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LINEARLY SINGLY BI-k-SPACES

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Abstract. We characterize and study the images, under two kinds of continuous mapings, of topological spaces which are the perfect pre-images of τ-metrizable spaces. We also give several results concerning strongly radial spaces.

0. Introduction and basic definitions

From the early 1960's, when P. S. Alexandroff suggested the classification of topological spaces via mappings, many topologists have become increasingly interested in nvestigating which classes of spaces are images or pre-images of "nice" spaces under "nice" mappings. Usually "nice" spaces were metrizable spaces. Many authors have obtained beautiful and important results of this sort, so that we now have an extensive knowledge about these matters (see, for example, [**4, 11**] and [**12**]). Clearly, these results have a cardinality restriction—countability. It is interesting to study to what extent these results can be translated in the theory of continuous images of τ-metrizable or (generalized) orderable spaces. As might be expected, many results in the theory of so-called "generalized metric spaces" have their analogous, "linearly" of "higher cardinality" generalizations (even with similar proofs), in the theory of continuous images of τ-metrizable or orberable spaces. This paper studies several problems of this type. For other investigations with such intentions see, for example, [**2, 5, 6, 8, 9, 15**] and [**16**].

Notation and terminology in this paper are standard [**3**]. All spaces are Hausdorff and all mappigs are continuous and onto, τ always denotes an infinite (usually regular) cardinal, which is the smallest ordinal of a certain cardinality. $t(X)$ denotes the tightness of a space X, the smallest cardinal τ such that for every $A \subset X$ and every $x \in \tilde{A}$ there exists a set $B \subset A$ with $\mid B \mid\leq \tau$ and $x \in \overline{B}$. For any cardinal τ, a τ-sequence $s = (x_\alpha : \alpha \in \tau)$ (or simply (x_α)) in a space X is a function from τ into X. We use $T(\tau)$ to denote a convergent τ-sequence (x_α) together with its limit x toplogized so that every x_α is isolated and a base at x is the collection of all sets of the form $\{x\} \cup \{x\beta : \beta \geq \alpha\}$, $\alpha \in \tau$.

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We now define some spaces and mappings which play an important role in this paper.

0.1. A mapping $f : X \to Y$ is called *pseudo-open* if for every $B \subset Y$ and every $y \in B$, $x \in \overline{f^{-1}B)}$ for some $x \in f^{-1}(y)$.

0.2. A space X is *radial* (or Fréchet chain net) if for any $A \subset X$ and any $x \in \tilde{A}$ there is a τ-sequence in A (for some τ) converging to x [**5**]. Radial spaces are exactly the pseudo-open images of orderable spaces ([**2, 5**]).

0.3. A space X is called *strongly radial* if there is some cardinal τ such that whenever $(A_\alpha : \alpha \in \tau)$ is a decreasing τ-sequence of subsets of X-accumulating at $x \in X$ (i. e. $x \in \tilde{A}_\alpha$ for each $\alpha \in \tau$), then there are $x_\alpha \in A_\alpha$, $\alpha \in \tau$, such that (x_α) converges to x [**8**].

0.4. If τ is a regular cardinal, then a completely regular space X is called τ-*metrizable* (or linearly uniformizable) if its topology can be induced by a uniformity having a well-ordered base of order type τ.

These spaces were introduced by Kurepa [**10**] under the name pseudo-distancial spaces; the present definition is one of the characterizations of these spaces (for details see [**7**]).

0.5. A space X is said to be an lob-space if every one of its points has a neighbourhood base which is linearly ordered by reverse inclusion.

1. Strongly radial spaces

In [**8**] the author has defined the notion of a τ-bi-quotient mapping as a direct generalization to higher cardinals of the notion of a countably bi-quotient mapping introduced in [**13**] (see also [**11**] and [**12**]).

Definition 1.1. Let τ be a cardinal. A mapping $f : X \to Y$ is called τ-*bi-quotient* if, whenever $(B_\alpha : \alpha \in \tau)$ is a decreasing τ-sequence of subsets of Y accumulating at $y \in Y$, then $(f^{-1}(B_\alpha))$ accumalates at some $x \in f^{-1}(y)$ [**8**].

In the same paper [**8**] strongly radial spaces were introduced and it was proved that they are precisely the τ-bi-quotient images (for suitable τ) of orderable (and lob-) spaces. In this section we shall give three further facts about these spaces, the first of which is similar to Proposition 4.D.5. of [**11**] (which states that a space X is strongly Fréchet iff $X \times [0, 1]$ is Fréchet) and the second is analogous to a result obtained independently by Arhangel'skii [**1**] and Olson [**13**] (it says that every countably compact Fréchet space is strongly Fréchet).

We begin with a few lemmas whose proofs can be obtained by minor modifications of the proofs of Propositions 4.3, 4.4, 8.1 and 8.6 of [**11**] concerning countably bi-quotient mappings.

LEMMA 1.2. *If $f : X \to Y$ is a τ-bi-quotient mapping and $g : A \to B$ is a bi-quotient mapping* [**11**] *from a space A to a biradial space* [**9**] *B of tightness τ, then the mapping $f \times g : X \times A \to Y \times B$ is also τ-bi-quotient.*

LEMMA 1.3. *A mapping $f : X \to Y$ is τ-bi-quotient if and only if the maping $f \times i_T : X \times T(\tau) \to Y \times T(\tau)$ is pseudo-open, where, i_T is the identity mapping on $T(\tau)$.*

LEMMA 1.4. *Let $f : X \to Y$ be a quotient mapping. If Y is strongly τ-radial, then f is τ-bi-quotient.*

LEMMA 1.5. *Let $f : X \to Y$ be a mapping and let $t(X) \leq \tau$. If the mapping $f_B = f \mid f^{-1}(B) : f^{-1}(B) \to B$ is τ-bi-quotient for every $B \subset Y$ of cardinality $\leq \tau$, then f is a τ-bi-quotient mapping.*

THEOREM 1.6. *If there exists a cardinal τ such that $X \times T(\tau)$ is a radial space of tightness τ, then X is strongly radial.*

Proof. We shall prove that X is strongly τ-radial. Let $(A_\alpha : \alpha\tau)$ be a decreasing τ- sequence of subset of X accumulating at $x \in X$. Define $M \subseteq X$ by $M = \cup\{A_\alpha \times \{\alpha\} : \alpha \in \tau\}$. Obviously, $(x, \tau) \in \overline{M}$. Since, by assumption, $X \times T(\tau)$ is radial and $t(X \times T(\tau)) = \tau$, there is a τ-sequence (p_α) in M converging to (x, τ). Without loss of generality we may assume that $p_\alpha \in A_\alpha \times \{\alpha\}$ for every $\alpha \in \tau$. If π denotes the projection of $X \times (\tau)$ onto X, then one easily sees that the τ-sequence $(\pi(p_\alpha))$ which converges to x witnesses that X is a strongly τ-radial space. The theorem is proved.

Recall now that a space X is initially τ-compact, τ is a cardinal, iff every subset $A \subset X$ with $\mid A \mid\leq \tau$ has a complete accumulation point in X (as usual, x is a complete accumulation point of $A \subset X$ if for each neighbourhood U of x, $\mid A \cap U \mid=\mid A \mid$).

THEOREM 1.7. *regular initially τ compact radial space X of tightness τ is strongly radial.*

Proof. In fact, we prove that X is strongly τ-radial. Let $(A_\alpha : \alpha \in \tau)$ be a decreasing τ-sequence of subsets of X accumulating at $p \in X$. Put

$$M = \{x \in X;\ \text{ there are } x_\alpha \in A_\alpha,\, \alpha \in \tau,\ \text{ such that } (x_\alpha),\ \text{ converges to } x\}.$$

We want to prove that $p \in M$ which will mean that X is strongly τ-radial.

Suppose, for a moment, that M is closed. Assuming, on the contrary, that $p \notin M$ we shall derive a contradiction. The regularity of X and closedness of M imply the existence of a neighbourhood U of p for which $\bar{U} \cap M = \varnothing$. Choose by simple transfinite induction distinct points $a_\alpha \in U \cap A_\alpha$, $\alpha \in \tau$. In this way we obtain a subset of X of cardinality τ. The initial τ-compactness of X guaraties that this set has a complete accumulation point, say a. Let $A = \{a_\alpha : \alpha \in \tau\}\backslash\{a\}$. Then $a \in \tilde{A}\backslash A$, and since X is a radial space of tightness τ, one can find a τ-sequence $(a_{\alpha_\nu} : \nu \in \tau)$ in A converging to a. Of course, we can choose the points a_{α_ν} in such a way that $\alpha_\nu \geq \nu$; therefore, $a_{\alpha_\nu} \in A_{\alpha_\nu} \subset A_\nu$ for every $\nu \in \tau$. Putting $a_{\alpha_\nu} = b_\nu$ we have a τ-sequence $(b_\nu : \nu \in \tau)$, $b_\nu \in A_\nu$, which converges to a. As M is closed (by our assumption) we have $a \in M$. But $a \in \hat{U}$ and we have a contradiction. So, $p \in M$.

It remains to prove that M is closed. Let $y \in \overline{M}$. Using the fact that X is radial and $t(X) = \tau$, choose a τ-sequence (y_α) in M which converges to y. If some $y_\alpha = y$, then clearly there is nothing to prove. So we asume $y_\alpha \neq y$ for every $\alpha \in \tau$. By definition of the set M, for every $\alpha \in \tau$ there are $z_{\alpha\beta} \in A_\beta,\ \beta \in \tau$, such that $(z_{\alpha\beta} : \beta \in \tau)$, converges ta y_α. It is understood that $y \in \overline{\{z_{\alpha\beta}; \beta \geq \alpha\}},\ \alpha \in \tau$. Again, radiality of X implies that there exists a τ-sequence $(z_{\alpha_\nu \beta_\nu} : \nu \in \tau),\ \beta_\nu \geq \alpha_\nu$, in $(z_{\alpha\beta}\beta \geq \alpha)$ which converges to x. We may assume $\alpha_\nu \geq \nu,\ \beta_\nu \geq \nu$, and thus we have $z_{\alpha_\nu \beta_\nu} \in A_{\beta_\nu} \subset A_\nu$; in other vords, for every $\nu \in \tau$ there is a point $z_{\alpha_\nu \beta_\nu} \in A$ such that $(z_{\alpha_n u}\beta_\nu)$ converges to y. By the definition of M we deduce $y \in M$, i. e. M is closed. The theorem is proved.

THEOREM 1.8 *Let X be a space that $t(X) \leq \tau$ and every one of its subspaces of cardinality $\leq \tau$ is strongly τ-radial; then X is also strongly τ-radial.*

Proof. For every τ-sequence $s = (x_\alpha : \alpha \in \tau)$ in X which converges to a point $x \in X$, let Y_s be a copy of $s \cup \{x\}$ toplogized in the same way as the space $T(\tau)$ (see the Introduction). Let Y be the topological sum of all such spaces Y_s and let $f : Y \to X$ be the natural surjection. Clearly, Y is an lob-space. If M is any subset of X of cardinality $\leq \tau$, then, by assumption, it is strongly τ-radial. Since $f^{-1}(M)$ is an lob-space, the mapping $f_M : f^{-1}(M) \to M$ is τ-bi-quotient according to Theorem 1.3 of [**8**]. By Lemma 1.5 the mapping $f : Y \to X$ is also τ-bi-quotinet. Again according to Theorem 1.3 of [**8**] X is strongly τ-radial, which completes the proof of the theorem.

2. Linearly singly bi-k-spaces

Definition 2.1. A space X is called a *linearly singly bi-k-space* if for any $A \subset X$ and any $x \in \hat{A}$ there exist a family $\mathcal{S} = \mathcal{S}(A, \tau)$ of subsets of X linearly ordered by (reverse) inclusion ($S_1 \leq S_2$ iff $S_1 \supset S_2$) such that:

(i) $x \in \cap\{S : S \in \mathcal{S}\}$ and $x \in \overline{A \cap S}$ for each $S \in \mathcal{S}$,

(ii) $\cap\{S : S \in \mathcal{S}\} = C$ is compact,

(iii) $\mathcal{S}$ converges to C, i.e. for every neighbourhood U of C there is some $S \in \mathcal{S}$ tuch shat $C \subset S \subset U$.

If in our definition above $\mathcal{S}$ is a countable collection, we obtain the definition of a singly bi-k-space [**11**]. It is clear from the definitions that every radial and every linearly bi-k-space [**8**] is linearly singly bi-k.

For a topological space X let

$$\psi_k(X) = \omega_0 \cdot \min\{\tau : \text{every compact subset of } X \text{ is the intersection of} \leq \tau \text{ open sets in } X\}.$$

THEOREM 2.2.[1] *If X is a non-discrete linearly singly bi-k-space, then the*

[1] The authord would like to thank the referee and Đ. Kurepa for several corrections concerning this theorem

family $\mathcal{S}$ in Definition 2.1 *can be chosen so that*

(*a*) $\mid \mathcal{S} \mid \leq \psi_k(X)$,

(*b*) $\mid \mathcal{S} \mid$ *is* $\begin{cases} 1, & \textit{if } A \textit{ is compact} \\ \textit{a regular infinite cardinal,} & \textit{if } A \textit{ is non-compact.} \end{cases}$

Proof. (a) Let $\psi_k(X) = \tau,\ A \subset X,\ x \in \tilde{A}$. If A is compact, one can select $\mathcal{S} = \{A\}$; conditions (a) and (b) are satisfied. If A is not compact, let $\mathcal{S}$ and C be as in Definition 2.1 and let $\{U_\alpha : \alpha \in \tau\}$ be a collection of open subsets of X such that $\cap\{U_\alpha : \alpha \in \tau\} = C$. For each $\alpha \in \tau$, by (iii), let S_α be a member of $\mathcal{S}$ for which $C \subset S_\alpha \subset U_\beta$. We claim that there exists an $\mathcal{S}$ such that the family $\mathcal{S} = \{S_\alpha : \alpha \in \tau\}$ satisfies the conditions of Definition 2.1 Condition (i) is obviously satisfied, so we need only check that (ii) and (iii) hold. We have

$$C = \cap\{S : S \in \mathcal{S}\} \subset \cap\{S_\alpha : \alpha \in \mathcal{S}\} \subset \cap\{U_\alpha : \alpha \in \tau\} = C$$

so that condition (ii) is satisfied by $\mathcal{S}$ for every $\mathcal{S}$. We now prove that $\mathcal{S}$ satisfies (iii). If $C \in \mathcal{S}$, all is done: C is the minimal element in $\mathcal{S}$. If $C \notin \mathcal{S}$, suppose, on the contrary, that there exists a neighbourhood U of C such that $S_\alpha \backslash U \neq \varnothing$, and thus $S_\alpha \backslash C \neq \varnothing$ (bacause $U \supset C$) for every $\alpha \in \tau$. By (iii) there is an $S \in \mathcal{S}$ with $C \subset S \subset U$. Since $\mathcal{S}$ is linearly ordered, S must be proper subset of S_α for each $\alpha \in \tau$, and consequently $S \subset \cap\{S_\alpha : \alpha \in \tau\} = C$. This relation together with $C \subset S$ implies $C = S$, which contradicts the assumption $C \notin \mathcal{S}$. Statement (a) is proved.

(b) Let $\tau = \min\{\mid \mathcal{P} \mid : \mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ and $\mathcal{P}$ is cofinal in $\mathcal{S}\}$. Then τ is a regular cardinal number and, as can easily be verified, the corresponding family $\mathcal{P}$ satisfies all the conditions of the definition of a linearly bi-k-space. The theorem is proved.

If τ is the smallest initial cardinal which can be used in the definition of a linearly singy bi-k-space X, we shall say that X is a τ-*singly bi-k-space.* So the class of all linearly singly bi-k-spaces may be decomposed in subclasses of τ-singly bi-k-spaces. Of course, ω_0-singly bi-k-spaces coincide with singly bi-k-spaces.

From Theorem 2.2. we obtain:

COROLLARY 2.3. *Every linearly bi-k-space X with $\psi_k(X) = \omega_0$ is singly bi-k*

THEOREM 2.4. *If X is a space, then* (1)$\Rightarrow$ (2) $\Rightarrow$ (3) *below:*

(1) *X is a linearly singly bi-k-space.*

(2) *For every $A \subset X$ and every $x \in \tilde{A}$ there exist a compact set $C \subset X$ and a set $B \subset A$ of regular cardinality such that $\mid B \backslash U \mid < \mid B \mid$ for every neighbourhood U of C.*

(3) *For every $A \subset X$ and every $x \in \tilde{A}$ there is a linearly ordered family $\mathcal{S}$ of subsets of X which converges to a compact subset of X and $S \cap A \neq \varnothing$ for every $S \in \mathcal{S}$.*

Proof. (1)$\Rightarrow$ (2) Let $\mathcal{S}$ be a family as in the definition of a linearly singly bi-k-space, and let $\mathcal{P}$ be a well-ordered cofinal subcollection of $\mathcal{S}$ of minimal cardinality $\tau : \mathcal{P} = \{S_\alpha : \alpha \in \tau\}$. Choosing points $x_\alpha \in A \cap S_\alpha$, $\alpha \in \tau$, $x_\alpha \neq x_\beta$, for $\alpha \neq \beta$, and putting $\mathbf{B} = \{x_\alpha : \alpha \in \tau\}$, we obtain a subset of A of regular cardinality τ. Let U be a neighbourhood of $C = \cap\{S : S \in \varphi\} = \cap\{S_\alpha : \alpha \in \tau\}$. Denote by S_α the first element of $\mathcal{P}$ for which $C \subset S_\alpha \subset U$ holds. Then $x_\alpha \in S_\alpha$ implies $x_\beta \in S_\alpha$ for each $\beta \geq \alpha$, and consequently $\{x_\beta : \beta \geq \alpha\} \subset U$; so that we have $\mid B\backslash U \mid=\mid \{x_\beta : \beta < \alpha\} \mid< \tau =\mid \mathbf{B} \mid$.

(2) $\Rightarrow$ (3) Let $A \subset X,\ x \in \tilde{A}$. By (2), one can find a compact set $C \subset X$ and a subset B of A of regular cardinality τ satisfying the conditions of (2). Letting $S_\alpha = C \cup \{x_\beta : \beta \geq \alpha\},\ \ \alpha \in \tau$ one obatins a linearly ordered family $\{S_\alpha : \alpha \in \tau\}$ having the property that $S_\alpha \cap A \neq \varnothing$ for each $\alpha \in \tau$, and converging to C. Let us check the last assertion. If U is an arbitrary neighbourhood of C, then the fact that $\mid B\backslash U \mid<\mid B \mid$ imlies the existence of some $\beta \in \tau$ for which one has $\{x_\alpha : \alpha < \beta\} \in B\backslash U$, and hence $\{x_\alpha : \alpha \geq \beta\} \subset U$. Therefore, $C \subset S_\beta = C \cup \{x_\alpha : \alpha \geq \beta\} \subset U$.

The following result is a characterization of linearly singly bi-k-spaces and may be considered as a higher cardinality version of the corresponding result about singly bi-k-spaces [**11**].

THEOREM 2.5. *A space X is a linearly singly bi-k-space if and only if it is a pseudo-open image of a space which admits a pefrect mapping onto a τ-metrizable space for some cardinal τ.*

Proof. ($\Rightarrow$) Let X be a τ-singly bi-k-space for some regular cardinal τ. Let $P(X)$ and $C(X)$ be the set of all subsets of X with the discrete topology and the collection of all compact subsets of X, recpectively. Identifying $S_{y_a} \in P(X)$ with y_α consider the set Y of all $y = (y_\alpha : \alpha \in \tau) \in P(X)^\tau$ for which $(S_{y_a} : \alpha \in \tau)$ satisfies the conditions of the definition of a linearly singly bi-k-space (for some $A \subset X$ and $x \in \tilde{A}$) with $\cap\{S_{y_a} : \alpha \in \tau\} = C_y \in C(X)$. The topology on Y is induced by the "natural topology" [**14**] on $P(X)^\tau$ defined by the base $\mathcal{V}$ consisting of the sets

$$V_\alpha(y) = \{p \in Y : p_\beta = y_\beta \text{ for } \beta \leq \alpha\},\ \ y \in Y,\ \alpha \in \tau$$

The collection $\mathcal{B}$ consisting of all sets of the form

$$B_\alpha = \{((p), (y_\alpha)) \in Y \times Y : p_\beta = y_\beta \text{ for } \beta \leq \alpha\},\ \alpha \in \tau,$$

is a well-ordered base of a unifority on Y which generates the "natural topology"; since $\mid \mathcal{B} \mid= \tau,\ Y$ is a τ-metrizable space. Put

$$Z = \{(x, y) \in X \times Y : x \in C_y,\ C_y \in C(X)\}$$

Let f and g be the projections of Z onto X and Y respecitively.

Claim. f is a pseudo-open surjection.

Let $A \in X,\ x \in \tilde{A}$. Since X is a τ-singly bi-k-space, there is a decreasing τ-sequence $(S_\alpha : \alpha \in \tau)$ of subsets of X satisfying conditions (i)—(iii) of Definition

2.1 (with $\cap\{S_\alpha : \alpha \in \tau\} = C$). Consider $y \in Y$ defined by $y = (y_\alpha \equiv S_\alpha : \alpha \in \tau)$. Then $\{V_\alpha(y) : \alpha \in \tau\}$ is a monotone base at $y \in Y$ for which one has

$$(*)\ \ fg^{-1}(V_\alpha(y)) = S_\alpha,\ \alpha \in \tau.$$

Indeed, if $p \in V_\alpha(y)$, then $fg^{-1} = C_p \subset S_{p\alpha} = S_{y\alpha} \equiv S_\alpha$, i. e. $fg^{-1}(V_\alpha(y)) \subset S_\alpha$). Conversely, let $q \in S_\alpha$. Then q belongs to some $K \in C(X)$ with $q \in K \subset S_\alpha$. Let $(S_{a_\nu} : \nu \in \tau)$ be a decreasing τ-sequence converging to K. We may assume $a_\nu = y_\nu$ for $\nu \leq \alpha$. If we take $a = (a_\nu : \nu \in \tau)$, we shall have $a \in V_\alpha(y)$, $q \in K = fg^{-1}(a)$ and hence $S_\alpha \subset fg^{-1}(v_\alpha(y))$. The equality $(*)$ is proved.

Let $z = (x, y)$. Then $z \in Z$, $f(z) = x$ and $z \in \overline{f^{-1}(A)}$, as can be verified without difficulty. This means that f is pseudo-open and because, by (*), it is a surjection, the proof of Claim 1 is complete.

Claim 2. g is a perfect surjection.

From the definition of g it follows that g is a compact surjection. Let us prove that it is closed. Let $y \in Y$ and let $W \subset Z$ be a neighbourhood of the set $g^{-1} = C_y \times \{y\}$. By a well-known theorem of Wallace, there are neighbourhoods U of C_y and $V_\alpha(y)$ of y such that $C_y \times \{y\} \subset U \times V_\alpha(y) \cap Z \subset W$. Let $(S_{y_\nu} : \nu \in \tau)$ be a decreasing τ- sequence converging to C_y. One can find $\beta \in \tau$, $\beta \geq \alpha$, for which $C_y \subset S_{y_\beta} \subset U$. Then, using (*), we have

$$g^{-1}(V_\beta(y)) = (S_{y_\beta} \times V_\beta(y)) \cap Z \subset (U \times V_\alpha(y)) \cap Z \subset W$$

which means that g is closed. The claim is proved.

($\Leftarrow$) Since very τ-metrizable space is linearly bi-k [**8**] (more precisely, τ-bi-k) and these spaces are preserved by perfect pre-images, we deduce that every perfect pre-images of a τ-metrizable space is τ-singly bi-k. But, as can easily be verified, linearly singly bi-k-spaces are preserved by psudo-open mappings, and the theorem is proved.

The preceding proof shows that the following result holds:

A space X is a linearly singly bi$-k$-space if and only if it is a pseudo-open image of a space which admits a perfect mapping onto an lob-space.

From the fact that every τ-metrizable space is a generalized orderable (GO) space we obtain:

COROLLARY 2.6. *Every linearly singly bi-k-space is a pseudo-open image of a space which admits a perfect mappiing onto a GO-space.*

3. Monotonically bi-k-spaces

In this section we shall define one subclass of the class of linearly singly bi-k-spaces. These space we call monotonically bi-k-spaces and they are a “linearly” generalization of countably bi-k-spaces [**11**]. Let us note that we use this terminology because the term “linearly bi-k-space” is reserved for the concept introduced in [**8**].

Definition. 3.1. A space X is called *monotonically bi-k* if there is a cardinal τ such that for every decreasing τ-sequence $(A_\alpha : \alpha \in \tau)$ of subsets of X and every $x \in \cap\{\tilde{A}_\alpha : \alpha \in \tau\}$ there exists a decreasing family $\mathcal{S} = \{S_\alpha : \alpha \in \tau\}$ of subsets of X having the following properties:

(i) $x \in \cap\{S_\alpha : \alpha \in \tau\}$ and $x \in \overline{A_\alpha \cap S_\alpha}$ for every $\alpha \in \tau$

(ii) $\cap\{S_\alpha : \alpha \in \tau\} = C$ is a compact set,

(iii) $\mathcal{S}$ converges to C.

If τ is the smallest cardinal which can be used in the definition above (it may be assumed to be regular; see Theorem 2.2), we say that the space is monotonically τ-bi-k. When $\tau = \omega_0$ we have the notion of a countably bi-k-space.

Every linearly bi-k-space ($=$ *a* bi-quotient image of a perfect pre-image of a τ-metrizable space [8]), and every finite product of such spaces, is monotonically bi-k. These spaces are preserved by closed subspaces and τ-biquotient images.

The proof of the following result is entirely analogous to the proof of Theorem 2.5, and thus it will be omitted.

THEOREM 3.2. *A space X is a monotonically bi-k-space if and only if it is a τ-bi-quotient image of a perfect pre-image of a τ-metrizable space for suitable τ.*

The next result gives an interesting connection between linearly singly bi-k and monotonically bi-k-spaces. At the same time it is a characterization of monotonically bi-k-spaces.

THEOREM 3.3. *A space X is a monotonically bi-k-space if and only if there exists a cardinal τsuch that $X \times T(\tau)$ is a linearly singly bi-k-space.*

Proof. ($\Rightarrow$) Let X be a monotonically τ-bi-k-space. According to Theorem 3.2 there is a space Z which admits a perfect mapping g onto a τ-metrizable space Y such that $X = f(Z)$, where f is a τ-bi-quotient mapping. Like any finite product of τ-metrizable spaces, the space $Y \times T(\tau)$ is τ-metrizable; on the other hand, the mapping $g \times i_Y : Z \times Y(\tau) \to T \times T(\tau)$ is perfect and, by Lemma 1.3, $f \times i_T$ is a pseudo-open mapping. According to Theorem 2.5, $X \times T(\tau)$ is a τ-singly bi-k-space.

($\Leftarrow$) Let now $X \times T(\tau)$ be a τ-singly bi-k-space. We are going to prove that X is monotonically τ-bi$-k$. Let $(A_\alpha : \alpha \in \tau)$ be any decreasing τ-sequence accumulating at $x \in X$. Let us define

$$M = \cup\{A_\alpha \times \{\alpha\} : \alpha \in \tau\} \subset X \times T(\tau).$$

It is clear that $(x, \tau) \in \overline{M}$. Since $X \times T(\tau)$ is τ-singly bi-k, there exists a family $\mathcal{P} = \{\mathcal{P}_\alpha : \alpha \in \tau\}$ satisfying the conditions of the definition of a τ-singly bi-k-space. A straightforward checking shows that $\{\pi(P_\alpha) : \alpha \in \tau\}$, π being the projection onto X, is the desired collection of subsets of X which witnesses that X is a monotonically τ-bi-k-space. This completes the proof of the theorem.

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