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ON A DECOMPOSITION OF NEAR-RINGS IN A SUBDIRECT SUM OF NEAR-FIELDS

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Abstract. We extend some results from [1] on one class of near-rings and we give a decomposition of near-rings from this class by a subdirect sum of near-fields.

First we give some basic notations and definitions. We recall that a (left zero symmetric) near-ring is a system $(R, +, \cdot)$ where:

(i) $(R, +)$ is a (not necessarily abelian) group;

(ii) $(R, \cdot)$ is a semigroup;

(iii) $x(y+z) = xy + xz$ for all x, y, z in R;

(iv) $0x = 0$ for all x in R, where 0 is the identity of $(R, +)$.

A near-ring R with more than one element is a near-field if the set of nonzero elements of R forms a multiplicative group. An element x in R is said to be distributive if $(y+z)x = yx + zx$ for all y, z in R. The set of all distributive elements of R forms a multiplicative semigroup. A distributively generated (d.g.) near-ring is a near-ring R which is additively generated by some subsemigroup S of distributive elements of R. Thus if R is distributively generated by S, then every element r in R can be expressed as a finite sum $r = \sum_{\pm s_i}$ $(s \in S)$.

A subgroup B of $(R, +)$ is an R-subgroup (right R-subgroup) if $b \in B$ and $r \in R$ implies $br \in B$. A right ideal of R is a subset B such that $(B, +)$ is a normal subgroup of $(R, +)$ and $(x+b)y - xy \in B$ for each $b \in B$, $x, y \in R$. A subset B of R is an ideal of R if it is a right ideal and $rb \in B$ for each $r \in R$, $b \in B$. A right ideal Q of R is called completely prime if and only if $Q \in R$ and $ab \in Q$ implies that $a \in Q$ or $b \in Q$.

Definition 1. We say that a right ideal P of R has a minimal strict extension if there exists an R-subgroup Q such that $P \subset Q$ and $P \subset T \subset Q$ implies $T = Q$, where T is an R-subgroup of R.

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A proper right ideal of R is called strictly maximal if it is maximal as an R-subgroup. It is evident that every strictly maximal right ideal of R is a right ideal which has a minimal strict extension. A partial converse is given by

LEMMA 1. *Let R be a near-ring. Then every completely prime right ideal of R which has a minimal strict extension is a strictly maximal right ideal of R.*

Proof. Let P be completely prime right ideal which has a minimal strict extension. Thus there is an R-subgroup Q of R such that $P \subset Q$. For all $a \in Q \setminus P$ we have $aQ \not\subset P$. Therefore, $P \subset P + aQ \subseteq Q$ and hence $Q = P + aQ$. For this we have $a = p + ae$ for suitable $p \in P$ and $e \in Q$. Then for $x \in R$, $ax = (p + ae)x - aex + aex$ and so $a(x - ex) = (p + ae)x - aex \in P$ since P is a right ideal. But $a \notin P$, $x - ex \in P \subseteq Q$, so $x \in Q$ and $P = Q$ as required.

A right ideal B of R is modular if and only if there is an element $e \in R$ with $ex - x \in B$ for each $x \in R$ (e is a left identity modulo B). A right ideal B of R is called 2-modular if B is modular and R/B is an R-group of type 2.

LEMMA 2. *If B is a right ideal of a near-ring R and e is a left identity modulo B, then $e + b$ for $b \in B$ is a left identity modulo B, too.*

Proof. Since $(u+b)v - uv \in B$, then for $u = e$, $v = x$ we have $(e+b)x - ex \in B$, thus $(e + b)x - x + x - ex \in B$. But $x + ex \in B$ and hence $(e + b)x - x \in B$.

Let A, B be subsets of R. Let us denote by $(B : A)$ the set $\{x \in B / Ax \subseteq B\}$. We write briefly $(B : q)$ instead of $(B : \{q\})$.

LEMMA 3. *If P is a strictly maximal right ideal of a near-ring R, then for $q \notin P$ $R = P + qR$ and $(P : q)$ is a 2-modular right ideal of R.*

Proof. The set $P + qR$ is an R-subgroup strictly containing P. But, P is a strictly maximal right ideal of R and consequently $R = P + qR$.

Taking $q = p + qe$ for suitable $e \in R$, $p \in P$ we get immediately that $q(x - ex) = gx - qex = (p + qe)x - qex \in P$ for all $x \in R$. Hence $x - ex \in (P : Q)$. We need to prove yet that $(P : q)$ is a strictly maximal right ideal of R. If $r \notin (P : q)$ then $r \notin P$ so $qrR \not\subseteq P$. It follows that $R = P + qrR$ and $qR \subseteq P + qrR$. For any $x \in R$, we have $qx = p + gry$ for some $p \in P$, $y \in R$. Thus $x - ry \in (P : q)$ and $R = (P : q) + rR$. Hence $(P : q)$ is strictly maximal in R.

In the following considerations we introduce a condition (D) as follows.

Definition 2. A near-ring R has a property (D) if for every strictly maximal right ideal P of R, $q \notin P$ implies $qR \not\subseteq P$.

There is a class of near-rings with property (D). For example, such a class form all near-rings with identity. Also, all d.g. near-rings with $R^2 \not\subseteq P$ have a property (D). Namely, if P is a strictly maximal right ideal of a d.g. near-ring R and $q \notin P$, then $R = (q)_R + P$, where $(q)_R$ is the R-subgroup generated by q. The elements of the R-subgroup $(q)R$ have the from $\sum(\pm qs_i + m_i q)$, where $s_i \in S$ and

$m_i \in Z$ (S is a multiplicative subsemigroup of distributive elements). Thus, for $s, t \in S$, $m \in Z$ we have

$$(\pm qs + mq + P)t = \pm qst + mgt + Pt = q(\pm st + mt) + Pt \in qR + P$$

and it follows that $R^2 \subseteq qR + P$. Since $R^2 \not\subseteq P$ we have $qR \not\subseteq P$ as required,

An ideal P of R ($P \neq R$) is called strictly prime if $A \subseteq P$ or $B \subseteq P$ for any two R-subgroups A and B of R such that $AB \subseteq P$. Call R a strictly prime near-ring if $\{0\}$ is a strictly prime ideal.

PROPOSITION 1. *If P is a strictly maximal right ideal of u near-ring R with property (D) such that for $x \in R$, $Rx \subseteq P$ implies $x \in P$, then P is a strictly prime right ideal of R.*

Proof. First we prove that if from $Rb \subseteq P$ follows $b \in P$, then $aRb \subseteq P$ implies $a \in P$ or $b \in P$. Let $aRb \subseteq P$ and $a \notin P$, then by property (D) $aR \in P$, i.e. $R = P + aR$. Hence, every r in R is of the form $r = p_1 + ar_1$ for some $r_1 \in R$, $p_1 \in P$. Thus,

$$rb = (p_1 + ar_1)b - ar_1 b + ar_1 b \in P + aRb \subseteq P$$

But $Rb \subseteq P$ implies $b \in P$ as required. Let for any two R-subgroups A and B of R, $AB \subseteq P$. Since $ARB \subseteq AB \subseteq P$, then for all $a \in A$ and $b \in B$ it follows that $aRb \subseteq P$ and that implies $a \in P$ or $b \in P$. Thus $A \subseteq P$ or $B \subseteq P$ and P is strictly prime.

COROLLARY. *If R is a near-ring with property (D), then every 2-modular right ideal of R is strictly prime or R is strictly prime.*

Proof. Let P be a 2-modular right ideal of R and let $e \in R$ be a left identity modulo P. If $Rx \subseteq P$ then from $ex - x \in P$ it follows that $x \in P$. Thus the conditions of Proposition 1 hold and hence P is a strictly prime right ideal of R.

Definition 3. An ideal T of a near-ring R is called a factor near-field ideal if and only if R/T is a near-field.

According to Theorem 8.3d of [2] for a factor near-field ideal T, R/T is a 2-primitive near-ring with a right identity. Thus T is a strictly maximal right ideal of R. Also, T is a modular right ideal of R, because in R/T there is an identity $\bar{e} = e + T$ $(e \in R)$. Thus $(e + T)(x + T) - x + T$, i.e. $ex - x \in T$ for all $x \in R$. Therefore, every factor near-field ideal is a 2-modular right ideal of R. In fact, for near-rings with property (D) we have

PROPOSITION 2. *Let R be a near-ring with property (D). The 2-modular right ideal P of R is a factor near field ideal if and only if for each left identity e modulo P, $re \in P$ $(r \in R)$ implies $rRe \subseteq P$.*

Proof. Suppose $re \in P$ implies $rRe \subseteq P$ for some 2-modular right ideal P of R where $r \in R$ and e is a left identity modulo P. We need to show only that P

is an ideal of R. If P is not an ideal, then $rp \notin P$ for some $r \in R$, $p \in P$ $(r \notin P)$ and thus $R^2 \nsubseteq P$. Since $rp \notin P$ it follows by condition (D) that $R = P + rpR$. Then $re = p_1 + rpr_1$ for some $r_1 \in R$, $p_1 \in P$ and so $r(e - pr_1) = p_1 \in P$. By Lemma 2, $e_1 = e - pr_1$ is a left identity modulo P and $re_1 \in P$ implies $rRe_1 \subseteq P$, i.e. $rRe_1R \subseteq P$. From the Corollary we have $rR \subseteq P$ or $e_1R \subseteq P$. But $rp \notin P$ so $e_1R \subseteq P$. However P is left modular and $e_1e - e \in P$ implies $e \in P$ which is false. Namely if e is a left identity modulo P, then $e \in P$ iff $P = R$ (Remarks 3.21, [2]). Hence P is an ideal of R.

The converse is immediate.

Proposition 3. *If P is a strictly maximal right ideal of a near-ring R with property (D), then the following assertions are equivalent:*

(i) P is a factor near field ideal;

(ii) P is a completely prime ideal;

(iii) There exists $q \in R$ for which $(P : q) \subseteq P$ and for every left identity e modulo P, $re \in P$ $(r \in R)$ implies $rRe \subseteq P$.

Proof. (i) $\Rightarrow$ (ii). This is obvious.

(ii) $Rightarrow$ (iii). Let P be a completely prime ideal of R. If $x \in (P : q)$ i.e. $qx \in P$, then for $q \notin P$ we have $x \in P$. Thus $(P : q) \subseteq P$. Let e be a left identity modulo P, then $e \notin P$ and therefore $re \in P$ implies $r \in P$. Consequently $rRe \subseteq P$.

(iii) $\Rightarrow$ (i). As a consequence of Lemma 3 it follows that $(P : q)$ is a 2-modular rigt ideal of R. Since $(P : q) \subseteq P$ we have $P = (P : q)$. Also, by the hypothesis $re \in P$ implies $rRe \subseteq P$. Using Proposition 2, it follows that P is a factor near-field ideal of R.

Theorem 1. *A right ideal P of a near-ring R with property (D) is a factor near field ideal if and only if P is a completely prime right ideal which has a minimal strict extension.*

Proof. Let P be a completely prime right ideal of R which has a minimal strict extension. By Lemma 1 P is a strictly maximal right ideal of R. Applying Proposition 3 it follows that P is a factor near-field ideal of R.

Conversely, if P is a factor near-field ideal of R then P is a 2-modular right ideal and hence a strictly maximal right ideal of R. It follows that P has a minimal strict extension. By Proposition 3, P is a completely prime right ideal of R.

Lemma 4. *Let B he a nonzero ideal of a near-ring R. If T_B is a factor near-field ideal of B, then $B \nsubseteq (T_B : B)$ and $(T_B : B)$ is a factor near field ideal of R.*

Proof. Since B/T_B is a near-field, so $B^2 \nsubseteq T_B$. Hence $B \subseteq (T_B : B)$.

The near-field B/T_B has an identity. Thus there is $e \in B$ such that $b - be \in T_B$ for all $b \in B$. Since $T_B \triangleleft B \triangleleft R$ it follows by Theorem 4.63 of [2] that T_B is an ideal of R. Hence $(b - be)x \in T_B$ for all $x \in R$. But $bx = (b - be + be)x - bex + bex \in T_B + bex$

and so $b(x - ex) \in T_B$. Hence $B(x - ex) \subseteq T_B$, i.e. $x - ex \in (T_B : B) \equiv T$. Consequently, $x \in T + ex \subseteq T + B$ for arbitrary $x \in R$, that is $R = T + B$. Since $T_B \subseteq B$ we have $T_B \subseteq B \cap T$. But T_B is a strictly maximal in B, so $T_B - B \cap T$. Now

$$\frac{R}{T} = \frac{T+B}{T} \simeq \frac{B}{T \cap B} = \frac{B}{T_B}$$

where B/T_B is a near-field. Thus, T is a factor near-field ideal of R.

Definition 4. A near-ring R has a strict property (D) if every nonzero ideal of R, used as a near-ring, has a property (D).

We say that a near-ring R is a subdirect sum of near-rings R_k if and only if there exist the ideals I_k of R with $\bigcap I_k = (0)$ and $R_k \simeq R/I_k$ as near-rings.

THEOREM 2. *A near-ring R with a strict property (D) is isomorphic to a subdirect sum of near fields if and only if every nonzero ideal of R, used as a near-ring, contains a completely prime right ideal which has a minimal strict extension.*

Proof. If a near-ring R is isomorphic to subdirect sum of near-fields R_k, then there exist ideals T_k with $\bigcap T_k = (0)$ and $R/T_k \simeq R_k$. Let B be a nonzero ideal of R, then there is a near-field T_k such that $B \not\subseteq T_k$ and hence $R - T_k + B$. If $T_B = T_k \cap B$, then

$$\frac{B}{T_B} = \frac{b}{T_k \cap B} \simeq \frac{T_k + B}{T_k} = \frac{R}{T_k} \simeq R_k$$

Thus, T_B is a near-field ideal of B. Hence T_B is a completely prime right ideal of B which has a minimal strict extension.

Conversely, let every nonzero ideal of a near-ring R with a strict property (D) contains a completely prime right ideal which has a minimal strict extension. Assume that the intersection $B = \bigcap T_k$ of all factor near-field ideals T_k of R is a nonzero ideal of R. By the hypothesis, B contains a completely prime right ideal T_B which has a minimal strict extension. According to Theorem 1, T_B is a factor near-field ideal of B. By Lemma 4, $(T_B : B) \equiv T$ is a factor near-field ideal of R and thus $B \subseteq T$. But this contradicts to the fact proved in Lemma 4 that $B \not\subseteq T$. Consequently, $B = \bigcap T_k = (0)$ and hence R is isomorphic to a subdirect sum of near-fields.

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