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# AN IMPROVED CONSTANT FOR THE MUNTZ-JACKSON THEOREM

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**Abstract**. We improve a Newman result [**2,3**] from 1974 concerning approximation of a continuous function by generalized polynomials. He proved that every $f \in C[0,1]$ there exists a generalized polynomial $P(x) = \sum_{k=0}^{N} c_k x^{\lambda k}$ such that

$$(1) \qquad \mid f(x) - P(x) \mid \leq A w_f(\varepsilon), \qquad x \in [0,1]$$

holds. Here $0 = \lambda_0 < \lambda_1 < \cdots \lambda_N$ are given numbers $w_f$ is the modulus of continuity of $f$, $\varepsilon = \max\{\mid B(z)/z \mid \Re z = 1\}$, $B(z)$ is the Blaschke product corresponding to the above set of $\lambda_k$'s and $A$ is a constant. Newman [**2**] proved that (1) holds with $A = 368$. We show that (1) is valid with $A = 66$. We prove this by slightly modifying Newman's proof and choosing the size of an interval, to which a suitable contradiction is extended, optimally.

Muntz's theorem gives a necessary and sufficient condition for the linear hull of the power functions

$$\{x^{p_0}, x^{p_1}, x^{p_2}, \dots\} \ (1 = p_0 < p_1 < \dots, p_n \to \infty \ \text{ as } \ n \to \infty)$$

to be dense in $C[0,1]$. Jackon's theorem determines the rate of approximation of a continuous function by polynomials in terms of the modulus of continuity. Several authors tried to combine the two theorems. Let $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N$ be given numbers, $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_N\}$ and denote by $P_*$ the set of all generalized polynomials, i. e.

$$P_* = \{P(x) = \sum_{k=0}^{N} C_k x^{\lambda_K} \mid \lambda_K \in \Lambda, \ C_k \ \text{ real}; \ \ k = 0, \dots N\}$$

How well can a function $f \in C[0,1]$ be approximated by elements of $p_*$? After several attempts the correct rate of approximation was found by Newman [**3**] in 1974.

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Theorem 1(Newman [**3, 2**]). *For every $f \in C[0,1]$ there exists a generalized $P \in P_*$ such that*

$$| f(x) - P(x) | \leq A w_f(\varepsilon) \tag{1}$$

*where $A$ is a universal constant, $w_f$ is the modulus of continuity of $f$ and $\varepsilon_*$ is given by*

$$\varepsilon_* = \max_{\Re z - 1} | B_*(z)/z |$$

*$B_*(z)$ being the Blaschke product corresponding to $\Lambda$, i.e.*

$$B_*(z) = \prod_{k=1}^{N} \frac{z - \lambda_k}{z + \lambda_k} \tag{3}$$

In [**3**] Newman proved that (1) holds with $A = 100$. This proof however seems to contain some minor errors which influence the value of $A$.

Let $S = S[a,b]$ the class of real valued functions $g$ defined on $[a,b]$ which satisfy $| g(x) - g(y) | \leq | x - y |$. Functions of $S$ will be called contraction on $[a,b]$. It can be shown [**2**, p. 122] that $g \in S[a,b]$ if end only if $g$ is absolutely continuous and $| g'(x) | \leq 1$ a. e. in $[a,b]$. In [**3**, p. 341] the formula (1) does not seem to hold with the constat 8 (but it certainly holds with $6\pi$ instead of 8). On p. 342 $N = 1/4\varepsilon$ is substituted. Since $N$ is an integer there (degree of a polynomial) only $N = [1/4\varepsilon]$ can be taken, but for small values of $N$, $1/N = [1/4\varepsilon]^{-1}$ can increase to $8\varepsilon$.

In [**2**] a slightly different proof, free of the above errors, was given for Thorem 1; however, the constant was increased to 368.

In this note we make an effort to reduce the value of $A$ in (1).

Theorem 2. *Theorem 1 is valid with $A = 66$.*

We follow the notations and proof of [**2**] with some changes. Let, for an $f \in C[0,1]$,

$$E_*(f) = \inf_{P \in P_*} \|f - P\|$$

be the distance of $f$ from the subspace $P_*$ and let

$$\rho_* = \sup_{g \in S[0,1]} E_*(g).$$

Instead of (1) we show that for every $g \in S = S[0,1]$:

$$E_*(g) \leq 33\varepsilon_*; \tag{4}$$

thus, $\rho_* \leq 33\varepsilon_*$. This implies (1) since by Theorem (1) of [**2**, p. 122] we have

$$E_*(f) \leq 2w_f(\rho_*) \leq 2w_f(33\varepsilon_*) \leq 66 w_f(\varepsilon_*). \tag{5}$$

To prove (4) let $g \in S[0,1]$, and extend $g$ to the interval $[-a,a]$, $a > 1$ such that it remains a contraction on $[-a,a]$. We need the following

LEMMA 1. *There exists an algebraic polynomial $P_M$ of degree $\leq M$ such that*

$$| g(x) - P_M(x) | \leq \frac{\pi^2}{2} \cdot \frac{a}{M+1}, \quad \textit{if } x \in [-a, a] \tag{6}$$

$$| P_M'(x) | \leq \frac{a}{\sqrt{a^2 - 1}} \quad \textit{if } x \in [-1, 1], \tag{7}$$

*Proof*. Let $f(u) = g(a \cos u),\ u \in [-\pi, \pi]$. Then

$$| f(u+t) - f(u) | = | g(a \cos(u+t)) - g(a \cos u) | \leq a \mid \cos(u+t) - \cos u | \leq | t | .$$

Let further

$$\sigma_M(u) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+t) K_M(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) K_M(u-y) dy,$$

where $K_M(t) = 1/2 + \sum_{k=1}^{N} \rho_{M_k}\ \cos\ kt$, be such that $K_M(t) \geq 0,\ \ t \in [-\pi, \pi]$ and $\rho_{M1} = \cos \pi / M + 2$.

The existence of such a $K_M$ is well-known [**1**, p. 337, Lemma 13.3.5]. Using Korovkin's estimate [**1**, pp. 335–336, Lemma 13.3.5] we get

$$\begin{aligned} | \sigma_M(u) - f(u) | &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} | f(u+t) - f(u) | K_M(t) dt \\ &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} a \mid t \mid K_M(t) dt \leq \frac{a\pi}{\sqrt{2}} \sqrt{1 - \rho_{M1}} \\ &= a\pi \mathrm{Sin} \frac{\pi}{2(M+2)} \leq \frac{\pi^2}{2} \frac{a}{M+1}. \end{aligned} \tag{8}$$

The function $f$ is even; thus $\sigma_M$ is $a$ cosine polynomial and

$$P_M(x) = \sigma_M\left(\arccos \frac{x}{a}\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} g\left[a \cos\left(\arccos \frac{x}{a} + t\right)\right] K_M(t) dt \tag{9}$$

is an algebraic plynomial of degree $\leq M$. From (8) we obtain (6) by substituting $u = \arccos x/a$. To prove (7) we notice that from (9)

$$P_M'(x) = -(a^2 - x^2)^{-1/2} \sigma_M'(\arccos x/a) \ \text{ a.e.}$$

Using (9) and the inequalities $| g'(x) | \leq 1$ a.e., $K_M(t) \geq 0$ we obtain

$$| P_M'(x) | \leq a(a^2 - x^2)^{-1/2} \leq a(a^2 - 1)^{-1/2} \quad \text{if} \quad x \in [-1, 1]$$

which proves (7).

Let us now return to the proof of Theorem 2. By (7) $\|P_M'\| C[-1, 1] \leq a(a^2 - 1)^{-1/2}$ therefore, applying Theorem 12 of [**2**, p. 21] for $P_M'$ we get

$$| P_M^{(k)}(0) | \leq (M-1)^{k-1} \|P_M'\| < a(a^2 - 1)^{-1/2} M^{k-1}.$$

Since $P_M(x) = \sum_{k=1}^{M} \frac{P_M^{(k)}(0)}{k!} x^k$ by Lemmas 5, 6 of [**2**, p. 125] for every $k = 1, 2, \ldots$ there is a $q_k \in P_*$ such that

$$| x^k - q_k(x) | \leq | B(k) | \leq (\varepsilon_* k)^k \quad x \in [0,1].$$

With $Q(x) = \sum_{k=0}^{M} \frac{|P_M^{(k)}(0)|}{k!} q_k(x) \quad (q_0(x) = 1)$ we have $Q \in P_*$ and

$$\begin{aligned} | P_M(x) - Q(x) | &\leq \sum_{k=1}^{M} \frac{| P_M^{(k)}(0) |}{k!} (\varepsilon_* k)^k \\ &\leq \sum_{k=1}^{M} \frac{a}{\sqrt{a^2-1}} \frac{M^{k-1}}{k} (\varepsilon_* k)^k \text{ if } x \in [0,1]. \end{aligned} \tag{10}$$

The inequality (6) implies that

$$| g(x) - P_M(x) | < \frac{\pi^2}{2} \frac{a}{M+1} \text{ if } x \in [0,1], \tag{11}$$

thus, by the triangle inequality (10), (11), we obtain

$$E_*(g) \leq | g(x) - Q(x) | \leq \frac{\pi^2}{2} \frac{a}{M+1} + \frac{1}{M} \frac{a}{\sqrt{a^2-1}} \sum_{k=1}^{M} \frac{1}{k!} (M\varepsilon_* k)^k. \tag{12}$$

LEMMA 2. (13) *$k! \geq \sqrt{2k/3e}(k/e)^k$ for $k = 1, 2, \ldots$*

*Proof*. Let $A_k$ be the area bounded by the curves $y = \ln x$, $y = 0$, $x = 1.5$ and $x = k + 0.5$. Then

$$A_k = \int_{1.5}^{k+0.5} \ln x dx = \ln(k + 1/2)^{k+1/2} - \ln(3/2)^{3/2} - (k-1)$$

and since the function $y = \ln x$ is concave, we have $A_k \leq T_k = t_2 + t_3 + \cdots + t_k t_n$ being the area bounded by $x = n - 1/2$, $x = n + 1/2$, $y = 0$ and $y - \ln x = 1/n(x-n)$ (the tangent line to $y = \ln x$ at $x = n$). We find that $T_k = \sum_{k=2}^{k} \ln n = \ln k!$ Using the inequality

$$(k + 1/2)^k > \binom{k}{0} k^k + \binom{k}{1} k^{k-1} \cdot 1/2 = 3/2 k^k$$

we easily obtain (13) from $A_k \leq T_k$.

By Lemma 2 our formula (12) can be written as

$$E_*(g) \leq \frac{\pi^2}{2} \frac{a}{M+1} + \frac{1}{e\sqrt{3/2}} \frac{a}{\sqrt{a^2-1}} \frac{1}{M} \sum_{k=1}^{M} (M\varepsilon_* k)^k. \tag{13}$$

Let

$$M = M(\varepsilon_*) = \begin{cases} [1/4\varepsilon_*], & \text{if } \ 0 < \varepsilon_* \leq 1/8 \\ 1, & \text{if } \ 1/8 < \varepsilon_* \end{cases}$$

First we show that

$$1/M + 1 < 4\varepsilon_*. \tag{14}$$

If $0 < \varepsilon_* \le 1/8$, then $1/4\varepsilon_* - 1 \le M \le 1/4\varepsilon_*$; therefore $1/4\varepsilon_* < M + 1$ which implies the inequality (14). If $1/8 < \varepsilon_*$, Then $M = 1$ and again $1/4\varepsilon_* < 2 = M + 1$

Secondly we show that

$$\frac{1}{M}\sum_{k=1}^{M}(e\varepsilon_* M)^k \le 5e/4 - e\varepsilon_* \tag{15}$$

We shall distinguish five cases.

Case 1. $0 < \varepsilon_* 1/20$ then $1/5\varepsilon_* \le 1/4\varepsilon_* - 1 < M \le 1/4\varepsilon_*$. Hence,

$$1/M < 5\varepsilon_* \text{ and } \sum_{k=1}^{M}(e\varepsilon_* k)^k \le \sum_{k=1}^{M}\left(\frac{e}{4}\right)^k \le \sum_{k=1}^{\infty}\left(\frac{e}{4}\right)^k - \frac{e}{4-e}$$

which implies (15).

Case 2. $1/20 < \varepsilon_* < 1/16$. Now $M = 4$ and $1/5\varepsilon_* = 20/5 = 4 < 1/4\varepsilon_*$. Hence $1/M < 5\varepsilon_*$ and, as in case 1, $\sum_{k=1}^{M}(e\varepsilon_* k)^k = e/4 - e$.

Case 3. $1/16 < \varepsilon_* < 1/12$. Now $M = 3$ and $\frac{1}{16/3\varepsilon_*} < \frac{16}{16/3} = M \le 1/4\varepsilon_*$. Hence, $1/M < 16/3\varepsilon_*$ and $\sum_{k=1}^{M}(e\varepsilon_* k)^k \le \sum_{k=1}^{3}(e/4)^k \le 1.46$. Since $16/3.1.46 < 5e/4 - e$, (2) is again valid.

Case 4. If $1/12 < \varepsilon_* \le 1/8$, then $M = 2$ and $1/6\varepsilon_* < 12/6 = M = 1/4\varepsilon_*$. Hence $1/M < 6\varepsilon_*$ and $\sum_{k=1}^{M}(e\varepsilon_* M)^k \le \sum_{k=1}^{2}(e/4)^k \le 1.15$. Since $6(1.15) < 5e/4 - e$, (2) is valid.

Case 5. $1/8 < \varepsilon$. Now $M = 1$, $1/M = 1$. On the other hand, $\sum_{k=1}^{M}(e\varepsilon_* M)^k = e\varepsilon_*$; thus, by $e < 5e/4 - e$, (2) is again valid. Thus for all values of $\varepsilon_* > 0$ we can use (1) and (2) to estimate $E_*(g)$. We obtain

$$E_*(g) \le \left[\left(4 \cdot \frac{\pi^2}{2}\right)a + \sqrt{\frac{3}{2}}\frac{5}{4-e}\frac{a}{\sqrt{a^2-1}}\right]\varepsilon_*.$$

Now we would like to choose $a > 1$ such that the expression in the bracket has a minimal value. Denote the function in bracket by $h = h(a)$. Since

$$\lim_{a\to 1+0} h(a) = \lim_{a\to\infty} h(a) = +\infty, \quad h(a) > 0$$

and $h$ is continuous, it a positive minimum for $a > 1$ The equation $h'(a) = 0$ gives

$$(a^2-1)^{3/2} = \sqrt{\frac{3}{2}}\frac{5}{4-e}\left[4 \cdot \frac{\pi^2}{2}\right]^{-2} \approx 0.2420442.$$

Hence, $a = a_1 \approx 1.1782978$. Since $h''(a) = \sqrt{3/2}5/4 - e3a(a^2-1)^{-5/2} > 0$, $h$ assumes its minimum at $a = a_1$, $\min h(a) = h(a_1) = 32.391968 \le 33$ and finally $E_*(g) \le 33\varepsilon_*$. This is exactly (4) which implies (5), proving Theorem 2.

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