $\hat{f}$ is finite). It follows from II that $\mathcal{P} \cap h_E(U) \subset X$. We denote by X° the closure $\mathcal{P} \cap h_E(U)$ in X.

Proposition 1.1. *For an A-space X the following assertions hold.*

(a) *X° is an A-space and $X(q,\varphi) = X^\circ(q,\varphi)$.*

(b) *$X(q,\varphi)$ is a complete A-space.*

(c) *If $f \in X(q,\varphi)$, $q < \infty$, then $\|f_\xi - f\|_{X(q,\varphi)} \to 0$ as $\xi \to 1$ ($|\xi| \le 1$).*

(d) *If $q < \infty$ then $X(q,\varphi)^\circ = X(q,\varphi)$.*

Proof. The proof of (a) is straightforward. It is also easy to see that $Y := X(q,\varphi)$ is a quasi-normed space satisfying Conaition III. Let $f \in h_E(U_R)$, $R > 1$. Then $f \in X$ and $\|f_r\|_X \le \|f\|_X (0 < r < 1)$. Using this and (0.1) we obtain $\|f\|_Y \le \|f\|_X q^{-1/q}\varphi(1)$ where $q^{-1/q} := 1$ for $q = \infty$. This implies $h_E(U_R) \subset Y$ with the continuous inclusion. On the other hand, if $f \in Y$ and $0 < \varrho < 1$ then $f_\varrho \in h_E(U_{1/\varrho}) \subset X$. Since $\|f_r\|_X$, $0 < r < 1$, is a non-decreasing function (by III) we have

$$\|f\|_Y \ge \left\{ \int_\varrho^1 [\varphi(1-r)\|f_\varrho\|_X]^q dm_\varphi(r) \right\}^{1/q} = \|f_\varrho\|_X q^{-1/q}[\varphi(1) - \varphi(\varrho)].$$

This implies $Y \subset h_E(U)$.

To prove that Y is complete let $\{f^n\}_0^\infty$ be a Cauchy sequence in $X = X(q,\varphi)$. In view of Conditions I, II and the completeness of $h_E(U)$, there exists $f \in h_E(U)$ such that $\|f^n - f_r\| \to 0$ as $n \to \infty$, for all r, $0 < r < 1$. Using Fatou's lemma we obtain $\|f^n - f\|_Y \le \lim_n \inf \|f^m - f^n\|_Y$, and this concludes the proof of (b).

For the proof of (c) observe that, whenever $0 < r < 1$, f_{ξ_r} tends to f_r in the topology of $h_E(U_{1/r})$ as $\xi \to 1$ ($|z| \le 1$). Hence, by II, $\|f_{\xi_r} - f_r\| \to 0$ as $\xi \to 1$. On the other hand, $\|f_{\xi_r} - f_r\|_\lambda \le C\|f_{\xi_r}\|_X + C\|f_r\|_X \le 2C\|f_r\|_X$, where $C < \infty$ is independent of r. Now the required result follows from the Lebesgue dominated convergence theorem.

The assertion (d) is a consequence of (c) and the following proposition.

PROPOSITION 1.2. *Let $f \in Y$, where Y is an A-space. Then the following statements are equivalent:* (i) *$f \in Y^\circ$;* (ii) *$\|f_\xi - f\| \to 0$ as $\xi \to 1$ ($\xi \in U$):* (iii) *$\|f_r - f\| \to 0$ as $r \to 1^-$.*

Proof. (iii) $\Rightarrow$ (i). Let $\varepsilon \to 0$ and choose r so that $\|f_r - f\| < \varepsilon$. Since $f_r \in Y^\circ$ (Proposition 1.1 (a)) there exists $g \in \mathcal{P} \cap Y$ such that $\|g - f_r\| < \varepsilon$. Hence $\|f - g\| \le C\|f - f_r\| + C\|f_r - g\| \le 2C\varepsilon$, where C is independent of ε.

(i) $\Rightarrow$ (ii). It is easily verified that (ii) holds for $f \in \mathcal{P}$. Let $f \in Y$, let $\varepsilon > 0$ and choose $g \in \mathcal{P} \cap Y$ so that $\|f - g\| < \varepsilon$. Since $\|g_\xi - f_\xi\| = \|(g-f)_\xi\| \le \|g - f\|$ we get

$$\|f - f_\xi\| \le 2C\varepsilon + C\|g - g_\xi\|, \quad |\xi| \le 1.$$

Finally, we choose $b > 0$ so that $|\xi - 1| < \delta$ implies $\|g - g_\xi\| < \varepsilon$. This concludes the proof because the implication (ii) $\Rightarrow$ (iii) is clear.

Next we consider some properties of dual spaces. The dual X' of a quasi-normed space X is the linear space of all bounded linear functionals on X, and is endowed with the norm

$$\|L\| = \sup\{|L(f)| : f \in X, \ \|f\|_X \le 1\}.$$

Let $X, Y \in h(U)$ be quasi-normed spaces. A function $g \in h(U)$ is said to be a multiplier from X to Y if the map $f \to f * g$ acts as a bounded linear operator from X to Y. If X, Y are A-spaces then

$$(X \to Y) := \{g \in s(X) \cap s(Y) : g \text{ is a multiplier from } X \text{ to } Y\}$$

is a quasi-normed space with the quasi-norm

$$\|g\|_{X,Y} = \sup\{\|f * g\|_Y : f \in X, \|f\|_X \leq 1\}.$$

Here $f * g$ stands for the convolution of $f, g \in h(U)$:

$$f * g(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(g)r^{|n|_e in\theta}, \quad z = re^{i\theta} \in U.$$

For an A-space X let $X^* = (X \to h^\infty)$. Let $h(\bar{U})$ be the subspace of h^∞ consisting of those functions which are continuous on the closed unit disc $\bar{U}$.

Proposition 1.3. *Let X be a complete A-space such that $X = X^\circ$. Then the following assertions hold.*

(a) *If $L \in X'$ then there exists a unique function $g \in (X \to h(\bar{U}))$ such that $L(f) = f * g(1)$, $f \in X$.*

(b) $X^* = (X \to h(\bar{U}))$.

(c) *If $L(f) = f * g(1)$, $f \in X$, where $g \in (X \to h(\bar{U}))$, then $L \in X'$ and $\|L\| = \|g\|_{X^*}$.*

Proof. (a) Let $s(X) = h_E(U)$ and define $h_n \in X$ by $\hat{h}_n = 0$ if $n \notin E$, and $h_n(re^{i\theta}) = r^{|n|}e^{in\theta}$ if $n \in E$. If $L \in X'$ define $g \in h_E(U)$ by $\hat{g}(n) = L(h_n)$. Let $\xi = \rho e^{it} \in U$. Since $f_\xi \in h_E(U_{1/|\xi|}) \subset X$ the series $\sum_{-\infty}^{\infty} \hat{f}(n)h_n\varrho^{|n|}e^{int}$ is convergent in X, whence $f * g(\xi) = L(f_\xi)$, for all $f \in X$. On the other hand, by Proposition 1.2 and the assumption $X = X^\circ$, $L(f_\xi) \to L(f_a)$ as $\xi \to a$, for all $a \in \bar{U}$. This implies $f * g \in h(\bar{U})$ (for all $f \in X$) and $L(f) = f * g(1)$. The function g belongs to $(X \to h(\bar{U}))$ because $|f * g(\xi)| = |L(f_\xi)| \leq \|L\|\|f_{|xi}\| = \|L\|\|f\|$ for $\xi \in U$. The uniqueness of g is obvious.

(b) Let $g \in X^*$ and define L_ξ, $|\xi| < 1$, by $L_\xi(f) = f * g(\xi)$, $f \in X$. Then $\{L_\xi : |\xi| < 1\}$ is a bounded subset of X'. On the other hand, if $f \in \mathcal{P} \cap X$ and $|a| = 1$ then the limit of $L_\xi(f)$ as $\xi \to a$ exists, because $f * g \in \mathcal{P}$. Since $\mathcal{P} \cap X$ is dense in X and X is complete we conclude that the above limit exists for all $f \in X$. Therefore g belongs to $(X \to h(\bar{U}))$. The inclusion $(X \to h(\bar{U})) \subset X^*$ is obvious.

The proof of (c) is straingtforward.

Since $(X^\circ)^* = X^*$ we have the following consequence of Proposition 1.3.

Proposition 1.4. *If X is a Banach A-space then*

$$\|f\|_X = \sup\{\|f * g\|_\infty : g \in X^*, \|g\|_{X^*} \leq 1\}, \quad f \in X^\circ.$$

As an application we have

PROPOSITION 1.5. *Let X be a Banach A-space and let g be a harmonic polynomial. Then $\|f * g\|_X \leq \|f\|_X \|g\|_1 \quad f \in X$.*

Proof. Since $f * g$ is a polynomial we may apply Proposition 1.4 to obtain

$$\|f_r * g\|_X = \sup\{\|f_r * g * h\|_\infty : \|h\|_{X^*} \leq 1\}, \quad 0 < r < 1.$$

It is well-known and easy to see that $\|(f_r * h) * g\|_\infty \leq \|f_r * h\|_\infty \|g\|_1$. Hence $\|f_r * g\|_X \leq \|f_r\|_X \|g\|_1 \leq \|f\|_X \|g\|_1$. Now the desired result is verified because $\|f_r * g\|_X = \|(f * g)_r\|_X \to \|f * g\|_X \quad (r \to 1^-)$.

2. On the sequence $\{w_n\}$

In what follows $A = \{\lambda_n\}_0^\infty$ is a lacunary sequence of positive integers $(\lambda_{n+1}/\lambda_n \geq c > 1)$ and N is a positive integer. We shall construct a sequence $W = \{w_n\}_0^\infty$ satisfying the conditions mentioned in Introduction. In fact, we can do somewhat more: this sequence satisfies

$$\left\|\sum_{j=0}^{n} w_j * f\right\|_X \leq C\|f\|_X, \quad f \in X, \ n \geq 0. \tag{2.1}$$

The letter "C" denotes an arbitrary positive real constant which need not be the same on each occurrence.

It is clear that (2.1) implies (0.6). And it is a consequence of (0.4) and (2.1) that

$$\left\|f - \sum_{j=0}^{n} w_j * f\right\|_X \to 0 \quad (n \to \infty) \quad \text{for} \quad f \in X^\circ.$$

THEOREM 2.1. *There exists a sequence $W = \{w_n\}_0^\infty$ satisfying the conditions (0.4), (0.5) and (2.1) for all Banach A-spaces X and for all $X \in \{H^p : p > 1/(N+1)\}$.*

In the case of H^p spaces the proof depends on the following theorem of Hardy and Littlewood [9]. An elegant elemetary proof is found by Colzani [2].

THEOREM HL. *If $X = H^p$, $p > 1/(N+1)$, then*

$$\|\sigma_n^N f\|_X \leq C\|f\|_X, \quad f \in X, \quad n \geq 0. \tag{2.2}$$

Here $\sigma_n^N f$, $n = 0, 1, 2, \ldots$, are the $(C.N)$ means of the series $\sum \hat{f}(k) r^{|k|} e^{ik\theta}$. They are defined for all $f \in h(U)$ by $\sigma_n^N f = K_n * K_{n+1} * \cdots * K_{n+N-1} * f$, where

$$K_n(re^{i\theta}) = \sum_{s=-n}^{n} \left(1 - \frac{|s|}{n+1}\right) r^{|s|} e^{is\theta}.$$

Using Proposition 1.5 and the well-known fact that $\|K_n\|_1 = 1$ we obtain the following variant of Fejer's theorem.

THEOREM 2.2. *If X is a Banach A-space then the condition* (2.2) *with $C = 1$ holds.*

For our purposes it is convenient to replace $\sigma_n^N f$ by $K_{n,N} * f$, where

$$K_{n,N}(re^{i\theta}) = \sum_{s=-n}^{n} \left(1 - \frac{|s|}{n+1}\right)^N r^{|s|} e^{is\theta}.$$

LEMMA 2.1. *If an A-space X satisfies* (2.2) *then*

$$\|K_{n,N} * f\|_X \leq C\|f\|_X, \quad f \in X, \ n \geq 0.$$

Proof. It is clear that there exist $A_{j,n}$ $(n \geq 0, \ 1 \leq j \leq N)$ such that

$$\left(1 - \frac{t}{n+1}\right)^N = \sum_{j=1}^{N} A_{j,n} \prod_{s=1}^{j} \left(1 - \frac{t}{n+s}\right)$$

for all real t. Taking $t = n+2, \dots, n+N$ we see that $|A_{j,n}| \leq C(n+1)^{j-N}$, where C depends only on N. This implies

$$\|K_{n,N} * f\|_X = \left\| \sum_{j=1}^{N} A_{j,n} \sigma_n^j f \right\|_x \leq C \sum_{j=1}^{N} (n+1)^{j-N} \|\sigma_n^j f\|_X, \quad f \in X,$$

where C depends only on N. This concludes the proof.

Proof of Theorem 2.1. For a fixed $n \geq 0$ define the matrix $B_n(t) = \{b_{mj}(t)\}$ $(0 \leq m, \ j \leq N)$ by

$$b_{mj}(t) = \begin{cases} \lambda_{n+m}^j, & 0 \leq j \leq N-1, \\ \lambda_{n+m}^N \xi_{n+m}(t), & j = N, \end{cases}$$

where

$$\xi_s(t) = \begin{cases} (1 - t/\lambda_s)^N, & 0 \leq t \leq \lambda_s \\ 0, & t \geq \lambda_s. \end{cases}$$

The function

$$\eta_n(t) := \det B_n(t) / \det B_n(0), \quad t \geq 0.$$

satisfies the condition

$$\eta_n(t) = 0 \quad \text{for} \quad t \geq \lambda_{n+N}, \quad \text{and} \quad \eta_n(t) = 1 \ \text{for} \ 0 \leq t \leq \lambda_n. \tag{2.3}$$

The first equality holds because $x_{n+m}(t) = 0$ for all $t \geq \lambda_{n+N}$ and $0 \leq m \leq N$. To prove the rest of (2.3) observe that the function η_n coincides on the segment $[0, \lambda_n]$ with a polynomial of degree $\leq N$. Thus, it suffices to show that $d^k \eta_n(0)/dt^k = 0$ for $1 \leq k \leq N$. And this is immediate from the identity

$$d^k b_{mN}(0)/dt^k = k! \binom{N}{k} (-1)^k \lambda_{n+m}^{N-k}$$

by using the usual differentiation rule for determinants.

Now we define the sequence $\{w_n\}$ by $\hat{w}_0(k) = \eta_0(|k|)$ and

$$\hat{w}_n(k) = \eta_n(|k|) - \eta_{n-1}(|k|), \quad n \ge 1, \quad -\infty < k < \infty.$$

It follows from (2.3) that the condition (0.5) is satisfied. We also have

$$\sum_{n=1}^{\infty} \hat{w}_n(k) = 1, \quad -\infty < k < \infty. \tag{2.4}$$

If f is a harmonic polynomial then (0.4) holds because of (2.4) and the fact that $w_n * f = 0$ for sufficiently large n. The general case of (0.4) is now obtained by using (2.1), $X = h^\infty$. It remains to verity the validity of (2.1).

LEMMA 2.2. *We have* $\eta_n(t) = \sum_{m=0}^{N} D_{n,m}\xi_{k+m}(t)$, $t \ge 0$, *where:* $\sup_{n,m} |D_{n,m}| \le \infty$.

Proof. For a finite sequence $x_0, x_1, \dots, x_s$ let $V(x_0, \dots, x_s)$ be the determinant of the matrix $\{x_m^j\}_{m,j=0}^s$ (Vandermonde's determinant). It is easily seen that the desired identity holds with

$$|D_{n,0}| = \lambda_n^N V(\lambda_{n+1}, \dots, \lambda_{n+N})/V_n, \ \ |D_{n,N}| = \lambda_{n+N}^N V(\lambda_n, \dots, \lambda_{n+N-1})/V_n,$$

and, for $1 \le m \le N-1$,

$$|D_{n,m}| = \lambda_{n+m}^N V(\lambda_n, \dots, \lambda_{n+m+1}, \lambda_{n+m-1}, \dots, \lambda_{n+N})/V_n,$$

where $V_n = V(\lambda_n, \dots, \lambda_{n+N})$. Using the formula

$$V(x_0, \dots, x_s) = \prod_{j,m}(x_m - x_j) \quad (0 \le j < m \le s)$$

we find

$$|D_{n,m}|^{-1} = \lambda_{n+m}^{-N} \prod_{\substack{j=0 \\ j \ne m}}^{N} |\lambda_{n+j} - \lambda_{n+m}|, \quad 0 \le m \le N.$$

Since the sequence $\{\lambda_n\}$ is lacunary there exists $c > 0$ such that

$$|\lambda_{n+j} - \lambda_{n+m}| \ge c\lambda_{m+m}$$

for all $j, m, n \ge 0$, $j \ne m$. This implies $|D_{n,m}|^{-1} \ge c^N$.

We return to the proof of Theorem 2.1. Since

$$\sum_{j=0}^{n} \hat{w}_j(k) = \eta_n(|k|) \quad \text{and} \quad \xi_{n+m}(|k|) = \hat{K}_{\lambda_{n+m}-1,N}(k)$$

we have

$$\sum_{j=0}^{n} w_j * f = \sum_{m=0}^{N} D_{n,m} K_{\lambda_{n+m}-1,N} * f.$$

Using Theorems HL and 2.2 and Lemmas 2.1 and 2.2 we obtain

$$\left\| \sum_{j=0}^{n} w_j * f \right\|_X \le C \sum_{m=0}^{N} D_{n,m} \|f\|_X,$$

where C is independent of $f \in X$ and $n \ge 0$. This concludes the proof of the theorem.

3. Quasi-normal functions

For the sake of convenience we suppose that all quasi-normal functions under discussion are defined, increasing and absolutely continous on $(0, \infty)$ and

$$\varphi(t)\varphi(1/t) \sim 1, \quad t > 0. \tag{3.1}$$

For non-negative functions F, G we write $F(s) \sim G(s)$, $s \in S$, if there is $C(0 < C < \infty)$ such that $G(s)/C \le F(s) \le CG(s)$ for all $s \in S$. If the condition (0.2) is satisfied then

$$\varphi(at) \sim \varphi(t), \quad t > 0, \tag{3.2}$$

for all $a > 0$. Furthermore, there are C, $\alpha > 0$ such that

$$\varphi(tu) \le Cu^{\alpha}\varphi(t), \quad u > 1, \quad t > 0. \tag{3.3}$$

The following proposition is an immediate consequence of [10], Theorem II.1.1.

PROPOSITION 3.1. If (3.2) *holds then there exists a concave function* φ_0 *on* $(0, \infty)$ *such that* $\varphi(t) \sim \varphi_0(t)^{\alpha}$, $t > 0$.

An increasing sequence $\{A_n\}_{n=0}^{\infty}$ of positive real numbers is said to be *normal* if there are positive constants C, c such that

$$C^{-1}(1+c)^j \le A_{n+j}/A_n \quad \text{and} \quad A_{n+1}/A_n \le C, \quad n, j \ge 0. \tag{3.4}$$

This is equivalent to the requirement that $1 + c \le A_{n+m}/A_n \le C$, $n \ge 0$, for some C, c and some integer $m > 0$.

PROPOSITION 3.2. *Let* $\{A_n\}_0^{\infty}$ *be a normal sequence and let* φ *be a quasi-normal function. Then there exists a lacunary sequence* $\{\lambda_n\}_0^{\infty}$ *of positive integers such that* $\varphi(\lambda_n) \sim A_n$, $n \ge 0$.

Proof. Let $B_n = (1+c)^n \sup\{A_j(1+c)^{-j} : 0 \le j \le n\}$, $n \ge 0$, where c satisfies (3.4). Then $B_n \sim A_n$, $n \ge 0$, and $B_{n+1}/B_n \ge 1 + c$. Define t_n by $\varphi_0(t_n) = B_n^{1/\alpha}$ $n \ge 0$, where φ_0 satisfies the conclusion of Proposition 3.1. We have

$$\varphi_0(t_{n+1}) = \varphi_0(t_{n+1}t_n^{-1}t_n) \le t_{n+1}t_n^{-1}\varphi(t_n)$$

because φ_0 is concave, $\varphi_0(0+) = 0$ and $t_{n+1}/t_n > 1$. Hence $t_{n+1}/t_n > (1+c)^{\alpha}$ and consequently there is an integer n_0 such that, for every $n \ge n_0$, the set $[t_n, t_{n+1})$

contains at least two integers. We define λ_n, $n \geq 0$, to be the smallest integer in $[t_{n+n_0}, t_{n+n_0+1})$. Since $t_{n+n_0} \leq \lambda_n \leq 1 + t_{n+n_0}$ we have $\lambda_{n+1}/\lambda_n \leq t_{n+n_0+1}/(1+t_{n+n_0})$, and this implies that $\{\lambda_n\}$ is lacunary. The relation $\varphi(\lambda_n) \sim A_n$ is obvious.

From now on we shall assume that $A := \{\lambda_n\}$ is a lacunary sequence of positive integers such that the sequence $\{\varphi(\lambda_n)\}$ is normal. (One can show that the existence of such a sequence is equivalent to the condition (0.2).)

THEOREM 3.1. *Let $\{t_n\}_0^\infty\}$ be an increasing sequence of positive numbers such*

$$\varphi(t_{n+j})/\varphi(t_n) \geq C^{-1}(1+c)^j, \quad n,\, j \geq 0. \tag{3.5}$$

Let $F(r) = \varphi(1-r)\sup_{n\geq 0} a_n r^{t_n}$ or $\varphi(1-r)\sum_{n=1}^{\infty} a_n r^{t_n}$, $0 < r < 1$, where $a_n \geq 0$ for all n. Then

$$C^{-1}\|\{\varphi(1/t_n)a_n\}\|_{l^q} \leq \|F\|_{L^{q(m_\varphi)}} \leq C\|\varphi(1/t_n)a_n\}\|_{l^q}, \tag{3.6}$$

where C is independent of $\{a_n\}$.

The case when φ is normal, $t_n = 2^n$ and $dm\,(r) = dr\,/(1-r)$ is discussed in [13]. Here we proceed in a similar way. Note that

$$dm_\varphi(r) = \varphi'(1-r)dr\,/\varphi(1-r).$$

LEMMA 3.1. *Let $\{t_n\}$ be as in Theorem 3.1. and $0 < s,\, \beta < \infty$. Then*

$$\sum_{n=0}^{\infty} \varphi(t_n)^\beta r^{st_n} \leq C\varphi(1-r)^{-\beta}, \quad 0 < r < 1.$$

Proof. Since φ^β and $\{st_n\}$ have the same properties as φ and $\{t_n\}$ the lemma reduces to the case $\beta = s = 1$. First we prove that

$$\sum_{n=0}^{\infty} \varphi(\lambda_n) r^{\lambda_{n-1}} \leq C(1-r)^{-1}, \qquad 0 < r < 1, \tag{3.7}$$

where $\lambda_{-1} = 0$. We may suppose $r = 1 - 1/\lambda_m$ for some m. Then

$$\sum_{n=0}^{\infty} \varphi(\lambda_n) r^{\lambda_{n-1}} \leq \sum_{n=0}^{m} \varphi(\lambda_n) + \sum_{n=m+1}^{\infty} \varphi(\lambda_n) e^{-\lambda_{n-1}/\lambda_m}.$$

Since $\{\varphi(\lambda_n)\}$ is normal we have $\varphi(\lambda_n) \leq C(1+c)^{n-m}\varphi(\lambda_m)$, $0 \leq n \leq m$, and therefore

$$\sum_{n=0}^{m} \varphi(\lambda_n) \leq C\varphi(\lambda_m) = C\varphi(1/(1-r)) \leq C\varphi(1-r)^{-1}.$$

Similarly, using the inequalities $\varphi(\lambda_n)/\varphi(\lambda_m) \leq C^{n-m}$ and $\lambda_{n-1} \geq (1+c)^{n-m-1}$ λ_m, $n \geq m+1$, we obtain

$$\sum_{n=m+1}^{\infty} \varphi(\lambda_n) e^{-\lambda_{n-1}/\lambda_m} \leq \varphi(\lambda_m) \sum_{n=m+1}^{\infty} C^{n-m} e^{-(1+c)^{n-m-1}}$$

$$= C\varphi(\lambda_m) \sum_{n=0}^{\infty} C^n e^{-(1+c)^n} \leq C\varphi(1-r)^{-1}.$$

This completes the proof of (3.7).

Suppose now that $\{t_n\}$ satisfies (3.5). Since $\varphi(\lambda_{k+1})/\varphi(\lambda_k) \le C$, $k \ge 0$, there exists $j > 0$ such that

$$\varphi(t_{n+j})/\varphi(t_n) \ge \varphi(\lambda_{k+1})/\varphi(\lambda_k), \quad n,\, k \ge 0.$$

This implies that there is $J > 0$ such that, for every $k > 0$, the set $E_k := \{n \ge 0 : \lambda_{n-1} \le t_n < \lambda_k\}$ contains at most J elements. (In fact, $J = \max\{j, \operatorname{card} E_0\}$.) Hence

$$\sum_{n=0}^{\infty}(t_n)r^{t_n} = \sum_{k=0}^{\infty}\sum_{n\in E_k}\varphi(t_n)r^{t_n} \le J\sum_{k=1}^{\infty}\varphi(\lambda_k)r^{\lambda_{k-1}} \le C\varphi(1-r)^{-1}.$$

(If $E_k = \emptyset$ we put $\sum_{E_k} = 0$). This concludes the proof of Lemma 3.1.

Lemma 3.2. *If ψ is a quasi-normal function then*

$$\int_0^1 \psi'(1-r)r^x\,dr \sim \psi(1/x), \qquad x \ge 1.$$

Proof. It is proved in [13] that if ψ_0 is normal then

$$\int_0^1 \psi_0(1-r)(1-r)^{-1}r^{x-1}dr \sim \psi_0(1/x), \quad x \ge 1.$$

The function $\psi_0(t) := t\psi(t)$ is normal and therefore

$$\int_0^1 \psi'(1-r)r^x\,dr = x\int_0^1 \psi(1-r)r^{x-1}dr \sim x\psi_0(1/x) = \psi(1/x).$$

Proof of Theorem 3.1. Consider first the case $q < \infty$. Let $s_n = t_n/2$. Then, by means of Lemma 3.1,

$$\begin{aligned} F(r)/\varphi(1-r) &\le \sum_{n=1}^{\infty}\varphi(t_n)^{1/2}r^{s_n} \\ &\le \sup_{n\ge 0} a_n\varphi(t_n)^{-1/2}r^{s_n}\sum_{n=0}^{\infty}\varphi(t_n)^{1/2}r^{s_n} \\ &\le C\sup_{n\ge 0} a_n\varphi(t_n)^{-1/2}r^{s_n}\varphi(1-r)^{-1/2}, \quad 0 < r < 1. \end{aligned}$$

Hence

$$F(r)^q \le C\varphi(1-r)^{q/2}\sup_{n\ge 0} a_n^q\varphi(t_n)^{-q/2}r^{qs_n} \le C\varphi(1-r)^{q/2}\sum_{n=1}^{\infty}a_n^q\varphi(t_n)^{-q/2}r^{qs_n}.$$

Integration yields

$$\int_0^1 F(r)^q\,dm_\varphi(r) \le C\sum_{n=0}^{\infty}a_n^q\varphi(t_n)^{-q/2}\int_0^1 \psi'(1-r)r^{qs_n}\,dr.$$

where $\psi(t) = \varphi(t)^{q/2}$. Using Lemma 3.2, (3.2) and (3.1) we obtain the right hand side inequality in (3.6).

On the other hand, by Lemma 3.1,

$$\int_0^1 F(r)^q dm_\varphi(r) \geq C \sum_{n=1}^{\infty} \varphi(t_u) \int_0^1 F(r)^q \varphi'(1-r) r^{t_n} dr.$$

Hence, by the inequality $F(r) \geq \varphi(1-r) a_n r^{t_n}$,

$$\int_0^1 F(r)^q dm_\psi(r) \geq C \sum_{n=0}^{\infty} \varphi(t_n) a_n^q \int_0^1 \psi_0'(1-r) r^{(q+1)t_n} dr,$$

where $\psi_0(t) = \varphi(t)^{q+1}$. The proof is finished using Lemma 3.2.

If $q = \infty$ let $A = \sup_{n \geq 0} a_n \varphi(1/t_n) < \infty$. Then

$$F(r) \leq C\varphi(1-r) \sum_{n=1}^{\infty} \varphi(t_n) A r^{t_n} \leq CA,$$

where Lemma 3.1 has been used.

THe rest follows from the inequality

$$F(1 - 1/t_n) \geq \varphi(1/t_n) a_n (1 - 1/t_n)^{t_n} \quad (t_n > 1).$$

As a consequence of Theorem 3.1 we have the following integrability theorem for power series with positive coefficients.

THEOREM 3.2. *Let $J_0 = [0, \lambda_0)$ and $J_n = [\lambda_{n+1}, \lambda_n)$, $n \geq 1$. The function $F(r) = \varphi(1-r) \sum_{n=0}^{\infty} a_n r^n$, $0 < r < 1$, where $a_n \geq 0$, belongs to the space $L^q(m_\varphi)$ if and only if*

$$\Big\{\varphi(1/\lambda_n) \sum_{J_n} a_k\Big\}_{n=0}^{\infty} \in l^q.$$

The case when $\varphi(t) = t^\alpha$ and $\lambda_n = 2^n$ is known [12].

4. Decomposition of $X(q, \varphi)$

The following theorem and Theorem 2.1 provide a sort of finite dimensional decomposition of $X(q, \varphi)$.

THEOREM 4.1. *Let X be a Banach A-space or $X = H^p$, $p > 0$, let be λ a quasi-normal function, and let $\{w_n\}_n^\infty$ satisfy (0.4), (0.5) and (0.6), where $N \geq 0$ and the sequence $\{\lambda_n\}_0^\infty$ is chosen so that $\{\varphi(\lambda_n)\}_0^\infty$ is normal. Then (0.7) holds.*

We shall deduce Theorem 4.1 from Theorem 3.1. For $g \in h(U)$ let g^j be defined by $g^\circ(z) = g(O)$ and if $j \geq 1$

$$\hat{g^j}(z) = \hat{g}(j) z^j + g(-j) \bar{z^j}.$$

Observe that $g(z) = \sum_0^\infty g^j(z), \quad z \in U.$

LEMMA 4.1. *Let* $g = \sum_m^n g^j \in X$ $(0 \le m < n)$, *where* X *is as in Theorem* 4.1. *Then*

$$3^{-1} r^{2n} \|g\|_X \le \|g_r\|_X \le 2r^{m/2} \|g\|_X, \quad 0 < r < 1.$$

Proof. For the case $X = H^p$ see [13]. Let X be a Banach space. After two summations by parts we find

$$g_r = \sum_0^\infty r^j g^j = (1-r)^2 \sum_0^\infty r^j (j+1) \sigma_j^1 g.$$

Taking into account that $\sigma_j^1 g = 0$ for $j < m$ and using the inequality $\|\sigma_j^1 g\| \le \|g\|$ (Theorem 2.2) we get

$$\|g_r\| \le (1-r)^2 \sum_m^\infty r^j (j+1) \|g\| = r^m (1 + m(1+r)).$$

Using the elementary inequality $m(1-r)+1 \le 2r^{-m/2}$ we prove half of the lemma.

To prove the rest let $R = 1/r > 1$ and $f = g_r$. Then two summations by parts give

$$g = \sum_0^n R^j f_j = \sum_0^{n-1} (R^j + R^{j+2} - 2R^{j+1})(j+1)\sigma_j^1 f + (R^n - R^{n+1}) n \sigma_{n-1}^1 f + R^n f.$$

Hence, by Theorem 2.2,

$$\|g\| \le (R-1)^2 \sum_0^{n-1} R^j (j+1) \|f\| + (R-1) R^n n \|f\| + R^n \|f\|.$$

Finally, we use the inequalities $n(R-1) \le R^n - 1 \le R^n$ and

$$\sum_0^{n-1} R^j (j+1) \le n \sum_0^{n-1} R^J = n(R^n - 1)(R-1)^{-1} nR \le nR^n (R-1)^{-1}.$$

We obtain $\|g\| \le 3R^{2n} \|f\|$, and this concludes the proof.

Remmark. If X is a Banach space of analytic functions then

$$r^n \|g\| \le \|g_r\| \le r^m \|g\|.$$

This follows from the special case $X = H^\infty$ [13] by using Proposition 1.4.

LEMMA 4.2. *Let* X *and* $\{w_n\}$ *be as in Theorem* 4.1. *Let* $\beta = p$ *if* $X = H^p$, $p < 1$, *and* $\beta = 1$ *if* X *is a Banach space. Then there exists a constant* $c > 0$ *such that*

$$c \sup_{n \ge 0} \|w_n * f\|_X r^{2\lambda_{n+N}} \le \|f_r\|_X \le 2 \left\{ \sum_{n=0}^\infty \|w_n * f\|_X^\beta r^{\beta \lambda_{n-1}/2} \right\}^{1/\beta}$$

for all $f \in s(X)$ $(= h_E(U)$ *for some* $E)$ *and* $0 < r < 1$.

Here we put $\lambda_{n-1} = 0$.

Proof. The first inequality followsfrom (0.6) and Lemma 4.1, because $g := w_n * f$ is of the form

$$g = \sum_{\lambda_{n-1}}^{\lambda_{n+N}} g^j.$$

on the other hand, it follows from (0.4) and the triangle inequality for $\|\cdot\|_X^\beta$ that

$$\|f_r\|_X^\beta \le \sum_{n=0}^{\infty} \|w_n * f_r\|_X^\beta.$$

Applying Lemma 4.1 we conclude the proof.

Proof of Theorem 4.1. The part "$\|f\| \ge \ldots$" of (0.7) follows immediately from Lemma 4.2, Theorem 3.1 ($t_n = 2\lambda_{n+N}$) and the relation $\varphi(2/\lambda_{n+N}) \sim \varphi(1/\lambda_n)$, $n \ge 0$. To prove the rest we put $s = q/\beta$ (where β is as in Lemma 4.2), $a_n = \|w_n * f\|_X$, $\psi(t)^\beta = \varphi(t)$ and

$$F(r) = \psi(1-r) \sum_{n=0}^{\infty} a_n r^{\beta\lambda_{n-1}/2}, \quad 0 < r < 1.$$

By Lemma 4.3

$$2^{-\beta}\|f\|_{X(q,\varphi)}^\beta \le \|F\|_{L^s(m_\varphi)} = \beta^{-1/s}\|F\|_{L^s(m)_\psi}.$$

Now we desired result is easily deduced from Theorem 3.1 (with ψ, s instead of φ, q).

As an application of Theorem 4.1 we have a generalization of Theorem A. Let $\{S_n\}_0^\infty$ be the unique sequence $\{w_n\}_0^\infty$ satisfying (0.4) and (0.5) with $N = 0$. We have

$$\hat{S}_n(j) = \begin{cases} 1, & \lambda_{n-1} \le |j| < \lambda_n \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 4.2. *If* $1 < p < \infty$ *and the sequence* $\{\varphi(\lambda_n)\}_0^\infty$ *is normal then*

$$\|f\|_{h(p,q,\psi} \sim \|\{\varphi(1/\lambda_n)\|S_n * f\|_p\}\|_{l^q}, \quad f \in h(U).$$

Proof. This follows from Theorem 4.1 and the well known Riesz theorem: If $1 < p < \infty$ then there is $C < \infty$ such that $\|\sum_0^n g^j\|_p \le C\|g\|_p$ for all $g \in h^p$, $n \ge 0$.

The following propostion shows that Theorem 4.2 is actually a generalization of Theorem A.

PROPOSITION 4.1. *Let* X *be an* A*-space and let* φ *be a normal function. Then*

$$\|f\|_{X(q,\varphi)} \sim \left\{\int_0^1 [\varphi(1-r)\|f_r\|_X]^q dr/(1-r)\right\}^{1/q}, \quad f \in s(X).$$

Proof. Only the case $q < \infty$ requires a proof. Let $f \in s(X)$ and $\xi(r) = \|f_r\|_X^q$, $0 < r < 1$. Integration by parts shows that

$$q\|f\|^q_{X(q,\varphi)} = \int_0^1 \varphi(1-r)^q d\xi(r).$$

Since ξ is non-decreasing we see that if $\psi(t) \sim \varphi(t)$ then $\|f\|_{X(q,\psi)} \sim \|f\|_{X(q,\varphi)}$. By Proposition 3.1 we may take $\psi(t) = \psi_0(t)^\alpha$, where ψ_0 is a concave function. Then the function $\psi_0(t)/t$, $t > 0$, is non-increasing and consequently $t\psi_0'(t) \le \psi_0(t)$ for almost all $t > 0$. On the other hand, from the concavity of ψ_0 it follows that

$$\psi_0'(t)(ut - t) \ge \psi_0(ut) - psi_0(t), \quad u > 1.$$

Using (0.3) we find b so that $\psi_0(bt) \ge 2\psi_0(t)$. Then $\psi_0'(t)t(b-1) \ge \psi_0(t)$, whence

$$c/(1-r) \le \psi'(1-r)/\psi(1-r) \le \alpha/(1-r), \quad 0 < r < 1,$$

where $c = \alpha/(b-1)$. This concludes the proof.

5. Duality theorems

Shields and Williams [18] found the predual of $h(\infty, \infty, \varphi)$ for a quasi-normal function φ satisfying the following condition.

(SW) There exist a positive finite Borel measure μ on [0, 1) and a constant $C < \infty$ such that

$$\varphi(n+1)^{-1} = \int_0^1 r^{2n} d\mu(r)$$

and

$$(n+1)|\varphi(n) + \varphi(n+2) - 2\varphi(n+1)| \le C(\varphi(n+1) - \varphi(n))$$

for all integers $n \ge 0$.

Our duality theorem does not depend on (SW).

It follows from the proof of Proposition 4.1 that if $\psi \sim \varphi$ then $X(q,\psi) = X(q,\varphi)$. Thus we may suppose $\varphi^{1/\alpha}$ is concave for some integer $\alpha > 0$ (Proposition 3.1).

A consequence is that $\varphi^{-1/\alpha}$ is convex, while this implies that

$$(5.1) \qquad (1/\varphi)^2 \text{ is convex on } (1,\infty).$$

In fact, if φ is defined on (0, 1] we may extend it by

$$b/\varphi(t) = \int_0^1 r^t \varphi'(1-r) dr, \quad t > 1,$$

where b is chosen so that $\varphi(1) = \varphi(1+)$. Then φ is increasing and absolutely continuous on $(0,\infty)$ and satisfies (5.1) and (3.1) (by Lemma 3.2).

For a quasinormal function ψ define $D^\psi : h(U) \to h(U)$ by

$$(D^\psi f)\hat{}(n) = \psi(|n|+1)\hat{f}(n), \quad -\infty < n < \infty.$$

Note that D^{ψ} is an isomorphism of $h(U)$ onto itself. For $q \in (0, \infty]$ let

$$q' = \begin{cases} \infty & \text{if } q \leq 1, \\ q/(q-1) & \text{if } 1 < q < \infty, \\ 1 & \text{if } q = \infty. \end{cases}$$

THEOREM 5.1. *Let X be a Banach A-space or $X = H^p$, $p > 0$, and let φ be a quasi-normal function satisfying* (5.1). *Then the operation D^{φ^2} acts as an isomorphism from $X(q,\varphi)^*$ onto $X^*(q',\varphi)$.*

Note that if $q < \infty$ then $X(q,\varphi)^* = (X(q,\varphi) \to h(\bar{U}))$ and the dual of $X(q,\varphi)$ is naturally identified with $X(q,\varphi)^*$. (See Propositions 1.1 (d) and 1.3).

Since $(h)^* = h^{p'}$, $1 \leq p \leq \infty$, we have a solution to the duality problem for $h(p,q,\varphi)$.

THEOREM 5.2. *Let $1 \leq p \leq \infty$ and let φ be as in Theorem* 5.1. *Then the space $h(p,q,\varphi)^*$ is isomorphic to $h(p',q',\varphi)$, via the operator D^{φ^2}.*

The analogous result for $H(p,q,\varphi)$ holds if $1 < p < \infty$. If $p = 1$ we use Fefferman's result that $(H^1)^* = BMOA$, the space of analytic functions of bounded mean oscilation [8]. If $0 < p < 1$ then $(H^p)^*$ is equal (up to an equivalent renorming) to the space M^p of those $f \in H(U)$ for which

$$\|f\|_{M^p} := \|D^{1/p} f\|_{H(\infty,\infty,1)} = \sup_{0<r<1} (1-r) M_\infty(r, D^{1/p} f) < \infty \tag{5.2}$$

This is a result of Duren, Romberg and Shields [6, 7]. The operator $D^s : h(U) \to h(U)$, $-\infty < s < \infty$, is defined by

$$(D^s f)\hat{}(k) = (k| + 1)^s \hat{f}(k).$$

Thus we have the following.

THEOREM 5.3. *If φ is as in Theorem* 5.1 *then the operator D^{ψ^2} is an isomorphism of $H(p,q,\varphi)^*$ onto $Y(q',\varphi)$, where: 1. $Y = H^{p'}$ if $1 < p < \infty$, 2. $Y = BMOA$ if $p = 1$, 3. $Y = M^p$ if $p < 1$.*

For the proof of Theorem 5.1 we define the space $l^q_s(X)$, $-\infty < s < \infty$, to be the class of all sequences $F = \{f_n\}_0^\infty$ such that $f_n \in X$, $n \geq 0$, and

$$\|F\|_{l_s q(X)} := \left\{ \sum_{n=0}^{\infty} [2^{-ns} \|f_n\|_X]^q \right\}^{1/q} < \infty.$$

Let $w = \{w_n\}_0^\infty$ be a sequence satisfying (0.4) and (0.5) for some lacunary $\{\lambda_n\}_0^\infty$ and $N \geq 0$. For an A-space X we denote by $W^q_s(X)$ the class of those $f \in s(X)$ $(= h_E(U)$ for some $E)$ for which

$$\|f\|_{W_s{}^q(X)} := \|\{w_n * f\}\|_{l_s{}^q(X)} < \infty.$$

If the condition (0.6) is satisfied then $W^q_s(X)$ is an A-space. Moreover, Proposition 1.1 remains true if we replace $X(q,\varphi)$ by $W^q_s(X)$. We omit the proof.

The following lemma will be used instead of the Hahn-Banach theorem. (The Hahn-Banach theorem does not hold for quasi-normed spaces.)

Lemma 5.1. *Let $\bar{l}^q_s(X)$ denote the subspace of $l^q_s(X)$ consisting of all $\{f_n\}$ such that $f_n = 0$ for sufficiently large n. If (0.6) holds then the operator V defined by*

$$VF = \sum_{n=1}^{\infty} w_N * f_n, \quad F = \{f_n\} \in \bar{l}^q_s(X),$$

is a bounded linear operator from $\bar{l}^q_s(X)$ to $W^q_s(X)$.

Proof. It follows from (0.5) that

$$w_n * w_j = 0 \quad \text{for} \quad |j-n| \geq N+1. \tag{5.3}$$

Applying now (0.4) we get

$$w_n * VF = \sum_{j=n-N}^{n+N} w_n * w_j * f_j, \quad n \geq 0,$$

where $w_j = f_j = 0$ for $j < 0$. Hence

$$\|w_n * VF\|_X \leq K^{2N} \sum_{j=n-N}^{n+N} \|w_n * w_j * f_j\|_X,$$

where K satisfies $\|f+g\|_X \leq K\|f\|_X + K\|g\|_X$. Using (0.6) we get

$$\|w_n * VF\| \leq C \sum_{j=n-N}^{n+N} \|f_j\|_X, \quad n \geq 0.$$

Now the desired result is obtained by the use of the following lemma.

Lemma 5.2. *Let m be a non-negative integer. Then the operator S defined on real sequences by*

$$Sx = \left\{ \sum_{j=n-m}^{n+m} \xi_j \right\}_{n=0}^{\infty}, \quad x = \{\xi_j\}_0^\infty \quad (\xi_j := 0 \text{ for } j < 0)$$

acts as a bounded operator from $l^q_s(R)$ to itself, where R is the real line.

Proof. It is easily seen that the operators S_j, $0 \leq j \leq 2m$, defined by $S_j x = \{\xi_{j+n-m}\}_{n=0}^\infty$ map continuously $l^q_s(R)$ into $l^q_s(R)$. The operator S the has same property because $S = \sum_0^{2m} S_j$.

Theorem 5.4. *If (0.6) holds then $W^q_s(X)^* = W^{q'}_{-s}(X^*)$ (with equivalent norms).*

Proof. Define the sequence $\{P_n\}_0^\infty$ by

$$P_n = \sum_{j=n-N}^{n+N} w_j \quad (w_j = 0 \text{ for } j < 0).$$

It follows from (0.4) and (5.3) that $P_n * w_n = w_n$ for $n \geq 0$. Hence

$$f * g = \sum_{n=0}^{\infty} w_n * f * g = \sum_{n=0}^{\infty} P_n * f * w_n * g$$

and consequently

$$\|f * g\|_\infty \leq \sum_{n=0}^{\infty} \|P_n * f\|_X \|w_n * f\|_{X^*},$$

where $f \in W_s^q(X)$, $g \in W_{-s}^{q'}(X^*)$. Using Hölder's inequality we get

$$\|f * g\|_\infty \leq \|\{P_n * f\}\|_{l s^q(X)} \|g\|_{W_{-s}^{q'}(X^*)}.$$

Since

$$\|P_n * f\|_X \leq C \sum_{j=n-N}^{n+N} \|w_j * f\|_X$$

we have, by Lemma 5.2 ($\xi_j = \|w_j * f\|_X$),

$$\|\{P_n * f\}\|_{l s^q(X)} \leq C\|f\|_{W s^q(X)},$$

and this concludes the proof of the inclusion $W_{-s}^{q'}(X^*) \subset W_s^q(X)^*$.

To prove the converse let $g \in W_s^q(X)^*$ and define the operator T on $\bar{l}_s^q(X)$ by

$$TF = (VF) * g = \sum_{n=1}^{\infty} f_n * g_n,$$

where $g_n = w_n * g$. It follows from Lemma 5.1 that T is a bounded operator from $\bar{l}_s^q(X)$ to h^∞ with

$$\|T\| \leq C\|g\|_{W s^q(X)^*},$$

where C is independent of g. Now it suffices to prove that

$$\|T\| \geq \|\{g_n\}\|_{l_{-s}^{q'}(X^*)}$$

Let $0 < \varepsilon < 1$ and, for every $n \geq 0$, choose $f_n \in X$ so that $\|f_n\|_X = 1$ and $f_n * g_n(1) = \|f_n * g_n\|_\infty \geq \varepsilon\|g_n\|_{X^*}$. If $\{a_n\} \in \bar{l}_s^q(R)$ then $\{a_n f_n\} \in \bar{l}_s^q(X)$,

$$\|\{a_n f_n\}\|_{l s^q(X)} = \|\{a_n\}\|_{l s^q(R)}$$

and

$$\|T\{a_n f_n\}\|_\infty \geq \varepsilon \sum_{0}^{\infty} \|g_n\| a_n.$$

This implies

$$\|T\| \geq \varepsilon\|\{\|g_n\|\}\|_{l_{-s}^{q'}(R)} = \varepsilon\|\{g_n\}\|_{l_{-s}^{q'}(X^*)}.$$

This concludes the proof.

Proof of THeorem 5.1. Choose $\{\lambda_n\}$ so that $\varphi(\lambda_n) \sim 2^n$, $n \geq 0$. Applying first Theorems 2.1 and 4.1 and then Theorem 5.4 we obtain $X(q,\varphi)^* = W^{q'}_{-1}(X^*)$. On the other hand, X^* is a Banach A-space so that

$$\|D^{\varphi^2} f\|_{X^*(q',\varphi)} \sim \|D^{\varphi^2} f\|_{W_1^{q'}(X^*)}, \quad f \in s(X) = s(X^*),$$

by Theorem 4.1. Thus it remains to prove the following.

LEMMA 5.3. *Let Y be a Banach A-space, let $1/\psi$ be convex on $(1,\infty)$, and let $\psi(\lambda_n) \sim \psi(\lambda_{n+1})$, $n \geq 0$. Then*

$$\|w_n * D^\psi f\|_Y \sim \psi(\lambda_n)\|w_n * f\|_Y, \quad n \geq 0, \ f \in s(Y).$$

Proof. For fixed f and $n \geq 1$ let $g = w_n * f$, $k = \lambda_{n-1}$, $m = \lambda_{n+N}$. Then

$$w_n * D^\psi f = \sum_{j=k}^{m} A_j g^j,$$

where $A_j = \psi(j+1)$, and the functions g^j are defined as in Section 4. We have

$$\sum_k^m A_j g^j = \sum_k^{m-1} (A_j + A_{j+2} - 2A_{j+1})(j+1)\sigma_j^1 g + (A_m - A_{m+1}) m \sigma_{m-1}^1 g + A_m g.$$

Hence, by Theorem 2.2,

$$\left\| \sum_k^{m-1} A_j g^j \right\| \leq \sum_k^{m-1} |A_j + A_{j+2} - 2A_{j+1}|(j+1)\|g\|(A_{m+1} - A_m)m\|g\| + A_m\|g\|,$$

where $\|\cdot\| = |\cdot\|_Y$. Letting $a_j = 1/A_j$ we have

$$A_j + A_{j+2} - 2A_{j+1} = -A_j A_{j+2}(a_j + (a_{j+2} - 2a_{j+1}) + 2A_j(A_{j+2} - A_{j+1})(a_j - a_{j+1}).$$

Since $1/\psi$ is convex the function $F_u(t) := (1/\psi(t) - 1/\psi(u))/(u-t)$, $1 \leq t < u$, is non-increasing. Therefore

$$a_j - a_{j+1} = F_{j+2}(j+1) \leq F_{j+2}(1 + j/2) \leq 2/(j+2)\psi(1+j/2) \leq Ca_j/(j+1).$$

On the other hand, we have $a_j + a_{j+2} - 2a_{j+1} \geq 0$ because 1ψ is convex. It follows that

$$\sum_k^{m-1} |A_j + A_{j+2} - 2A_{j+1}|(j+1) \leq A_{m-1}A_{m+1} \sum_k^{m-1} (a_j + a_{j+2} - 2a_{j+1})(j+1)$$
$$+C \sum_k^{m-1} A_j(A_{j+2} - A_{j+1})a_j = A_{m-1}A_m((a_k - a_{k+1})k - (a_m - a_{m+1})m + a_k - a_m)$$
$$+C(A_{m+1} - A_{k+1}) \leq A_{m-1}A_m(Ca_k + a_k) + CA_{m+1} \leq C\psi(\lambda_n).$$

In the last step we used the estimates $A_{m+1} = \psi(\lambda_{n+N} + 1) \leq C\psi(\lambda_n)$, $a_k = \psi(\lambda_{n-1})^{-1} \leq C\psi(\lambda_n)^{-1}$. In the same way we get

$$m(A_{m+1} - A_m) = A_m A_{m+1}(a_m - a_{m+1})m \leq CA_{m+1}A_m a_m = CA_{m+1} \leq C\psi(\lambda_n).$$

Thus $\| \sum_k^m A_j g^j \| \le C\psi(\lambda_n)\|g\|$.

In the other direction, let $h = \sum_k^m A_j g^j$. Then $g = \sum_k^m a_j h^j$.

Now we have

$$\begin{aligned}\|g\| &\le \sum (a_j + a_{j+2} - 2a_{j+1})(j+1)\|h\| + (a_m - a_{m+1})_m\|h\| + a_m\|h\| \\ &= ((a_k - a_{k+1})k - (a_m - a_{m+1})m + a_k - a_m)\|h\| + (a_m - a_{m+1})m\|h\| \\ &+ a_m\|h\| \le C(a_k + a_m)\|h\| \le C\psi(\lambda_n)^{-1}\|h\|.\end{aligned}$$

This completes the proof of Theorem 5.1.

6. More on the dual of $X(q, \varphi)$

For a function φ (not necessarilly quasi-normal) define the measure $M = M_\varphi$ on [0, 1) by

$$dM_\varphi(r) = \varphi(1-r)^2 dm_\varphi(r) = \varphi(1-r)\varphi'(1-r)dr.$$

For $f,\ g \in h(U)$ let

$$(f, g) = \int_0^1 f_r * g_r(1)dM(r), \tag{6.1}$$

provided that the integral exists. For example, an application of Hölder's inequality shows that if $f \in X\ (q, \varphi) =: Y$, $q \ge 1$, and $g \in X^*(q', \varphi) =: Z$, then the function $r \mapsto f_r * g_r(1)$ belongs to $L^1(M_\varphi)$ and

$$|(f, g)| \le \|f\|_Y \|g\|_Z.$$

The analogous fact for $q < 1$ holds as well, but with $|(f, g)| \le C\|f\|\|g\|$. This shows that if $g \in X^*\ (q', \varphi)$ then $(\cdot, g)$ is a bounded linear functional on $X(q, \varphi)$. In some cases the converse holds too.

Proof. We have only to prove that if $L \in X(q, \varphi)'$ then there is $g \in X^*(q', \varphi)$ such that $L(f) = (f, g)$ for all $f \in X(q, \varphi)$. We extend φ to $(0, \infty)$ by

$$\varphi(t)^{-2} = c\int_0^1 r^{2(t-1)}dM(r), \quad t > 1,$$

where c is chosen so that $\varphi(1+) = \varphi(1)$. Applying Lemma 3.2 with $\psi = \varphi^2$ we see that (3.1.) holds. The condition (5.1) is obviously satisfied. It follows that if $L \in X(q, \varphi)'$, $q < \infty$, then there exists a unique h such that $D^{\varphi^2}h \in X^*(q', \varphi)$ and $L(f) = f * h(1)$. (See Theorem 5.1 and Proposition 1.1 (d) and 1.3.) Letting $g = D^{\varphi^2}h$ we have

$$\begin{aligned}L(f) = f * h(1) &= \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)\int_0^1 r^{2|n|}dM(r) \\ &= \int_0^1 \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)r^{2|n|}dM(r) = (f, g)\end{aligned}$$

for all harmonic polynomials $f \in X\ (q, \varphi)$. Since such polynomilas are dense in $X(q, \varphi)$, we have $L(f) = (f, g)$ for all $f \in X(q, \varphi)$. This completes the proof.

Let η be a positive finite Borel measure on [0, 1], and let $h^1(\eta)$ be the subspace of $L^1(d\eta(r)d\theta/2\pi)$ consisting of harmonic functions. In [18] Shields and Williams proved that if a quasi-normal function φ satisfies the condition (SW) (mentioned at the beginning of Section 5) then there exists a measure η such that $h(\infty, \infty, \varphi)$ is isomorphic to the dual of $h^1(\eta)$ in the pairing (6.1) with $dM(r) = \varphi(1-r)d\eta(r)$. However, the measure d is not given in an explicit form. Theorem 6.1 shows that one can take $d\eta(r) = \varphi'(1-r)dr$ without additional restrictions on φ. Moreover, our proofs show that any of the measures $d\eta(r) = \varphi(1-r)^{\beta-1}\varphi'(1-r)dr$ with $\beta > 0$ can be used.

It should be remarked that if $f \in h(p, q, \varphi)$, p, $q \geq 1$, and $g \in h(p', q', \varphi)$ then the function $f(z)g(\bar{z})$, $z \in U$, belongs to $L^1(\mu)$, where the measure μ on U is defined by

$$d\mu(re^{i\theta}) = dM(r)d\theta/2\pi.$$

The bilinear form (6.1) can be written as

$$(f, g) = \int_U \int f(z)g(\bar{z})d\mu(z)$$

As further application of Theorem 5.1 we prove a result concerning the space

$$X(0, \varphi) := \{f \in X(\infty, \varphi) : \lim_{r \to 1^-} \varphi(1-r)\|f_r\|_X = 0\}.$$

THEOREM 6.2. *Let X be a Banach A-space and let φ be a quasi-normal function. Then the second dual of $X(0, \varphi)$ is isometrically isomorphic to $X(\infty, \varphi)$.*

In the case $X = H^\infty$ a slightly more general result is proved by Rubel and Shields [15].

Proof. It is easily seen that $Y := X(0, \varphi)$ is the closure of the harmonic polynomials in $Z := X(\infty, \varphi)$, and that $Y^* = Z^*$. By Proposition 1.3 the dual of Y is canonically isometric to Y^*. Since Z^* is isomorphic to $X^*(1, \varphi)$ we see that the harmonic polynomials are dense in Y^* as well. This implies that the second dual of Y is isometric to the space $(Y^*)^*$. Thus it remains to prove that $(Y^*)^* = Z$ with equality of the norms. To see this let

$$S(f) = \sup\{\|f_\xi\|_Y^{\cdot} : |\xi| < 1\}$$

for $f \in h_E(U) = s(Y) = s(X)$. It is clear that $Z = \{f \in h_E(U) : S(f) < \infty\}$ and $\|f\|_Z = S(f)$. On the other hand, if $f \in h_E(U)$ and $|\xi| < 1$ we have

$$\|f_\xi\|_{(Y^*)^*} = \sup\{\|f_\xi * g\|_\infty : \|g\|_{Y^*} \leq 1\} = \|f_\xi\|_Y^{\cdot},$$

where Proposition 1.4 (with Y instead of X) has been used. This implies

$$S(f) == \sup\{\|f_\xi * g\|_\infty : \|g\|_{Y^*} \leq 1,\ |\xi| < 1\} = \sup\{\|f * g\|_\infty : \|g\|_{Y^*} \leq 1\},$$

and this concludes the proof.

7. On the dual of $H(p,q,\varphi)$ when φ is normal

Theorem 5.3 may be simplified if φ is supposed to be normal. We may assume that for some $\alpha > 0$ the function $\varphi(t)/t^\alpha$, $t > 0$, is non-increasing (Proposition 3.1). Then the function φ_γ, $\gamma > \alpha$, defined by

$$\varphi(t)\varphi_\gamma(t) = t^\gamma, \quad t > 0, \tag{7.1}$$

is normal.

THEOREM 7.1. *If $p \le 1$ and (7.1) holds, where φ and φ_γ are normal functions, then the operator $D^{\gamma+1/p-1}$ acts as an isomorphism of $H(p,q,\varphi)^*$ onto $H(\infty, q', \varphi_\gamma)$.*

Some special cases of this theorem have been discussed by Shields and Williams [17] ($p = q = 1$) and Mateljević and Pavlović [14] ($p = 1$, $q \ge 1$). The case $p < 1$, $q > 1$, is new even if $\varphi(t) = t^\alpha$. For further information see [1, 7].

Proof. Consider first the case $p < 1$. Let $\{w_n\}$ be a sequence described by Theorem 2.1 with $N > 1/p - 1$ and $\lambda_n = 2^n$. An obvious modification of Theorem 5.4 shows that the norm in $H(p,q,\varphi)^*$ is equivalent to

$$\left\{\sum_0^\infty [\varphi(2^{-n})^{-1}\|w_n * f\|_p^*]^{q'}\right\}^{1/q'},$$

where $\|\cdot\|_p^*$ stands for the norm in $(H^p)^*$. (This also follows from Theorems 5.1 and 4.1 and Lemma 5.3). By the Duren-Romberg-Shields theorem we have

$$\|w_n * f\|_p^* \sim \sup_{0<r<1} (1-r) M_\infty(r, w_n * D^{1/p} f), \quad f \in H(U),\ n \ge 0;$$

see (5.2). Using Lemma 4.1 we find

$$\|w_n * f\|_p^* \sim 2^{-n}\|w_n * D^{1/p} f\|_\infty.$$

Now Lemma 5.3 gives

$$\|w_n * D^s f\|_\infty \sim 2^{n\gamma}\|w_n * f\|_p^*,$$

where $s = \gamma + 1/p - 1$. Hence

$$\varphi_\gamma(2^{-n})\|w_n * D^s f\|_\infty \sim \varphi(2^{-n})^{-1}\|w_n * f\|_p^*.$$

This implies that the norm in $H(p,q,\varphi)^*$ is equivalent to

$$\left\{\sum_0^\infty [\varphi_\gamma(2^{-n})\|w_n * D^s f\|_\infty]^{q'}\right\}^{1/q'}.$$

Now the desired result follows fromTheorem 4.1.

If $p = 1$ we proceed in a similar way as in the case $p < 1$. We have only to prove that

$$\|w_n * f\|_1^* \sim \|w_n * f\|_\infty, \quad f \in H(U),\ n \geq 0.$$

It is clear that $\|w_n * f\|_1^* \leq \|w_n * f\|_\infty$. Let $g(z) = (1-z)^{-2}$. By the definition of $\|\cdot\|_1^*$ we have

$$\|w_n * f * g_r\|_\infty \leq \|w_n * f\|_1^* \|g_r\|_1 = \|w_n * f\|^* * (1-r^2)^{-1}.$$

Taking $r = 1 - 2^{-n}$ and using Lemma 4.1 we get

$$\|w_n * D^1 f\|_\infty = \|w_n * f * g\|_\infty \leq C2^n \|w_n * f\|_1^*.$$

Hence, by Lemma 5.3, $\|w_n * f\|_\infty \leq C\|w_n * f\|_1^*$. This completes the proof of the theorem.

It should be noted that if $p < 1$, $q \leq 1$ then the requirement that φ is normal is not necessary for the validity of Theorem 7.1. Namely, for any quasi-normal function φ such that $\varphi(t)/t^\alpha$ is non-increasing the function $\varphi_\gamma(t) := t^\gamma/\varphi(t)$ $(\gamma > \alpha)$ is normal, and we have the following.

THEOREM 7.2. *If φ is a quasi-normal function then the space $H(p,q,\varphi)^*$, where $p < 1$ and $q \leq 1$, is isomorphic to $H(\infty,\infty,\varphi_\gamma)$ via the operator $D^{\gamma+1/p-1}$.*

Proof. Let $Z = H(p,q,\varphi)^*$, $p < 1$, $q \leq 1$ and $\psi(t) = \varphi(t)^2 t^{1/p}$. It follows from Theorem 5. that

$$\|f\|_Z \sim \sup\{\varphi(1-r)(1-s) M_\infty(rs,\, D^\psi f) : 0 < r,\ s < 1\}.$$

This easily gives

$$\|f\|_Z \sim \sup\{\varphi(1-r)(1-r) M_\infty(r,\, D^\psi f) : 0 < r < 1\}.$$

Since the function $t\varphi(t)$, $t > 0$, is normal we have, by Theorem 4.1,

$$\|f\|_Z \sim \sup_n 2^{-n} \varphi(2^{-n}) \|w_n * D^\psi f\|_\infty,$$

where $\{w_n\}$ is as in the proof of Theorem 7.1. Using now Lemma 5.3 we see that

$$\|f\|_Z \sim \sup_n \varphi_\gamma(2^{-n}) \|w_n * D^{\gamma+1/p-1}\|_\infty.$$

Theorem 4.1 concludes the proof.

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(Received 25 12 1985)