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ON SUMS INVOLVING RECIPROCALS OF CERTAIN LARGE ADDITIVE FUNCTIONS

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Abstract. Let $\beta(n)=\sum_{p|n} p$, $B(n)=\sum_{p^\alpha\|n}\alpha p$, $B_1(n)=\sum_{p^\alpha\|n}p^\alpha$. Sums of reciprocal of these functions are evaluated asymptotically. Asymptotic formulas for some related sums, involving the function $\Omega(n)$ and $\omega(n)$ (the number of distinct and total number of prime factors of n) are also derived.

1. Introduction. Let $p(n)$ denote the largest prime factor of an integer $n\geq 2$, and let $p(1)=1$. Let $\beta(n)$, $B(n)$ and $B_1(n)$ denote the additive functions

$$\beta(n)=\sum_{p|n} p,\ \ B(n)=\sum_{p^\alpha\|n}\alpha p,\ \ B_1(n)=\sum_{p^\alpha\|n}p^\alpha,$$

where as usual p denotes primes and $p^\alpha\|n$ means that p^α divides n but $p^{\alpha+1}$ does not.

In 1981, Ivić [**7**] proved that

$$\sum_{n\leq x}1/p(n)=x\exp\{-(2\log x\cdot\log_2 x)^{1/2}+O((\log x\cdot\log_3 x)^{1/2})\},\tag{1.1}$$

where $\log_2 x=\log\log x$, $\log_3 x=\log\log\log x$. The formula above remains true if $p(n)$ is replaced by $\beta(n)$ or $B(n)$.

In 1984, Ivić and Pomerence [**9**] proved that

$$\sum_{n\leq x}1/p^r(n)=x\exp\{-(2r\ \log x\log_2 x)^{1/2}(1+g_{r-1}(x)+O(\log_3^3 x/\log_2^3 x))\},$$

where $r>0$ is fixed and

$$g_r(x)=\frac{\log_3 x+\log(1+r)-2-\log 2}{2\log_2 x}\left(1+\frac{2}{\log_2 x}\right)-\frac{(\log_3 x+\log(1+r)-\log 2)^2}{8\log_2^2 x}.$$

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The proofs of results above depend on estimates for $\psi(x, y)$. the number of positive integers not exceeding x all of whose prime factors do not exceed y.

Recently Hildebrand [**4**] and Maier (unpublished) obtained independently much better results concerning $\psi(x, y)$ (see Lemma 2 below). With the help of these results, Erdös, Ivić and Pomerance [**3**] obtained a precise estimate for the sum in (1.1), where it was shown that

$$\sum_{n\le x}\frac{1}{p(n)} = \left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\delta(x), \tag{1.2}$$

where

$$\delta(x) = \int_2^x \rho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^2},$$

and the function $\rho(u)$ is defined for $u \ge 0$ as the continuous solution of the equations

$$\begin{aligned} \rho(u) =&1, \qquad (0 \le u \le 1),\\ u\rho'(u) =& -\rho(u-1), \qquad (u > 1). \end{aligned} \tag{1.3}$$

It is well-known that (see [**6**] or [**2**])

$$\rho(u) = \exp\left\{-u(\log u + \log_2 u - 1 + \frac{\log_2 u}{log\, u} + O\left(\frac{1}{\log u}\right)\right)\right\}. \tag{1.4}$$

In [**3**], it was shown that

$$\delta(x) = \exp\{-(2\log x\log_2 x)^{1/2}(1+g_0(x)+O(\log_3^3 x/\log_2^3 x))\}$$

Very recently, Ivić [**8**] proved that

$$\sum_{n\le x}\frac{\omega(n)}{p(n)} = \left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}, \tag{1.5}$$

$$\sum_{n\le x}\frac{\Omega(n)-\omega(n)}{p(n)} = \left(\sum_p \frac{1}{p^2-p} + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\sum_{n\le x}\frac{1}{p(n)}, \tag{1.6}$$

and

$$\sum_{n\le x}\frac{\mu^2(n)}{p(n)} = \left(\frac{6}{\pi^2} + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\sum_{n\le x}\frac{1}{p(n)} \tag{1.7}$$

where $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors of n and the total number of prime factors of n, respectively, and $\mu(n)$ is the Moebius function.

Moreover, it was shown in [**10**] that

$$\left(\frac{1}{2}+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)} \le \sum_{2\le n\le x}\frac{1}{\beta(n)} \le \left(\log 2+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}. \tag{1.8}$$

The following result occurs as a remark in [**3**]:

$$\sum_{2\le n\le x} 1/\beta(n) = (1+\exp\{-C(\log x\log_2 x)^{1/2}\})\sum_{x\le x} 1/p(n), \tag{1.9}$$

From (1.8) we known that (1.9) is not true.

The purpose of this paper is to give estimates for the analogous sums in (1.2), (1.5), (1.6) and (1.7) with $p(n)$ replaced by $\beta(n)$.

2. Statement of results.

THEOREM 1. $\sum_{2\le n\le x}\frac{1}{\beta(n)} = \left(D+O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}$,
where $1/2 < D < 1$ *denotes an absolute constant which will be described precisely in section* 4.

THEOREM 2.

$$\sum_{2\le n\le x}\frac{\omega(n)}{\beta(n)} = D\left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1+O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)},$$

where D *is as in Theorem* 1.

The last formula remains true if $\omega(n)$ is replaced by $\Omega(n)$.

THEOREM 3.

$$\sum_{2\le n\le x}\frac{\Omega(n)-\omega(n)}{\beta(n)} = \left(D\left(\sum_p\frac{1}{p^2-p}\right)+O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}.$$

THEOREM 4.

$$\sum_{2\le n\le x}\frac{\mu^2(n)}{\beta(n)} = \left(\frac{6}{\pi^2}D+O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}.$$

Moreover, it was shown in [**11, 12**] that

$$\sum_{2\le n\le x}\left(\frac{1}{\beta^r(n)}-\frac{1}{B^r(n)}\right) = x\exp\Big\{-\big(2(r+1)\log x\log_2 x\big)^{1/2} - \left(\frac{r+1}{2}\frac{\log x}{\log_2 x}\right)^{1/2}\log_3 x + O\left(\left(\frac{\log x}{\log_2 x}\right)^{1/2}\right)\Big\},$$

and

$$\sum_{2\le n\le x}\left(\frac{1}{\beta^r(n)}-\frac{1}{B_1^r(n)}\right) = x\exp\Big\{-\big(2(r+1)\log x\log_2 x\big)^{1/2} - \left(\frac{2r+1}{4}\frac{\log x}{\log_2 x}\right)^{1/2}\log_3 x + O\left(\left(\frac{\log x}{\log_2 x}\right)^{1/2}\right)\Big\},$$

respectively, where r is any fixed positive number. From the two results above, we known that Theorems 1, 2, 3, and 4 remain true if $\beta(n)$ is replaced by $B(n)$ or $B_1(n)$.

3. Several Lemmas.

LEMMA 1. *Let*
$$L_1 = \exp\left\{\left(\tfrac{1}{2}\log x\log_2 x\right)^{1/2}\left(1-2\tfrac{\log_3 x}{\log_2 x}\right)\right\},$$
$$L_2 = \exp\left\{\left(\tfrac{1}{2}\log x\log_2 x\right)^{1/2}\left(1-2\tfrac{\log_3 x}{\log_2 x}\right)\right\},$$
Then we have
$$\sum_{n\le x}\frac{1}{p(n)} = \left(1+O\left(\frac{1}{\log^A x}\right)\right)\sum_{L_1\le p\le L_2}\frac{1}{p}\psi\left(\frac{x}{p},p\right),$$
for any fixed $A>0$.

Proof. See [**8**, formula (4.3)].

LEMMA 2. [**4**]. *For any fixed $\varepsilon>0$ and $x\ge 3$, $\exp\{(\log_2 x)^{5/3+\epsilon}\}\le y\le x$, we have uniformly*
$$\psi(x,y) = x\rho(u)\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right),\quad u=\frac{\log x}{\log y}.$$

LEMMA 3.[**1, 5**]. *Uniformly for $u\ge 1$ and $0\le t\le 1$ we have*
$$\rho(u-t) = \rho(u)e^{t\xi(u)}\big(1+O(1/u)\big), \tag{3.1}$$
where $\xi=\xi(u)$ denotes the positive solution of the equation
$$e^{\xi} = u\xi+1, \tag{3.2}$$
and satisfies
$$\xi(u) = \log u + O\big(\log_2(u+2)\big),\quad u\ge 2. \tag{3.3}$$

LEMMA 4. [**5**]. *Uniformly for $u\ge 1$ and $0\le t\le u$ we have*
$$\rho(u-t)\ll \rho(u)e^{t\xi(u)}.$$

LEMMA 5. *For any fixed $\varepsilon>0$ and $1\le d\le y$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\}\le y\le x^{1/2}$, we have uniformly*
$$\psi\left(\frac{x}{d},y\right) = \psi(x,y)d^{-\beta}\left(1+O\left(\frac{1}{u}\right)+O\left(\frac{\log(u+1)}{\log y}\right)\right),$$
where
$$\beta=\beta(x,y) = 1-\xi\big((\log x)/(\log y)\big)/(\log y), \tag{3.4}$$

and $\xi(u)$ is defined as in Lemma 3.

Proof. Let $t = \log d/\log y$. In view of Lemmas 2 and 3 we have

$$\begin{aligned}\psi\Big(\frac{x}{d}, y\Big) &= \frac{x}{d}\rho\Big(\frac{\log(x/d)}{\log y}\Big)\Big(1 + O\Big(\frac{\log(u+1)}{\log y}\Big)\Big)\\ &= \frac{x}{d}\rho(u)e^{t\xi(u)}\Big(1 + O\Big(\frac{1}{u}\Big) + O\Big(\frac{\log(u+1)}{\log y}\Big)\Big)\\ &= \psi(x,y)d^{-\beta}\Big(1 + O\Big(\frac{1}{u}\Big) + O\Big(\frac{\log(u+1)}{\log y}\Big)\Big).\end{aligned}$$

The Lemma is proved.

LEMMA 6. *For any fixed $\varepsilon > 0$, and $1 \le d \le x/y$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x$*, we have uniformly $\psi(x/d, y) \ll \psi(x,y)d^{-\beta}$. where $\xi = \xi(x,y)$ is given by (3.4).

Proof. Using Lemma 4 instead of Lemma 3, the proof of this result is analogous to the proof of Lemma 5.

4. Proof of Theorem 1. By Lemma 1 we have

$$G(x) := \sum_{2\le n\le x}\frac{1}{\beta(n)} = \sum_{2\le n\le x, L_1<p(n)\le L_2}\frac{1}{\beta(n)} + O(R), \tag{4.1}$$

where $R = \frac{1}{\log^A x}\sum\limits_{n\le x}\frac{1}{p(n)}$, for any fixed $A > 0$. Writing

$$\sum_{2\le n\le x, L_1<p(n)\le L_2}\frac{1}{\beta(n)} = \sum_{p(n)\|n} + \sum_{p^2(n)|n},$$

we then obtain

$$\begin{aligned}G(x) = &\sum_{L_1<p_1\le L_2}\ \sum_{m_1\le x/p_1, p(m_1)<p_1}\frac{1}{p_1+\beta(m_1)} +\\ &+O\Big(\sum_{L_1<p_1\le L_2}\frac{1}{p_1}\psi\Big(\frac{x}{p_1^2}, p_1\Big)\Big) + O(R).\end{aligned}$$

Again, writing

$$\sum_{m_1\le x/p_1 p(m_1)<p_1}\frac{1}{p_1+\beta(m_1)} = \sum_{L_1<p(m_1)<p_1, p(m_1)\|m_1} + \sum_{L_1<p(m_1)<p_1, p^2(m_1)|m_1} + \sum_{p(m_1)\le L_1},$$

we have

$$\begin{aligned}G(x) = &\sum_{L_1<p_1\le L_2}\ \sum_{L_1<p_2<p_1}\ \sum_{m_2\le x/p_1p_2, p(m_2)<p_2}\frac{1}{p_1+p_2+\beta(m_2)}\\ &+O\Big(\sum_{L_1<p_1\le L_2}\frac{1}{p_1}\sum_{L_1<p_2<p_1}\psi\Big(\frac{x}{p_1^2}, p_1\Big)\Big) + O\Big(\sum_{L_1<p_1<L_2}\frac{1}{p_1}\psi\Big(\frac{x}{p_1^2}, p_1\Big)\Big)\\ &+O\Big(\sum_{L_1<p_1\le L_2}\frac{1}{p_1}\sum_{p_2\le L_1}\psi\Big(\frac{x}{p_1p_2}, p_2\Big)\Big) + O(R).\end{aligned}$$

Proceeding as before, finally we have

(4.2)

$$G(x)=\sum_{L_1<p_1\le L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_s<p_{s-1}}\sum_{m_s\le x/p_1\cdots p_s,p(m_s)<p_s}\frac{1}{p_1+\cdots+p_s+\beta(m_s)}$$
$$+O\Big(\sum_{j=1}^{s}G_{1j}\Big)+O\Big(\sum_{j=2}^{s}G_{2j}\Big)+O(R),$$

where $2\le s\le \log_3 x$ is a large number which will be chosen latter, and

$$G_{1j}=\sum_{L_1<p_1\le L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_j<p_{j-1}}\frac{1}{p_1}\psi\Big(\frac{x}{p_1\cdots p_{j-1}p_j^2},p_j\Big), \tag{4.3}$$

$$G_{2j}=\sum_{L_1<p_1\le L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_{j-1}<p_{j-2}}\sum_{p_j\le L_1}\frac{1}{p_1}\psi\Big(\frac{x}{p_1\cdots p_j},p_j\Big). \tag{4.4}$$

Further, from (4.2) we have

$$\begin{aligned}G(x)&=\sum_{L_1<p_1\le L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_s<p_{s-1}}\frac{1}{p_1+\cdots+p_s}\psi\Big(\frac{x}{p_1\cdots p_s},p_s\Big)\\&+O\Big(\sum_{L_1<p_1\le L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_s<p_{s-1}}\sum_{m\le x/p_1\cdots p_s,p(m_s)<p_s}\frac{\beta(m_s)}{p_1^2}\Big)\\&+O\Big(\sum_{j=1}^{s}G_{1,j}\Big)+O\Big(\sum_{j=2}^{s}G_{2j}\Big)+O(R)\\&=G_3+(G_4)+O\Big(\sum_{j=1}^{s}G_{1,j}\Big)+O\Big(\sum_{j=2}^{s}G_{2j}\Big)+O(R),\ \text{say.}\end{aligned} \tag{4.5}$$

Now we come to the estimation of G_3. Changing the order of summation gives

$$G_3=\sum_{L_1<p_s\le L_2}\sum_{p_s<p_{s-1}\le L_2}\cdots\sum_{p_2<p_1\le L_2}\frac{1}{p_1+\cdots+p_s}\psi\Big(\frac{x}{p_1\cdots p_s},p_s\Big). \tag{4.6}$$

Nothing that $p_1>p_2>\cdots>p_s$, we get

$$\frac{1}{p_1+\cdots+p_s}=\sum_{k_1=0}^{\infty}\frac{(p_1+\cdots+p_{s-1}-p_s)^{k_1}}{2^{k_1+1}(p_1+\cdots+p_{s-1})^{k_1+1}}=$$
$$=\sum_{k_1=0}^{\infty}\frac{1}{2^{k_1+1}}\sum_{r_1=0}^{k_1}(-1)^{r_1}C_{k_1}^{r_1}\frac{p_s^{r_1}}{(p_1+\cdots+p_{s-1})^{r_1+1}},$$

and

$$\frac{1}{(p_1+\cdots+p_{s-1})^{r_1+1}}=\sum_{k_2=r_1}^{\infty}C_{k_2}^{r_1}\cdot\frac{1}{2^{k_2+1}}$$
$$\sum_{r_2=0}^{k_2-r_1}(-1)^{r_2}C_{k_2-r_1}^{r_2}\frac{p_{s-1}^{r_2}}{(p_1+\cdots+p_{s-2})^{r_1+r_2+1}},$$

where we used the following two formulas:

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k, \quad (-1<x<1), \tag{4.7}$$

$$\frac{1}{(1-x)^{r+1}}=\sum_{k=r}^{\infty}C_k^r x^{k-r}, \quad (-1<x<1). \tag{4.8}$$

Proceeding as before, finally we have

$$\begin{aligned}\frac{1}{p_1+\cdots+p_s}&=\sum_{k_1=0}^{\infty}\frac{1}{2^{k_1+1}}\sum_{r_1=0}^{k_1}(-1)^{r_1}C_{k_1}^{r_1}\sum_{k_2=r_1}^{\infty}C_{k_2}^{r_1}\cdot\frac{1}{2^{k_2+1}}\\ &\sum_{\gamma_2=0}^{k_2-r_1}(-1)^{r_2}C_{k_2-\gamma_1}^{r_2}\cdots\sum_{k_{s-1}=r_1+\cdots+r_{s-2}}^{\infty}C_{k_{s-1}}^{r_1+\cdots+r_{s-2}}\frac{1}{2^{k_{s-1}+1}}\\ &\sum_{r_{s-1}=0}^{k_{s-1}-r_1-\cdots-r_{s-2}}(-1)^{r_{s-1}}C_{k_{s-1}-r_1-\cdots-r_{s-2}}^{r_{s-1}}\cdot\frac{p_s^{r_1}\cdots p_2^{r_{s-1}}}{p_1^{r_1+\cdots+r_{s-1}+1}}=\\ &\sum_{k,r}F(k_1,\ldots,k_{s-1},r_1,\ldots,r_{s-1})p_s^{r_1}\cdots p_2^{r_{s-1}}p_1^{-(r_1+\cdots+r_{s-1}+1)}, \text{ say.}\end{aligned} \tag{4.9}$$

From (4.6) and (4.9) we get

$$\begin{aligned}G_3=\sum_{k,r}F(k_1,\ldots,k_{s-1},r_1,\ldots,r_{s-1})\sum_{L_1<p_s\le L_2}p_s^{r_1}\sum_{p_s<p_{s-1}\le L_2}p_{s-1}^{r_2}\cdots\\ \cdot\sum_{p_3<p_2\le L_2}p_2^{r_{s-1}}\sum_{p_2<p_1\le L_2}\frac{1}{p_1^{r_1+\cdots+r_{s-1}+1}}\psi\left(\frac{x}{p_1\cdots p_s},p_s\right).\end{aligned} \tag{4.10}$$

Let

$$u_i=\frac{\log(x/p_{i+1}\cdots p_s)}{\log p_s}, \quad \delta_i=\frac{\xi(u_i)}{\log p_s}, \quad i=1,2,\ldots,s-1,$$

Note that $s\le\log_3 x$ and $L_1<p_i\le L_2$. We then get

$$u_i=\left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right), \quad \log u_i=\frac{1}{2}\log_2 x+O(\log_3 x). \tag{4.11}$$

Thus from this and (3.3) we obtain

$$\xi(u_1)=(1/2)\log_2 x+O(\log_3 x), \qquad i=1,2,\ldots,s-1 \tag{4.12}$$

From (3.2), (4.11) and (4.12) we have $\exp(\xi(u_i)-\xi(u_{s-1}))=1+O(\log_3 x/\log_2 x)$. So $\xi(u_i)=\xi(u_{s-1})+O(\log_3 x/log_2 x)$, and therefore $\delta_i=\delta+O(\log_3 x/(\log p_s\cdot\log_2 x))$, where $\delta=\delta_{s-1}$. Hence for $L_1<p\le L_2$ we have

$$p^{\delta_i}=p^{\delta}(1+O(\log_3 x/\log_2 x)) \tag{4.13}$$

By Lemma 5 and (4.13) we have

$$\sum_{p_2<p_1\le L_2}\frac{1}{p_1^{r_1+\cdots+r_{s-1}+1}}\psi\Big(\frac{x}{p_1\cdots p_s},p_s\Big)$$

$$=\psi\Big(\frac{x}{p_2\cdots p_s},p_s\Big)\sum_{p_2<p_1\le L_2}\frac{1}{p_1^{r_1+\cdots+r_{s-1}+1}}\cdot\frac{1}{p_1^{1-\delta_1}}\Big(1+O\Big(\Big(\frac{\log_2 x}{\log x}\Big)^{1/2}\Big)\Big)$$

$$=\psi\Big(\frac{x}{p_2\cdots p_s},p_s\Big)\int_{p_2}^{L_2}\frac{1}{\xi^{r_1+\cdots+r_{s-1}+2-\delta}}d(\pi(\xi))\Big(1+O\Big(\Big(\frac{\log_2 x}{\log x}\Big)^{1/2}\Big)\Big)$$

$$=\psi\Big(\frac{x}{p_1\cdots p_s},p_s\Big)\frac{1}{r_1+\cdots+r_{s-1}+1}\cdot\frac{1}{p_2^{r_1+\cdots+r_{s-1}+1-\delta}M}\cdot$$

$$\cdot\Big(1+O\Big(\frac{\log_3 x}{\log_2 x}\Big)+O\Big(\Big(\frac{p_2}{L_2}\Big)^{r_1+\cdots+r_{s-1}+1-\delta}\Big)\Big),$$

where $M=((1/2)\log x\cdot\log_2 x)^{1/2}$, So (4.10) becomes

$$G_3=\sum_{k,r}\frac{F(k_1,\dots,k_{s-1},r_1,\dots,r_{s-1})}{r_1+\cdots+r_{s-1}+1}\sum_{L_1<p_s\le L_2}p_s^{r_1}\sum_{p_s<p_{s-1}\le L_2}$$

$$p_{s-1}^{r_2}\cdots\sum_{p_4<p_3\le L_2}p_3^{r_{s-2}}\sum_{p_3<p_2\le L_2}\frac{1}{p_2^{r_1+\cdots+r_{s-2}+1-\delta}M}\psi\Big(\frac{x}{p_2\cdots p_s},p_s\Big)\cdot$$

$$\cdot\Big(1+O\Big(\frac{\log_3 x}{\log_2 x}\Big)+O\Big(\Big(\frac{p_2}{L_2}\Big)^{r_1+\cdots+r_{s-1}+1-\delta}\Big)\Big).$$

Proceeding analogously, finally we have

$$G_3=\sum_{k,r}\frac{F(k_1,\dots,k_{s-1},r_1,\dots,r_{s-1})}{(r_1+\cdots+r_{s-1}+1)\cdots(r_1+1)}\sum_{L_1<p_s\le L_2}\frac{1}{p_s}\psi\Big(\frac{x}{p_s},p_s\Big)\cdot$$

$$\cdot\frac{p_s^{(r-1)\delta}}{M^{s-1}}\Big(1+O\Big(s\frac{\log_3 x}{\log_2 x}\Big)\Big)+O\Big(\sum_{j=1}^{s-1}G_{3j}\Big),$$

where

(4.14) $$G_{3j}=\sum_{k,\gamma}\frac{F(k_1,\dots,k_{s-1},r_1,\dots,r_{s-1})}{(r_1+\cdots+r_{s-1}+1)\cdots(r_1+\cdots+r_{s-j}+1)M^j}$$

$$\cdot\sum_{L_1<p_s\le L_2}p_s^{r_1}\sum_{p_s<p_{s-1}\le L_2}p_{s-1}^{r_2}\cdots\sum_{pj+3<pj+2\le L_2}p_{j+2}^{r_{s-j}+1}$$

$$\sum_{pj+2<pj+1\le L_2}\frac{1}{p_{j+1}^{r_1+\cdots+r_{s-j-1}+1-j\delta}}\psi\Big(\frac{x}{p_{1+1}\cdots p_s},p_s\Big)\cdot(r_{j+1}/L_2)^{r_1+\cdots+r_{s-j}+1-j\delta}.$$

From (3.2) and (3.3) we have $p_s^{(s-1)\delta}\cdot M^{-(s-1)}=1+O(s\log_3 x/\log_2 x)$, so that

(4.15) $$G_3=D_s\sum_{L_1<p_s\le L_2}\frac{1}{p_s}\psi\Big(\frac{x}{p_s},p_s\Big)\Big(1+O\Big(s\frac{\log_3 x}{\log_2 x}\Big)\Big)+O\Big(\sum_{j=1}^{s-1}G_{3j}\Big),$$

where

$$(4.16)\quad D_s=\sum_{k_1=0}^{\infty}\frac{1}{2^{k_1+1}}\sum_{r_1=0}^{k_1}\frac{(-1)^{r_1}C_{k_1}^{r_1}}{r_1+1}\sum_{k_2=r_1}^{\infty}\frac{C_{k_2}^{r_1}}{2^{k_2+1}}\sum_{r_2=0}^{k_2-r_1}\frac{(-1)^{r_2}C_{k_2-r_1}^{r_2}}{r_1+r_2+1}\cdots$$
$$\cdot\sum_{k_{s-1}=r_1+\cdots+r_{s-2}}^{\infty}\frac{C_{k_{s-1}}^{r_1+\cdots+r_{s-2}}}{2^{k_{s-1}+1}}\sum_{r_{s-1}=0}^{k_{s-1}-r_1-\cdots-r_{s-2}}\frac{(-1)^{r_{s-1}}C_{k_{s-1}-r_1-\cdots-r_{s-2}}^{r_{s-1}}}{r_1+\cdots+r_{s-1}+1}.$$

Next we show

$$(4.17)\qquad G_{3j}\ll R,\ \ (j=1,2,\dots,s-1)$$

Let $L_2'=\exp\Big\{\Big(\frac12\log x\log_2 x\Big)^{1/2}\Big(1+1.9\frac{\log_3 x}{\log_2 x}\Big)\Big\}$,

$D=\{(p_s,\dots,p_{j+1})\mid L_1<p_s<\cdots<P_{j+1}\le L_2\}$,

$D_1=\{(p_s,\dots,p_{j+1})\mid L_1<p_s<\cdots<p_{j+1}\le L_2'\}$,

$D_{2t}=\{(p_s,\dots,p_{j+1})\mid L_1<p_s<\cdots<p_{s-t+1}\le L_2', L_2'<p_{s-t}<\cdots$
$<P_{j+1}\le L_2\}$,

$D_3=\{(p_s,\dots,p_{j+1})\mid L_2'<p_s<\cdots<p_{j+1}\le L_2\}$.

So we may put

$$G_{3j}:=\sum_{(D)}=\sum_{(D_1)}+\sum_{t=1}^{s-j-1}\sum_{(D_{2t})}+\sum_{(D_3)}.$$

Now we come to the estimation of $\sum_{D_{2t}}$. Since $p_{j+1}/L_2\le 1$, by using Lemma 6 we obtain

$$\sum_{(D_{2t})}\ll\sum_{k,r}\frac{F(k_1,\dots,k_{s-1},r_1,\dots,r_{s-1})}{(r_1+\cdots+r_{s-1}+1)\cdots(r_1+\cdots+r_t+1)M^{s-t}}\sum_{L_1<p_s\le L_2'}p_s^{r_1}\cdots$$
$$\sum_{p_{s-t+2}<p_{s-t+1}\le L_2'}\frac{1}{p_{s-t+1}^{r_1+\cdots+r_{t-1}+1-(s-t)\delta}}\psi\Big(\frac{x}{p_{s-t+1}\cdots p_s},p_s\Big)$$
$$\cdot(p_{s-t+1}/L_2')^{r_1+\cdots r_t+1-(s-t)\delta}.$$

From this we know that

$$\sum_{(D_{21})}\ll\sum_{k,r}\frac{F(k_1,\dots,k_{s-1},r_1,\dots,r_{s-1})}{(r_1+\cdots+r_{s-1}+1)\cdots(r_1+1)N^{s-1}}\cdot$$
$$\cdot\sum_{L_1<p_s\le L'2}\frac{1}{p_s^{1-(s-1)\delta}}\psi\Big(\frac{x}{p_s},p_s\Big)\Big(\frac{p_s}{L_2'}\Big)^{r_1+1-(s-1)\delta}$$
$$\ll D_s\sum_{L_1<p_s\le L_2''}\frac{1}{p_s}\psi\Big(\frac{x}{p_s},p_s\Big)\Big(\frac{L_2''}{L_2'}\Big)^{r_1+1-(s-1)\delta}+D_s\sum_{L_2''<p_s\le L_2'}\frac{1}{p_s}\psi\Big(\frac{x}{p_s},p_s\Big)\ll R,$$

where $L_2'' = \exp\Big\{\Big(\frac{1}{2}\log x \log_2 x\Big)^{1/2}\Big(1 + 1.89\frac{\log_3 x}{\log_2 x}\Big)\Big\}$.

By the same argument as before, we obtain that $\sum_{(D_{2t})} \ll R$. Also, it is evident that

$$\sum_{(D_1)} \ll D_s \sum_{L_1 < p_s \le L_2'} \frac{1}{p_s}\psi\Big(\frac{x}{p_s}, p_s\Big)\Big(\frac{L_2'}{L_2}\Big)^{r_1+\cdots+r_{s-j}+1-j-\delta} \ll R,$$

$$\sum_{(D_3)} \ll D_s \sum_{L_2' < p_s \le L_2} \frac{1}{p_s}\psi\Big(\frac{x}{p_s}, p_s\Big) \ll R.$$

Thus it follows that (4.17) is true. So (4.15) becomes

$$G_3 = D_s \sum_{L_1 < p_s \le L_2} \frac{1}{p_s}\psi\Big(\frac{x}{p_s}, p_s\Big)\Big(1 + O\Big(s\frac{\log_3 x}{\log_2 x}\Big)\Big). \tag{4.18}$$

Next we come to the estimation of G_4 in (4.5). By the definition of $\beta(m)$, we have

$$G_4 = \sum_{L_1 < p_s \le L_2}\sum_{p<p_s} p \sum_{p_s < p_{s-1} \le L_2} \cdots \sum_{p_3 < p_2 \le L_2}\sum_{p_2 < p_1 \le L_2} \frac{1}{p_1^2}\psi\Big(\frac{x}{p_1\cdots p_s, p}, p_s\Big).$$

Let

$$u_i' = \frac{\log(x/p_{i+1}\cdots p_s p)}{\log p_s}, \quad \delta_i' = \frac{\xi(u_i')}{\log p_s}, \quad \delta' = \delta_{s-1}'.$$

As for p^{δ_i} of (4.13) we similarly obtain $p^{\delta_i'} = p^{\delta'}(1 + O(\log_3 x/\log_2 x))$, for $L_1 < p \le L_2$. By this and Lemma 5 we have

$$\begin{aligned}\sum_{p_2 < p_1 \le L_2} \frac{1}{p_1^2}\psi\Big(\frac{x}{p_1\cdots p_s p}, p_s\Big) &= \psi\Big(\frac{x}{p_2\cdots p_s p}, p_s\Big)\sum_{p_2 < p_1 \le L_2}\frac{1}{p_1^{3-\delta_1'}}(1+o(1))\\ &= \psi\Big(\frac{x}{p_2\cdots p_s p}, p_s\Big)\cdot\frac{1}{2p_2^{2-\delta'}M}(1+o(1)).\end{aligned}$$

Proceeding as before, we have finally

$$\begin{aligned}G_4 &= \frac{1}{2^{s-1}}\sum_{L_1 < p_s \le L_2}\frac{1}{p_s^{2-(s-1)\delta}M^{s-1}}\sum_{p<p_s} p\psi\Big(\frac{x}{p_s p}, p_s\Big)(1+o(1))\\ &= \frac{1}{2^{s-1}}\sum_{L_1 < p_s \le L_2}\frac{1}{p_s^2}\psi\Big(\frac{x}{p_s}, p_s\Big)\sum_{p<p_s} p^{\delta'}(1+o(1))\\ &\le \frac{1}{2^{s-1}}\sum_{L_1 < p_s \le L_2}\frac{1}{p_s}\psi\Big(\frac{x}{p_s}, p_s\Big)(1+o(1)).\end{aligned} \tag{4.19}$$

Now we show

$$G_{2j} \ll R, \qquad (j = 2, 3, \ldots, s). \tag{4.20}$$

We have

$$G_{22} = \sum_{L_1^{1/10} < p_2 \le L_1} \sum_{L_1 < p_1 \le L_2} \frac{1}{p_1} \psi\left(\frac{x}{p_1 p_2}, p_2\right)$$
$$+ \sum_{p_2 \le L_1^{1/10}} \sum_{L_1 < p_1 \le L_2} \frac{1}{p_1} \psi\left(\frac{x}{p_1 p_2}, p_2\right) = \sum_1 + \sum_2$$

By Lemma 6 we get

$$\sum_1 \ll \sum_{L_1^{1/10} < p_2 \le L_1} \psi\left(\frac{x}{p_2}, p_2\right) \sum_{L_1 < p_1 \le L_2} \frac{1}{p_1^{2-\Delta_1}} \ll \sum_{L_1^{1/10} < p_2 \le L_1} \psi\left(\frac{x}{p_2}, p_s\right) \frac{L_2^{\Delta_1}}{L_1 \log L_1},$$

where $\Delta_1 = \frac{1}{\log p_2} \xi\left(\frac{\log(x/p_2)}{\log p_2}\right)$. From (3.2) and (3.3) we have

$$\frac{L_2^{\Delta_1}}{\log L_1} \ll \frac{1}{\log L_1} \left(\frac{\log x}{\log p_2} \cdot \frac{1}{2} \log_2 x\right)^{(\log L_2)/(\log p_2)} \ll (\log_2 x)^{C_1}.$$

By Lemma 1 we have

$$\sum_1 \ll (\log_2 x)^{C_1} \sum_{L_1^{1/10} < p_2 \le L_1} \frac{1}{p_2} \psi\left(\frac{x}{p_2}, p_2\right) \ll R.$$

Using Lemma 2 and (1.7) we obtain

$$\sum_2 \ll \sum_{p_2 < L_1^{1/10}} \sum_{L_1 < p_1 \le L_2} \frac{1}{p_1} \psi\left(\frac{x}{p_1 p_2}, L_1^{1/10}\right)$$
$$\ll x \exp\{-4(\log x \log_2 x)^{1/2}\} \sum_{p_2 \le L_1^{1/10}} \frac{1}{p_2} \sum_{L_1 < p_1 \le L_2} \frac{1}{p_1^2} \ll R.$$

Hence, we have

$$G_{22} \ll R. \tag{4.21}$$

Let $\Delta_j = \frac{1}{\log p_j} \xi\left(\frac{\log(x/p_2 \cdots p_j)}{\log p_j}\right)$. By Lemma 6 we have

$$\begin{aligned} G_{2j} &= \sum_{p_j \le L_1} \sum_{p_j < p_{j-1} \le L_2} \cdots \sum_{p_2 < p_1 \le L_2} \frac{1}{p_1} \psi\left(\frac{x}{p_1 \cdots p_j}, p_j\right) \\ &\ll \sum_{p_j \le L_1} \sum_{p_j < p_{j-1} \le L_2} \cdots \sum_{p_3 < p_2 \le L_2} \psi\left(\frac{x}{p_2 \cdots p_j}, p_j\right) \sum_{p_2 < p_1 \le L_2} \frac{1}{p_1^{2-\Delta_j}} \\ &\ll \sum_{p_j \le L_1} \sum_{p_j < p_{j-1} \le L_2} \cdots \sum_{p_3 < p_2 \le L_2} \frac{1}{p_2} \psi\left(\frac{x}{p_2 \cdots p_j}, p_j\right) (\log_2 x)^{C_1} \\ &\ll (\log_2 x)^{C_1} G_{2,j-1}. \end{aligned} \tag{4.22}$$

From (4.21) and (4.22). we can derive (4.20).

Next we come the estimation of G_{1j}. Changing the order of summation gives

$$G_{1j} = \sum_{L_1 < p_j \le L_2} \sum_{p_j < p_{j-1} \le L_2} \cdots \sum_{p_2 < p_1 \le L_2} \frac{1}{p_1} \psi\left(\frac{x}{p_1 \cdots p_{j-1} p_j^2}, p_j\right).$$

Using Lemma 6 repeatedly, we get

$$\begin{aligned} G_{1j} &\ll (\log_2 x)^{C_1 S} \sum_{L_1 < p_j \le L_2} \frac{1}{p_j} \psi\left(\frac{x}{p_j^2}, p_j\right) \\ &\ll (\log_2 x)^{C_1 S} \sum_{L_1 < p_j \le L_2} \frac{1}{p_j} \psi\left(\frac{x}{p_j}, p_j\right) \cdot \frac{1}{p_j^{1/2}} \ll R. \end{aligned} \tag{4.23}$$

From (4.5), (4.18), (4.20) and (4.23), and noting that $1/2 < D_s < 1$ (see (4.30) below), we obtain

$$G(x) = D_s \sum_{L_1 < p_s \le L_2} \frac{1}{p_s} \psi\left(\frac{x}{p_s}, p_s\right) \left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right) + O\left(\frac{1}{2^s}\right)\right).$$

Now if we put $s = s_0 = [\log_3 x / \log 2]$, we have

$$G(x) = D_{s_0} \sum_{L_1 < p \le L_2} \frac{1}{p} \psi\left(\frac{x}{p}, p\right) \left(1 + O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right), \tag{4.24}$$

where D_s is defined as in (4.16).

To finish the proof of the theorem is remains to simplify the expression for D_{s_0}. We shall use the following three formulas:

$$\sum_{r=0}^{k-h} (-1)^r C_{k-h}^r \frac{\xi^{h+r+1}}{h+r+1} = \int_0^{\xi} (1-x)^{k-h} x^h dx, \tag{4.25}$$

$$\sum_{r=0}^{k-h} (-1)^r C_{k-h}^r \frac{1}{h+r+1} = (k+1)^{-1} (C_k^h)^{-1}, \tag{4.26}$$

$$\sum_{k=r}^{\infty} \frac{1}{2^{k+1}(k+1)} = \int_0^{\frac{1}{2}} \frac{x^r}{1-x} dx. \tag{4.27}$$

By (4.26) and (4.27), we have

$$\begin{aligned} &\sum_{k_{s-1}=r_1+\cdots+r_{s-2}}^{\infty} \frac{C_{k_{s-2}}^{r_1+\cdots+r_{s-2}}}{2^{k_{s-1}+1}} \sum_{r_{s-1}=0}^{k_{s-1}-r_1-\cdots-r_{s-2}} \frac{(-1)^{r_{s-1}} C_{k_{s-1}-r_1-\cdots-r_{s-2}}^{r_{s-1}}}{r_1+\cdots+r_{s-1}+1} \\ &= \sum_{k_{s-1}=r_1+\cdots+r_{s-2}}^{\infty} \frac{1}{2^{k_{s-1}+1}(k_{s-1}+1)} = \int_0^{\frac{1}{2}} \frac{x_1^{r_1+\cdots+r_{s-2}}}{1-x_1} dx_1. \end{aligned}$$

By (4.25) and (4.8) we further have

$$\sum_{k_{s-2}=r_1+\cdots+r_{s-3}}^{\infty} \frac{C_{k_{s-2}}^{r_1+\cdots+r_{s-3}}}{2^{k_{s-2}+1}} \sum_{r_{s-2}=0}^{k_{s-2}-r_1-\cdots-r_{s-3}} \frac{(-1)^{r_{s-2}} C_{k_{s-2}-r_1-\cdots-r_{s-3}}^{r_{s-2}}}{r_1+\cdots+r_{s-2}+1} \int_0^{\frac{1}{2}} \frac{x_1^{r_1+\cdots+r_{s-2}}}{1-x_1} dx_1$$
$$\sum_{k_{s-2}=r_1+\cdots+r_{s-3}}^{\infty} \frac{C_{k_{s-2}}^{r_1+\cdots+r_{s-3}}}{2^{k_{s-2}+1}} \int_0^{\frac{1}{2}} \frac{dx_1}{x_1(1-x_1)} \int_0^{x_1} (1-x_2)^{k_{s-2}-r_1-\cdots-r_{s-3}} x_2^{r_1+\cdots+r_{s-3}} dx_2$$
$$= \int_0^{\frac{1}{2}} \frac{dx_1}{x_1(1-x_1)} \int_0^{x_1} \frac{x_2^{r_1+\cdots+r_{s-3}}}{(1+x_2)^{r_1+\cdots+r_{s-3}+1}} dx_2 .$$

Proceeding as before, finally we have

$$D_s = \int_0^{\frac{1}{2}} \frac{dx_1}{x_1(1-x_1)} \int_0^{x_1} \frac{dx_2}{x_2} \int_0^{\frac{x_2}{1+x_2}} \frac{dx_3}{x_3} \cdots \int_0^{\frac{x_{s-3}}{1+x_{s-3}}} \frac{dx_{s-2}}{x_{s-2}} \int_0^{\frac{x_{s-2}}{1+x_{s-2}}} \frac{dx_{s-1}}{1+x_{s-1}}, \; s \geq 3.$$

If we put

$$x_1 = x_1', x_2 = x_1'x_2', x_3 = \frac{x_1'x_2'x_3'}{1+x_1'x_2'}, \cdots, x_{s-1} = \frac{x_1' \cdots x_{s-1}'}{1+x_1'x_2'+\cdots+x_1'\cdots x_{s-2}'},$$

we then have

(4.28)
$$D_s = \int_0^{\frac{1}{2}} \frac{dx_1}{1-x_1} \int_0^1 dx_2 \cdots \int_0^1 dx_{s-2} \int_0^1 \frac{dx_{s-1}}{1+x_1x_2+\cdots+x_1\cdots x_{s-1}}, \; s \geq 3.$$

From this, it is easy to see that

$$0 < D_{s+1} < D_s < D_2 = \int_0^{\frac{1}{2}} \frac{dx}{1-x} = \log 2, \quad (s = 3, 4, \cdots,).$$

Hence

(4.29) $$D = \lim_{s\to\infty} D_s$$

exists. Obviously, we have

$$D_s - D_{s+1} \leq \int_0^{\frac{1}{2}} \frac{dx_1}{1-x_1} \int_0^1 dx_2 \cdots \int_0^1 dx_{s-2} \int_0^1 x_1 \cdots x_{s-1} dx_{s-1} = \left(\log 2 - \frac{1}{2}\right) 2^{-(s-1)}$$

Hence

$$D_s > D_3 - (\log 2 - (1/2)) \cdot 2^{-1}.$$

Since

$$D_3=\int_0^{\frac{1}{2}}\frac{\log(1+x_1)}{x_1(1-x_1)}dx_1\geq\int_0^{\frac{1}{2}}\frac{1}{x_1(1-x_1)}\left(x_1-\frac{x_1^2}{2}+\frac{x_1^3}{3}-\frac{x_1^4}{4}+\frac{x_1^5}{5}-\frac{x_1^6}{6}\right)dx=0.6140,$$

so that

$$0.5174<D<D_s<\log 2,\quad (s\geq 3). \tag{4.30}$$

Also, it is evident that

$$0<D_{s_0}-D=\sum_{s=s_0}^{\infty}(D_s-D_{s+1})\leq\sum_{s=s_0}^{\infty}\left(\log 2-\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)^{s-1}$$
$$=(4\log 2-2)\cdot 2^{-s_0}.$$

Recalling that $S_0=[\log_3 x/\log 2]$ we obtain

$$D_{s_0}=D+O(1/\log_2 x). \tag{4.31}$$

From this and (4.24), the theorem follows.

5. Proofs of Theorems 2, 3 and 4. *Proof of Theorem*2. We shall only sketch the proof of Theorem 2. As for $G(x)$ in (4.5) we obtain similarly

$$W(x):=\sum_{2<n\leq x}\frac{\omega(n)}{\beta(n)}=\sum_{L<p_1\leq L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_s<p_{s-1}}\frac{1}{p_1+\cdots+p_s}\cdot$$
$$\cdot\sum_{m\leq x/p_1\cdots p_s,p(m)<p_s}\omega(m)+O\Bigg(\sum_{L_1<p_1\leq L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_s<p_{s-1}}\frac{1}{p_1^2}\cdot$$
$$\cdot\sum_{m\leq x/p_1\cdots p_s,p(m)<p_s}\omega(m)\beta(m)\Bigg)+O(SR=W_1+O(W_2)+O(SR),\ \text{say},$$

where R is defined in Section 4. By[**8**], Lemma 6, we have

$$W_1=\sum_{L<p_1\leq L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_s<p_{s-1}}\frac{1}{p_1+\cdots+p_s}\cdot$$
$$\cdot\psi\left(\frac{x}{p_1\cdots p_s},p_s\right)\left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1+O\left(\frac{\log_3 x}{log_2 x}\right)\right)=G_3\left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1+O\left(\frac{\log_3 x}{log_2 x}\right)\right).$$

By the definition of $\beta(m)$ and by Lemma 6 of [**8**], we have

$$W_2\ll\sum_{L_1<p_1\leq L_2}\sum_{L_1<p_2<p_1}\cdots\sum_{L_1<p_s<p_{s-1}}\frac{1}{p_1^2}\sum_{p<p_s}p\cdot \tag{5.3}$$
$$\cdot\sum_{m'\leq x/p_1\cdots p_s p,p(m')<p_s}\omega(m')\ll G_4\left(\frac{\log x}{\log_2 x}\right)^{1/2}.$$

From (5.1), (5.2) and (5.3) the theorem follows.

Proofs of Theorems 3 and 4 are similar to the proof of Theorem 2, but they use Theorem 1 of [**8**] and Lemma 5 of [**8**] instead of Lemma 6 of [**8**], recpectively.

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