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A REVERSED KAKUTANI'S FIXED POINT THEOREM

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Abstract. A many-to-one version of the Kakutani's fixed point theorem is proved.

Well known Kakutani's fixed point theorem states that for every multivalud function from a compact convex set $K \subset R^n$ to itself there exists a point $x \in K$ such that $x \in f(x)$, provided $f(x) \subset K$ is compact and convex for all $x \in K$ and the graph of f, $\Gamma(f) := \{(x,y) \mid y \in f(x)\} \subset K \times K$ is closed.

In this short note we want to show that a many-to-one version of this theorem still holds. More precisely, let C be a compact convex set in the convex cone of all compact convex subsets of R^n. The topology on C is induced by the Hausdorff metric. Since both the Brouwer's and Kakutani's fixed point theorem hold only for self maps of a compact convex set we need an appropriate concept for many-to-one maps.

Definition. A map $f : C \to R^n$ is called a "selfmap" if $(\forall x \in \mathrm{Ran}\,(f))(\exists K \in C)K \ni x$ or equivalently if $\mathrm{Ran}\,(f) \subset \cup\{K \mid K \in C\}$.

THEOREM. *Let $f : C \to R^n$ be a continuous "selfmap". Then there exists a set $K \in C$ such that $f(K) \in K$.*

Proof. Let us define a multi-valued function $g : C \to C$ as follows $g(K) := \{L \in C \mid f(K) \in L\}$. Obviously, $g(K)$ is a nonempty convex subset of C for all $K \in C$. Let us show that the graph of g is closed. $\Gamma(g) = \{(K,L) \mid f(K) \in L\} = (f \times 1)^{-1}\{(z,L) \mid z \in L\}$ is closed since the relation $\in$ is closed and $f \times 1 : C \times C \to R^n \times C$ is continuous. Since, Kakutani's theorem is still true for compact convex subsets of linear topological spaces (Ky Fan [1]) there exists $K \in C$ such that $K \in g(K)$ or in other words $f(K) \in K$. Q.E.D.

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