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ALL REPRODUCTIVE SOLUTIONS OF FINITE EQUATIONS

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Summary. An equation, the solution set of which is a subset of a given finite set, is called a *finite equation*. Applying some kind of algebraic structure we effectively determine all reproductive solutions of such equations (*Theorem* 1 and *Theorem* 2).

1. Let E be a given non-empty set and $f\colon\ E \to E$ a given function. An x-equation

$$f(x) = x \tag{1}$$

is called *reproductive* [4] if the function f satisfies the identity

$$f(f(x)) = f(x) \tag{2}$$

All solutions of a reproductive equation can be found in a trivial way. Namely, the formula

$$x = f(p) \quad (p \text{ is an arbitrary element of } E) \tag{3}$$

determines all solution of (1) provided this equation is a reproductive one. The formula is an example of the so-called general reproductive solutions formula ([2], [3], [7]).

Next, any x-equation

$$eq(x) \quad (x \text{ is an unknown element of } E;\ eq \text{ is a given unary relation of } E) \tag{4}$$

which has at least one solution is equivalent to a reproductive equation [4].

Accordingly, to solve a given x-equation (4) it suffices to find any reproductive equation equivalent to (1). In this paper we are concerned with finding all such reproductive equations in case of a finite equation.

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2. Let $B = \{\sigma_0, \sigma_1, \dots, \sigma_n\}$ be a given set of $n+1$ elements and $S = \{0,1\}$. Define the operation x^y by

$$x^y = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases} \qquad (x, y \in B \cup S) \tag{5}$$

The standard Boolean operations $+$ and $\cdot$ ("or" and "and") are described by the following tables

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

Extend these operations to partial operations of the set $B \cup S$ in the following way

(6) $x + 0 = x,\ \ 0 + x = x,\ \ x \cdot 0 = 0,\ \ 0 \cdot x = 0,\ \ x \cdot 1 = x,\ \ 1 \cdot x = x \quad (x \in B \cup S)$

We consider the following x-equation

$$s_0 \cdot x^{b_0} + s_1 \cdot x^{b_1} + \cdots + s_n \cdot x^{b_n} = 0 \tag{7}$$

where $s_i \in \{0,1\}$ are given elements and $x \in B$ is unknown. Obviously the equation (7) is possible iff the condition

$$s_0 \cdot s_1 \cdots s_n = 0 \tag{8}$$

holds.

In the sequel, assuming the condition (8), we are going to determine all reproductive solutions of the equation (7).

First we introduce the following definition.

Let $(\sigma_0, \sigma_1, \dots, \sigma_n) \in S^{n+1}$ be any element. Then the set $Z(\sigma_0, \dots, \sigma_n)$ ("the zero-set of $(\sigma_0, \dots, \sigma_n)$") is defined as folows

$$b_i \in Z(\sigma_0, \dots, \sigma_n) \Leftrightarrow \sigma_i = 0 \qquad (i = 0, 1, \dots, n) \tag{9}$$

For instance, if $n = 3$ we have

$$Z(1,0,1,0) = \{b_1 b_3\},\ \ Z(1,1,1,1) = \emptyset,\ \ Z(0,0,0,0) = \{b_0, b_1, b_2, b_3\}$$

Let now $s_0, \dots, s_n$ be any elements of S satisfying the condition (8). With respect to the sequence $s_0, \dots, s_n$ we define the so-called *repro-function*[1] $A : B \to B$. This is any function defined by a certain formula of the form

$$A(x) = A_0(s_0, \dots, s_n)x^{b_0} \cdots + A_n(s_0, \dots, s_n)x^{b_n} \tag{10}$$

where each coefficient $A_k(s_0, \dots, s_n)$ is determined by some equality of the form

$$A_k(s_0, \dots, s_n) = b_k s_k^0 + \sum_{\sigma_k \neq 0,\ \sigma_0 \cdots \sigma_n = 0} F_k(\sigma_0, \dots, \sigma_n) s_0^{\sigma_0} \cdots s_n^{\sigma_n} \tag{11}$$

[1] As a matter of fact, A is a function of the type $A : S^{n+1} \times B \to B$.

assuming that coefficients $F_k(\sigma_0,\dots,\sigma_n)\in B$ satisfy the condition

$$F_k(\sigma_0,\dots,\sigma_n)\in Z(\sigma_0,\dots,\sigma_n) \tag{12}$$

For instance, if $n=2$, $k=1$ the equality (11) reads

$$A_1(s_0,s_1,s_2)=b_1s_1^0+F_1(0,1,0)s_0^0s_1^1s_2^0+F_1(0,1,1)s_0^0s_1^1s_2^1+F_1(1,1,0)s_0^1s_1^1s_2^0,$$

where the coefficients $F_1(0,1,0)$, $F_1(0,1,1)$, $F_1(1,1,0)$ can be any elements of B satisfying the conditions (of type (12))

$$F_1(0,1,0)\in\{b_0,b_2\},\ \ F_1(0,1,1)=b_0,\ \ F_1(1,1,0)=b_2$$

Note that generally, according to the condition $s_0\cdots s_n=0$, there exists at least one repro-function with respect to the sequence $s_0,\dots,s_n$.

THEOREM 1. *Let $s_0\cdots s_n=0$, then the equation of the form*

$$x=A(x)$$

is a reproductive equation and equivalent to the equation (7) *if and only if A is a repro-function.*

Proof. Denote the equation (7) by $g(x)=0$. Firstly, we prove the following fact

(p) Let $A(x)$ be determined by a certain equality of type (10) assuming only that $A_0(s_0,\dots,s_n)$, ..., $A_n(s,\dots,s_n)\in B$. Then the implication $g(x)=0\Rightarrow x=A(x)$ holds if and only if for each coefficient $A_k(s_0,\dots,s_n)$ an equality of the form (11) is satisfied, where $F_k(\sigma_0,\dots,\sigma_n)$ can be any elements[2] of B.

The proof immediately follows from the following equivalences

$$\begin{aligned}(p)\Leftrightarrow(\forall x\in B)[s_0x^{b_0}+\cdots+s_nx^{b_n}=0\Rightarrow x=A_0(s_0,\dots,s_n)x^{b_0}+\cdots+\\ +A_n(s_0,\dots,s_n)x^{b_n}]\end{aligned}$$

$\Leftrightarrow(\forall k\in\{0,\dots,n\})(s_k=0\Rightarrow A_k(s_0,\dots,s_n)=b_k)$

$\Leftrightarrow A_k(s_0,\dots,s_n)$ is determined by means of a certain equality (11), where $F_k(\sigma_0,\dots,\sigma_n)$ may be any elements of B.

Next we introduce the condition

$$(\forall x\in B)s_0(A(x))^{b_0}+\cdots+s_n(A(x))^{b_n}=0. \tag{q}$$

Obviously the sentence "$x=A(x)$ *is a reproductive equation, equivalent to the equation* $f(x)=0$" is logiccally equivalent to the conjuction $(p)\wedge(q)$. Accordingly, the remaining part of the proof reads:

$x=A(x)$ is a reproductive equation, equivalent to the equation $f(x)=0$

[2]Thus condition (8) is not assumed.

$\Leftrightarrow$ $(p) \wedge (q)$

$\Leftrightarrow$ $(\forall x \in B) s_0(A(x))^{b_0} + \cdots + s_n(A(x))^{b_n} = 0$, and $A(x)$ is determined by means of some equalities of the form (10), (11), where $F_k(\delta_0, \ldots, \delta_n)$ are certain elements of B

$\Leftrightarrow$ $(\forall x \in B)\ s_0(A_0(s_0, \ldots, s_n)x^{b_0} + \cdots + A_n(s_0, \ldots, s_n)x^{b_n})^{b_0} + \cdots +$
$$+s_n(A_0(s_0, \ldots, s_n)x^{b_0} + \cdots + A_n(s_0, \ldots, s_n)x^{b_n})^{b_n} = 0$$
and $A_k(s_0, \ldots, s_n)$, with $k \in \{0, \ldots, n\}$, are determined by some equalities of the form (11).

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})\ s_i A_k^{b_i}(s_0, \ldots, s_n) = 0$ and $A_k(s_0, \ldots, s_n)$, with $k \in \{0, \ldots, n\}$, are determined by some equalities of the form (11). This part of the proof is based on the following general facts: If $a_0, \ldots, a_n$, b are any elements of B then:

1° $(a_0 x^{b_0} + \cdots + a_n x^{b_n})^b = a_0^b x^{b_0} + \cdots + a_n^b x^{b_n}$ (for all $x \in B$)

2° $(\forall x \in B)\ a_0 x^{b_0} + \cdots + a_n x^{b_n} = 0 \Leftrightarrow (\forall i \in \{0, \ldots, n\}) a_i = 0$

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})\ s_i \cdot \big(\sum_{\sigma_k \neq 0,\ \sigma_0 \cdots \sigma_n = 0} F_k^{b_i}(\sigma_0, \ldots, \sigma_n) s_0^{\sigma_0} \cdots s_n^{\sigma_n}\big) = 0$, where $F_k(s_0, \ldots, s_n)$ are certain elements of B. We have used the equality of the form (11) and the identity $s_i \sigma_k^{b_i} s_k^0 = 0$

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})$
$$\Big(\sum_{(\sigma_0, \ldots, \sigma_n) \in S^{n+1}} \sigma_i s_0^{\sigma_0} \cdots s_n^{\sigma_n} \cdot \sum_{\sigma_k \neq 0,\ \sigma_0 \cdots \sigma_n = 0} F_k^{b_i}(\sigma_0, \ldots, \sigma_n) s_0^{\sigma_0} \cdots s_n^{\sigma_n} = 0$$
(For the identity $s_i = \sum_{(\sigma_0, \ldots, \sigma_n) \in S^{n+1}} \sigma_i s_0^{\sigma_0} \cdots s_n^{\sigma_n}$ holds).

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\}) \sum_{\sigma_k \neq 0,\ \sigma_0 \cdots \sigma_n = 0} \sigma_i F_k^{b_i}(\sigma_0, \ldots, \sigma_n) s_0^{\sigma_0} \cdots s_n^{\sigma_n} = 0$

$\Leftrightarrow$ $(\forall i,\ k \in \{0, \ldots, n\})\ (\forall \sigma_0, \ldots, \sigma_n \in S)(P \Rightarrow \sigma_i F_k^{b_i}(\sigma_0, \ldots, \sigma_n) = 0)$
where the condition $\sigma_k \neq 0$, $\sigma_0 \cdots \sigma_n = 0$ is denoted by P.
From $F_k \in B$ it follows that $(\forall k)(\exists j) F_k = b_j$. Hence we conclude the equality $F_k = b_{\varphi(k)}$ where $\varphi : \{0, \ldots, n\} \to \{0, \ldots, n\}$ is a certain function[3].

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall i,\ k \in \{0, \ldots, n\}(P \Rightarrow \sigma_i b_{\varphi(k)}^{b_i} = 0)$

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall k \in \{0, \ldots, n\}(P \Rightarrow \sigma_{\varphi(k)} = 0)$
For $x \neq y \Rightarrow x^y = 0$.

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall k \in \{0, \ldots, n\})(P \Rightarrow b_{\varphi(k)} \in Z(\sigma_0, \ldots, \sigma_n))$
Using definition (9).

$\Leftrightarrow$ $(\forall \sigma_0, \ldots, \sigma_n \in S)(\forall k \in \{0, \ldots, n\})(P \Rightarrow F_k \in Z(\sigma_0, \ldots, \sigma_n))$

$\Leftrightarrow$ A is a repro-function

[3] φ also depends on $\sigma_0, \ldots, \sigma_n$.

The proof is complete.

From Theorem 1 immediately follows the following result on the reproductive solutions.

THEOREM 2. *If* (7) *is a possible equation then a formula*

$$x = A(p) \qquad (p \text{ is any element of } B)$$

represents a general reproductive solution of the equation (7) *if and only if the function* A *is a repro-function.*

Example 1. Let $B = \{0, 1, 2\}$. Consider the x-equation

$$s_0 x^0 + s_1 x^1 + s_2 x^2 = 0 \qquad (s_i \text{ are given and } x \text{ is unknown})$$

This equation is possible if and only if $s_0 s_1 s_2 = 0$. Any general reproductive solution has the following form

$$x = A_0 p^0 + A_1 p^1 + A_2 p^2 \tag{14}$$

where A_i are defined by

$$A_0 = (1 \text{ or } 2) s_0^1 s_1^0 s_2^0 + 1 s_0^1 s_1^0 s_2^1 + 2 s_0^1 s_1^1 s_2^0, \quad A_1 = 1 s_1^0 + (0 \text{ or } 2) s_0^0 s_1^1 s_2^0 + 2 s_0^1 s_1^1 s_2^0$$

$$A_2 = 2 s_2^0 + (0 \text{ or } 1) s_0^0 s_1^0 s_2^1 + 1 s_0^1 s_1^0 s_2^1$$

In these equalities a symbol of the form (p or q) denotes an element which may be p or q. Consequently there are exactly 8 formulas of the form (14).

3. Now we state an aplication of Theorems 1 and 2. Let n be a given natural number and $B = S^n$. Then according to the definition (5) we have the following identity

$$(x_1, \dots, x_n)^{(i_1, \dots, i_n)} = x_1^{i_1} \cdots x_n^{i_n}$$

In connection with it the equation of the type (7) may be written in the following form

$$\sum a_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n} = 0 \tag{15}$$

where $a_{i_1 \dots i_n} \in S$ are given, and $x_j \in S$ are unknown elements. In other words, (14) is the standard Boolean equation in $x_1, \dots, x_n$. Theorems 1 and 2 can be directly applied to the equation (15); consequently one effectively finds all general reproductive solutions of it.

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