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ON A QUESTION OF LJUBIŠA KOČINAC

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Abstract. We give an example in ZFC of a weakly perfect space that is not perfect. Assuming the continuum hypothesis we give an example that is compact. Finally, we construct, in ZFC, a compact weakly perfect space that is not perfect.

In [**4**] Ljubiša Kočinac introduces the notion of weak perfectness and shows, among other things, that the property is inherited by both open and closed subspaces, as well as by finite topological sums. Making use of an example due to A.V. Ivanov [**2**], Kočinac estabilishes that, in a model of set theory in which $\diamondsuit$ holds, there is a weakly perfect space which is not perfect (and which is a compact S space).

In this note we give an example in ZFC of a weakly perfect space that is not perfect. Our example has the Lindelöf property and is quasi-developable (a quasi-developable space is developable iff it is perfect [**1**]), but our example is neither hereditarily separable nor compact. Assuming the continuum hypothesis we give an example that is compact. Finally, we use a λ-set as defined in [**5**] to construct, in ZFC, a compact weakly perfect space that is not perfect (the author wishes to thank Gary Gruenhage for calling his attention to the existence in ZFC of λ'-sets and J. Vaughan for a reference to same).

Definition (Kočinac). A space X is *weakly perfect* if every closed set in X contains a dense set that is a G_δ-set in X. We shall call X a *Kočinac space* if X is weakly perfect but not perfect.

THEOREM 1. *There exists a weakly perfect space that is not perfect, and it is a quasi-developable space which has the Lindelöf property.*

Proof: Let the subset L of $[0,1]$ be a λ-set, as defined in [**5**] or in [**3**] (that is, L is an uncountable dense-in-itself set every countable subset of which is a G_δ-set in L). The space X (which is a subspace of the familiar compact "double arrow"

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space of Alexandroff) is the space $L \times \{0,1\}$ with a basis consisting of all sets of the forms

(i) $\{(t,1)\}$ for t in L

(ii) $(S \times \{0,1\}) \setminus \{(p,1)\}$ for S an open interval in L, p in S.

Clearly X has the Lindelöf property but is not hereditarily Lindelöf; hence X is not perfect. (Or, X is not perfect because it is quasi-developable but not developable). To see that X is weakly perfect, one need check only closed subsets of $L \times \{0\}$ and note that each such set has a countable dense subset that is easily seen to be a G_δ-set in X.

Note: A linearly ordered topological space that is Kočinac can be embedded in the lexicographically ordered unit square in a similar fashion. Also if one takes X to be the whole "double arrow" space $[0,1] \times \{0,1\}$ and takes the non-degenerate members of the basis to be all sets of the form $\big(S \times \{0,1\}\big) \setminus \big(N \times \{1\}\big)$ for S an open interval, $N \subseteq S$ of measure zero then X is another Kočinac space; but this X is not first countable nor does it have the Lindelöf property.

Now we use the continuum hypothesis to modify the λ-set L and the space X of Theorem 1 to make X a compact Kočinac space.

THEOREM 2. (CH) *There exists a weakly perfect, non-perfect, compact Hausdorff space.*

Proof: Assume the Continuum Hypothesis. Using a λ-set L constructed as below, let

$$X = \big([0,1] \times \{0,1\}\big) \setminus \big(([0,1] \setminus L) \times \{1\}\big) .$$

To construct L: First, put into one-to-one correspondence with the countable ordinals both $[0,1]$ and the set F of all closed dense-in-themselves subsets of $[0,1]$ (i.e. write both $[0,1]$ and F as ω_1-sequences). Then, inductively define ω_1-sequences L and L^* in $[0,1]$, and an ω_1-sequence S of G_δ-sets in $[0,1]$ each of measure zero, as follows.

Let the initial infinite sequence, both in L and in L^* be initial infinite sequence of $[0,1]$, and let every one of the first infinitely many terms of S equal the set $\{L_1, L_1^*\}$. For each limit ordinal α, pick $L^*_{\alpha+1}, L^*_{\alpha+2}, L^*_{\alpha+3}, \ldots$ to be some simple ordering of the union of countable dense subsets, one from each of the members of F that precedes F_α; pick S_α to be a G_δ-set of measure zero that contains all points L_β and L^*_β and all sets S_β for $\beta < \alpha$; pick $L_{\alpha+1}, L_{\alpha+2}, L_{\alpha+3}, \ldots$ to be the first infinitely many members of $[0,1]$ outside S_α; and, for each natural number k, let $S_{\alpha+k} = S_\alpha$.

If the basis is then chosen as in the proof of Theorem 1, it follows (almost as easily) that X is weakly perfect.

A λ'-set L in $[0,1]$ is an uncountable subset of $[0,1]$ such that every countable subset of $[0,1]$ is a G_δ-set relative to L. By a considerably more complicated process than that used in the proof of Theorem 2, it has been shown (evidently by Hausdorff and Sierpiński — see [**5**]) that iz ZFC there exists such a λ'-set L. Theorem 3 follows at once.

THEOREM 3. *There exists (in ZFC) a compact Hausdorff Kočinac space.*

Questions. Do there exist interesting classes of spaces (such as cetain symmetrizable spaces, or all quasi-developable generalized ordered spaces) that are all weakly perfect but not necessarily perfect? Conversely, what implications heretofore established for perfect spaces also hold in weakly perfect spaces?

REFERENCES

[**1**] H. R. Bennett, *On quasi-developable spaces*, Gen. Top. Appl. **1** (1971), 253–262. MR 44#5921.

[**2**] A. V. Ivanov, *Bicompacta all of whose finite powers are hereditarily separable*, Dokl. Akad. Nauk. SSSR **243** (1978), 1109–1112 (Russian). MR 81k:54026.

[**3**] F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. **43** (1937), 671–676.

[**4**] Lj. Kočinac, *An example of a new class of spaces*, Mat. vesnik **35** (1983), 145–150 (Russian). MR 85j:54028.

[**5**] K. Kunen and J. E. Vaughan, *Handbook of Set-theoretic Topology*, North Holland 1984, pp. 212–213.

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