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CONVERGENCE OF SUBSERIES OF THE HARMONIC SERIES AND ASYMPTOTIC DENSITIES OF SETS OF POSITIVE INTEGERS

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Abstract. We investigate the relation between the convergence of subseries $\sum_{n=1}^{\infty} m_n^{-1}$ of the harmonic series $\sum_{n=1}^{\infty} n^{-1}$ and the asymptotic densities $d(M)$ of sets $M = \{m_1 < m_2 < \dots < m_n < \dots\}$ of positive integers. Here, $d(M) = \lim_{x\to\infty} M(x)/x$, where $M(x) = \sum_{a\in M,\, a\le x} 1$.

It is known that if $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$, then $d(M) = 0$. We show that this relation cannot be substantially improved. In particular, we give two counterexamples to the previous assertion (contained in Theorem 3 of [**3**]) that if $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$, then $\lim_{x\to\infty} M(x)\log x/x = 0$.

Furthermore, we proceed to prove, more generally, in Theorems 1 and 2 herein that if $\limsup_{x\to\infty} g(x) = +\infty$, where $g : (0,+\infty) \to (0,+\infty)$, then there exists an infinite set $M \subset N$ such that $\sum_{m\in M} m_n^{-1} < +\infty$ and simultaneously $\limsup_{x\to\infty} M(x)g(x)/x = +\infty$.

Whereas, in Theorems 3, 4, and 5 we prove that if $\sum_{m\in M} m_n^{-1} < +\infty$, then $L(M,g) = \liminf_{x\to\infty} M(x)g(x)/x = 0$ for certain functions $g(x)$, in particular, $g(x) = \log x \cdot \log\log x$.

In Theorem 7 we generalize Theorems 3, 4, and 5 by proving that if $\lim_{x\to\infty} g(x) = +\infty$ and $\sum_{n=1}^{\infty} 1/(ng(n)) = +\infty$, then $L(M,g) = 0$ for the sets M referred to above.

In Theorem 6, in contrast to Theorem 7, we prove that if $g(x)$ is a nondecreasing function on $(0,+\infty)$, and $\sum_{n=1}^{\infty} 1/(ng(n)) < +\infty$, then there exists a set M (as defined above) such that $L(M,g) > 0$.

In Theorem 8 we give a new proof of the known result that $\sum_{m\in M} m^{-1} < +\infty$ if and only if $\sum_{n=1}^{\infty} M(n)/n^2 < +\infty$.

We thus give new formulations of well-known principles of analytic number theory.

Numerous remarks and examples are provided throughout the paper in supplement to and clarification of the main Theorems.

There exists a relation between the convergence of subseries

$$\sum_{n=1}^{\infty} k_n^{-1} \qquad (k_1 < k_2 < \dots < k_n < \dots) \tag{1}$$

of the harmonic series $\sum_{n=1}^{\infty} n^{-1}$ and the asymptotic densities of sets

$$K = \{k_1 < k_2 < \dots < k_n < \dots\} \tag{1'}$$

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(see Theorem A). We shall show that this relation cannot be substantially improved.

If $M \subset N = \{1, 2, \dots, n, \dots\}$, then $d(M)$ denotes the asymptotic density the set M, i.e. $d(M) = \lim_{x\to\infty} M(x)/x$ if the limit on the right-hand side exists, here

$$M(x) = \sum_{a\in M,\, a\le x} 1$$

(cf. [**1**, p. xix]).

The following theorem expresses the mentioned relation between the convergence of subseries (1) and the asymptotic densities of sets (1′).

THEOREM A. *If* $\sum_{n=1}^{\infty} k_n^{-1} < +\infty$, *then* $d(K) = 0$.

For the proof of Theorem A see e.g. [**5**, Theorem 1]. Theorem A can be easily deduced also from the following result:

Let $\sum_{n=1}^{\infty} a_n$ be a series with real terms, let $a_1 \ge a_2 \ge \dots \ge a_n \ge \dots$, $a_n \to 0$, $\sum_{n=1}^{\infty} a_n < +\infty$. Denote by $N(x)$ the number of n's for which an $a_n \ge x > 0$. Then

$$\lim_{x\to 0+} xN(x) = 0 \tag{2}$$

(cf. [**4**], [**8**]).

If we put $a_n = k_n^{-1}$ $(n = 1, 2, \dots)$, then we have for $x > 0$:

$$N(x) = \#\{n : a_n \ge x\} = \#\{n : k_n \le 1/x\} = K(1/x).$$

Hence according to (2) we get

$$0 = \lim_{x\to 0+} xN(x) = \lim_{x\to 0+} \frac{K(1/x)}{1/x} = \lim_{y\to\infty} \frac{K(y)}{y} = d(K), \qquad d(K) = 0.$$

In [**3**] the following theorem is introduced (see Theorem 3 in [**3**]).

THEOREM B. *If* $M \subset N$ *and* $\sum_{m\in M} m^{-1} < +\infty$, *and if* $c_M = \lim_{x\to\infty} M(x)\log x/x$ *exists, then* $c_M = 0$*.

The following two examples show that Theorem B is not valid if the existence of the limit c_M is not assumed. (cf. [**6**]).

Example 1. Put $M = \bigcup_{n=2}^{\infty} M_n$, where $M_n = \{n^{n^2}+1, n^{n^2}+2, \dots, n^{n^2} + n^{n^2-2}\}$ $(n = 2, 3, \dots)$. Then it can be easily shown (cf. [**6**]) that $\sum_{m\in M} m^{-1} < +\infty$ and $\limsup_{x\to\infty} M(x)/x = +\infty$.

Example 2. Let $\{p_n\}_{n=1}^{\infty}$ be the increasing sequence of all prime numbers. We shall write $p(k)$ instead of p_k $(k = 1, 2, \dots)$. Put $Q = \bigcup_{n=1}^{\infty} Q_n$, where

$$Q_n = \{p(n^{n^2}+1), p(n^{n^2}+2), \dots, p(n^{n^2}+t_n)\},$$
$$t_n = [n^{-2}\cdot p(n^{n^2})], \qquad (n = 1, 2, \dots).$$

*In [**3**] the notation $v_M(x)$ is used instead of $M(x)$.

A detailed computation shows (cf. [**6**]) that $\sum_{q\in Q} q^{-1} < +\infty$ and

$$\limsup_{x\to\infty} \frac{Q(x)\log x}{x} \geq \frac{a}{2b^2} > 0,$$

where a, b are positive constants occurring in the Tchebysheff's inequalities

$$an\log n < p_n < bn\log n \qquad (n = 2, 3, \dots)$$

(cf. [**6**]). For example, if $Q(x)$ represents the number of twin primes $\leq x$, the result $\lim_{x\to\infty}(Q(x)/\Pi(x)) = 0$ established in [**3**] does not follow from the fact that the sum of the reciprocals of the twin primes converges.

Remark 1. In [**5**] the following result is proved (see Theorem 2 in [**5**]).

Let

$$d_1 \geq d_2 \geq \ldots \geq d_n \geq \ldots, \qquad \sum_{n=1}^{\infty} d_n = +\infty$$

and let $\sum_{k=1}^{\infty} \varepsilon_k(x) d_k < +\infty$, where $\varepsilon_k(x)$ $(k = 1, 2, \dots)$ are dyadic digits of the number $x \in (0, 1]$ (i.e. $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$ is the nonterminating dyadic expansion of x). Then we have $p_1 = \liminf_{n\to\infty} p(n, x)/n = 0$, where $p(n, x) = \sum_{k=1}^{n} \varepsilon_k(x)$ $(n = 1, 2, \dots)$.

If we apply this result to the subseries of the series $\sum_{n=1}^{\infty} n^{-1}$ we see that the convergence of such subseries implies that "the lower density" of this subseries in $\sum_{n=1}^{\infty} n^{-1}$ is zero. An analogous consideration can be made also for subseries of the series $\sum_{n=1}^{\infty} p_n^{-1}$.

The foregoing examples 1, 2 suggest the formulation and the proof of the following theorem which shows that the result obtained in Theorem A cannot be substantially improved. In what follows we shall give the proof of Theorem 1 published in [**6**] without the proof.

THEOREM 1. *Let* $g : (0, +\infty) \to (0, +\infty)$ *and* $\lim_{x\to\infty} g(x) = +\infty$ *(arbitrarily slowly). Then there exists an infinite set* $M \subset N$ *such that* $\sum_{m\in M} m^{-1} < +\infty$ *and simultaneously*

$$\limsup_{x\to\infty} M(x)g(x)/x = +\infty. \tag{3}$$

Proof. We can assume without loss of generality that $g(t) \geq 1$ for each $t \geq 1$.

We can construct (by induction) two sequences $\{x_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$, of positive integers with the following properties:

(a) $x_n \geq n^3$ $\quad (n = 1, 2, \dots)$,

(b) $\forall_{t\geq x_n}\ g(t) \geq n^3$ $\quad (n = 1, 2, \dots)$,

(c) $t_n = [n^{-2} x_n]$ $\quad (n = 1, 2, \dots)$,

(d) $x_n > x_{n-1} + t_{n-1}$ $\quad (n = 2, 3, \dots)$.

Put

$$M_n = \{x_n + 1, x_n + 2, \dots, x_n + t_n\}, \quad (n = 1, 2, \dots,); \qquad M = \bigcup_{n=1}^{\infty} M_n.$$

According to (d) the sets M_n, $(n = 1, 2, \dots)$ are mutually disjoint. A simple estimation gives

$$\sum_{m \in M_n} m^{-1} \le t_n x_n^{-1} \le n^{-2} \qquad (n = 1, 2, \dots),$$

hence $\sum_{m \in M} m^{-1} < +\infty$.

Putting $y_n = x_n + t_n$ $(n = 1, 2, \dots)$ we have

$$M(y_n) \ge t_n > n^{-2} x_n - 1, \quad y_n \le (1 + n^{-2}) x_n \qquad (n = 1, 2, \dots).$$

Using (a), (b) we get

$$\frac{M(y_n) g(y_n)}{y_n} \ge n^3 \frac{n^{-2} x_n - 1}{(1 + n^{-2}) x_n} \ge \frac{1}{2} n^3 \left(\frac{1}{n^2} - \frac{1}{x_n} \right) \ge \frac{1}{2}(n-1) \to +\infty \quad (\text{as } n \to \infty).$$

Hence (3) holds and the proof is finished.

A little modification of the construction of the set M in the proof of Theorem 1 leads to the following more general result.

THEOREM 2. *Let* $g : (0, +\infty) \to (0, +\infty)$, *and*

$$\limsup_{x \to \infty} g(x) = +\infty. \tag{4}$$

Then there exists an infinite set $M \subset N$ *such that* $\sum_{m \in M} m^{-1} < +\infty$ *and simultaneously we have* $\limsup_{x \to \infty} M(x) g(x)/x = +\infty$.

Remark 2. Condition (4) cannot be omitted. If $\limsup_{x \to \infty} g(x) < +\infty$ holds, then it follows from Theorem A that $\lim_{x \to \infty} M(x) g(x)/x = 0$ for each set $M \subset N$ with $\sum_{m \in M} m^{-1} < +\infty$.

Proof of Theorem 2. Construct by induction a sequence

$$\{x_n\}_{n=1}^{\infty}, \qquad 2 \le x_1 < x_2 < \dots < x_n < \dots$$

of real numbers such that (a) $x_n \ge n^3$ $(n = 1, 2, \dots)$, (b) $x_n > (x_{n-1} + 1)(1 - n^{-2})^{-1}$ $(n = 2, 3, \dots)$, (c) $g(x_n) \ge n^3$ $(n = 1, 2, \dots)$.

This is possible since (4) holds. Let us remark that from (b) we have

$$\begin{aligned} x_{n-1} + 1 &< x_n (1 - n^{-2}) && (n \ge 2), \\ x_{n-1} + x_n n^{-2} &< x_n - 1 && (n \ge 2), \\ x_{n-1} + [n^{-2} x_n] &< x_n - 1 (< [x_n]) && (n \ge 2). \end{aligned}$$

Hence

$$x_{n-1} + [n^{-2} x_n] < [x_n] \qquad (n \ge 2). \tag{5}$$

Put $M = \bigcup_{n=2}^{\infty} M_n$, where

$$\begin{aligned} M_n &= \{[x_n] - t_n, [x_n] - t_n + 1, \dots, [x_n] - 1\}, \\ t_n &= [n^{-2} x_n] \qquad (n = 2, 3, \dots). \end{aligned}$$

Let us remark that according to (5) the sets M_n $(n = 2, 3, \dots)$ are mutually disjoint. By a simple estimation we get

$$\sum_{m \in M_n} m^{-1} \le \frac{1}{[x_n] - t_n} \qquad (n = 2, 3, \dots).$$

But we have $[x_n] - t_n \ge x_n - 1 - n^{-2}x_n = x_n(1 - n^{-2}) - 1$ $(n \ge 2)$ and therefore

$$\sum_{m \in M_n} m^{-1} \le \frac{1}{x_n(1 - n^{-2}) - 1} n^{-2} x_n$$

$$\le n^{-2} \frac{1}{1 - n^{-2} - x_n^{-1}} \le n^{-2} \frac{1}{1 - 4^{-1} - 8^{-1}} = \frac{8}{5} \frac{1}{n^2}.$$

Thus $\sum_{m \in M} m^{-1} < +\infty$.

Put $A_n = M(x_n)g(x_n)/x_n$ $(n = 2, 3, \dots)$. We have

$$M(x_n) \ge t_n \ge n^{-2}x_n - 1 \qquad (n = 2, 3, \dots).$$

Using (a) and (c) we obtain

$$A_n \ge \frac{(n^{-2}x_n - 1)n^3}{x_n} = (n^{-2} - x_n^{-1})n^3$$

$$= n - n^3 x_n^{-1} \ge n - 1 \to +\infty \quad \text{as } n \to \infty.$$

Hence $\limsup_{x \to \infty} M(x)g(x)/x = +\infty$. This ends the proof.

Note that the converse of Theorem A is false. For example, if K represents the set of all prime numbers, $d(K) = 0$, while $\sum p^{-1}$ diverges.

Professor A. Schinzel remarked** in connection with Theorems A and B that the following result holds.

THEOREM 3. *Let* $M \subset N$ *and* $\sum_{m \in M} m^{-1} < +\infty$. *Then we have* $\liminf_{x \to \infty} M(x) \log x / x = 0$. *Hence* $\liminf_{x \to \infty} M(x)/\Pi(x) = 0$.

Remark 3. If $c_M = \lim_{x \to \infty} M(x) \log x / x$ exists, then $c_M = 0$.

Proof. We have from Theorem 3 above

$$c_M = \lim_{x \to \infty} \frac{M(x) \log x}{x} = \liminf_{x \to \infty} \frac{M(x) \log x}{x} = 0. \quad \text{Q.E.D.}$$

This is the result actually proved in Theorem 3 of [**3**].

We shall not give the proof of Theorem 3 because it is an easy consequence of Theorem 4. In what follows, we put for brevity $\log_k x = \underbrace{\log \log \dots \log}_{k \text{ times}} x$

THEOREM 4. *Suppose that the function* $g : (0, +\infty) \to (0, +\infty)$ *satisfies the condition*

$$g(x) = O(\log x \log_2 x) \qquad (x \to +\infty).$$

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If $M \subset N$ *and* $\sum_{m\in M} m^{-1} < +\infty$, *then* $\liminf_{x\to\infty} M(x)g(x)/x = 0$.

Proof. Assume that there are $a > 0$ and $x_0 > 0$ such that

$$M(x)g(x)/x \geq a > 0 \qquad \text{for } x > x_0. \tag{6}$$

According to the assumption there exists a $K > 0$ and $x_1 > 0$ such that

$$g(x) \leq K \log\log_2 x \tag{7}$$

for $x > x_1$.

Choose an $n_1 \in N$ such that $m_n > \max\{x_0, x_1\}$ for $n > n_1$, $M = \{m_1 < m_2 < \ldots < m_n < \ldots\}$. Then putting $x = m_n$ in (6) we get

$$ng(m_n)/m_n \geq a > 0 \qquad \text{for } n > n_1. \tag{8}$$

Using (7), (8) we get for $n > n_1$

$$a/n \leq K \log m_n \log_2 m_n / m_n. \tag{9}$$

But $\log m_n \log_2 m_n < \sqrt{m_n}$ for each $n > n_2 > n_1$ (n_2 is a suitable number). Then

$$a/n \leq K/\sqrt{m_n}, \quad m_n \leq (K/a)^2 n^2,$$
$$\log m_n \leq 2\log n + C_1, \quad C_1 = 2\log(K/a),$$
$$\log_2 m_n \leq \log_2 n + \log 2 + \sigma(1) \quad (n \to \infty).$$

We obtain by (9)

$$d_n = \frac{a}{K}\,\frac{1}{n(2\log n + C_1)(\log_2 n + \log 2 + \sigma(1))} \leq m_n^{-1}$$

for $n > n_2$. Since $\sum_{n>n_2} d_n = +\infty$, we have $\sum_{n=1}^{\infty} m_n^{-1} = +\infty$ — a contradiction.

In an analogous way the following more general result can be proved.

THEOREM 5. *Suppose that the function* $g : (0, +\infty) \to (0, +\infty)$ *satisfies the condition*

$$g(x) = O(\log x \log_2 x \ldots log_k x) \qquad (x \to \infty).$$

If $M \subset N$ *and* $\sum_{m\in M} m^{-1} < +\infty$, *then* $\liminf_{x\to\infty} M(x)g(x)/x = 0$.

Observe that the conditions satisfied by g in the Theorems 4 and 5 imply that $\sum_{n=1}^{\infty} 1/(ng(n)) = +\infty$. In the following theorem we shall investigate the behavior of

$$L(M, g) = \liminf_{x\to\infty} \frac{M(x)g(x)}{x}$$

for sets $M = \{m_1 < m_2 < \ldots < m_n < \ldots\} \subset N$ with $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$. In the first place we shall do it under the assumption that $\sum_{n=1}^{\infty} 1/(ng(n)) < +\infty$.

THEOREM 6. *Let* $g : (0, +\infty) \to (0, +\infty)$ *be a nondecreasing function. Suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{ng(n)} < +\infty. \tag{10}$$

Then there exists a set $M = \{m_1 < m_2 < \dots m_n \dots\} \subset N$ *with* $\sum_{n=1}^{\infty} m^{-1} < +\infty$ *such that* $L(M, g) > 0$.

Proof. Since the function g is nondecreasing, it follows from (10) that $\lim_{x\to\infty} g(x) = +\infty$. In the contrary case, if $g(n) \leq K$, $n = 1, 2, \dots$ we have $1/(ng(n)) \geq 1/(Kn)$ and so $\sum_{n=1}^{\infty} 1/(ng(n)) = +\infty$ by the comparison test, a contradiction to (10).

Define $\{m_n\}_{n=1,2,\dots}$ as follows:

$$\begin{aligned} m_1 &= 1, \qquad m_2 = 2, \\ m_n &= n, && \text{if } n > 2 \text{ and } g(n-1) \leq 2 \\ m_n &= [(n-1)g(n-1)], && \text{if } n > 2 \text{ and } g(n-1) > 2. \end{aligned}$$

If i is the first integer > 2 for which $g(i-1) > 2$, we have

$$\begin{aligned} m_i &= [(i-1)g(i-1)] > (i-1)g(i-1) - 1 \\ &> 2(i-1) - 1 = 2i - 3 > i - 1 = m_{i-1}. \end{aligned}$$

Therefore $m_i > m_{i-1}$. Furthermore, for $j \geq 1$,

$$\begin{aligned} m_{i+j} &= [(i+j-1)g(i+j-1)] > (i+j-1)g(i+j-1) - 1 \\ &\geq (i+j-1)g(i+j-2) - 1 > (i+j-2)g(i+j-2) \\ &\geq m_{i+j-1}, \end{aligned}$$

therefore $m_{i+j} > m_{i+j-1}$, $j \geq 1$, and so $m_1 < m_2 < m_3 < \dots < m_n < \dots$.

Since $\lim_{n\to\infty} g(n-1) = +\infty$, we have $m_{n+1} = [ng(n)]$, $n > T$ for some $T \in N$.

Since $\lim_{n\to\infty}(ng(n))/[ng(n)] = 1$, we have $\sum_{n=T+2}^{\infty} m_n^{-1} < +\infty$ by the limit comparison test. Therefore $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$. As $\sum_{n=1}^{\infty} n^{-1} = +\infty$, and from Theorem A, $\lim_{n\to\infty} M(n)/n = \lim_{n\to\infty} n m_n^{-1} = 0$, so that $m_n > n$ for $n > J$ for some positive integer J.

Thus for $\max\{J, T\} < n$, we have $n < m_n$, and hence $m_n \leq x < m_{n+1}$ implies that

$$\frac{M(x)g(x)}{x} > \frac{ng(m_n)}{m_{n+1}} \geq \frac{ng(n)}{[ng(n)]} \geq 1 > 0,$$

since $g(x)$ is nondecreasing, and thus $g(x) \geq g(m_n) \geq g(n)$ for $x \geq m_n > n$. Thus $M(x)g(x)/x > 1$ for $x > m_J, m_T$. Therefore $L(M, g) \geq 1 > 0$. Q.E.D.

Example 3(a). The function g, $g(x) = \max\{1, (\log x)^\alpha\}$ $(\alpha > 1)$ or more generally $g(x) = \max\{1, \log x \log_2 x \dots (\log_k x)^\alpha\}$ $(\alpha > 1)$ satisfies Theorem 6, i.e. g is nondecreasing and $\sum_{n=1}^{\infty} 1/(ng(n)) < +\infty$. Hence there exists a set $M = \{m_1 < m_2 < \dots < m_n < \dots\} \subset N$ with $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$ such that $L(M, g) > 0$ (compare this fact with Theorems 4, 5).

Example 3(b). The function g, $g(x) = \max\{1, x^x\}$ $(x > 0)$ also satisfies Theorem 6 — g is nondecreasing and $\sum_{n=1}^{\infty} 1/(ng(n)) < +\infty$. Hence there exists again a set $M = \{m_1 < m_2 < \ldots < m_n < \ldots\} \subset N$ with $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$ such that $L(M,g) > 0$.

The foregoing Theorem 6 can suggest the conjecture that in general if $\sum_{n=1}^{\infty} 1/(ng(n)) < +\infty$, then there is a set $M = \{m_1 < m_2 < \ldots < m_n < \ldots\} \subset N$ with $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$ such that $L(M,g) > 0$. The following example shows that such conjecture is false.

Example 4. Let $f : (0,+\infty) \to (0,+\infty)$ where $\sum_{n=1}^{\infty} 1/(ng(n)) < +\infty$ and $\lim_{x\to\infty} f(x) = +\infty$. Choose the function $g : (0,+\infty) \to (0,+\infty)$ in the following way: Put $g(j^2) = \log j^2$ $(j = 2,3,\ldots)$ and $g(x) = f(x)$ for each $x \in (0,+\infty)$, $x \neq j^2$ $(j = 2,3,\ldots)$. Then evidently

$$\sum_{n=1}^{\infty} \frac{1}{ng(n)} \leq \sum_{n=1}^{\infty} \frac{1}{nf(n)} + \sum_{j=2}^{\infty} \frac{1}{j^2 \log(j^2)} < +\infty.$$

We shall show that for each set $M = \{m_1 < m_2 < \ldots\} \subset N$ with $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$ we have $L(M,g) = 0$. Let M be such a set. Then according to Theorem 3 we have

$$\liminf_{x\to\infty} M(x) \log x/x = 0.$$

Hence there exists a sequence $x_1 < x_2 < \ldots < x_n < \ldots$, $x_n \to +\infty$ of real numbers such that

$$\liminf_{k\to\infty} M(x_k) \log x_k/x_k = 0. \tag{11}$$

For each $x_k \in R$ there exists a $j = j(x_k) \in N$ such that $j^2 < x_k \leq (j+1)^2$. But then by a simple estimation we get

$$\frac{M(j^2)\log j^2}{(j+1)^2} \leq \frac{M(x_k)\log x_k}{x_k}. \tag{12}$$

According to (11) for each $\varepsilon > 0$ there exists a k_0 such that for each $k > k_0$ we have

$$M(x_k) \log x_k/x_k < \varepsilon. \tag{13}$$

But then for $j = j(x_k)$ we get from (12) and (13)

$$\frac{M(j^2)\log j^2}{(j+1)^2} < \varepsilon.$$

For such j we have

$$\frac{j^2}{(j+1)^2} \cdot \frac{M(j^2)\log j^2}{j^2} < \varepsilon. \tag{14}$$

Since $\lim_{n\to\infty} n^2/(n+1)^2 = 1$, it is evident from (14) that for each sufficiently large k (say for $k > k_1 > k_0$) we have (for $j = j(x_k)$)

$$M(j^2) \log j^2/j^2 < \varepsilon. \tag{15}$$

Hence for an infinite number of j's we have (15). From this the equality $L(M,g)=0$ follows at once.

In this example $f(x)=x^x$ would suffice to disprove the conjecture.

Remark 4. Let $g:(0,+\infty)\to(0,+\infty)$ and let $\liminf_{x\to\infty} g(x)<+\infty$. If $M=\{m_1<m_2<\dots\}\subset N$ and $\sum_{n=1}^{\infty} m_n^{-1}<+\infty$, then according to Theorem A we have

$$\liminf_{x\to\infty} M(x)g(x)/x=0$$

holds. This shows that by investigation of the behavior of $L(M,g)$ we can restrict ourselves to the case if $\lim_{x\to\infty} g(x)=+\infty$. The following theorem is a generalization of Theorems 4, 5.

THEOREM 7. *Let $g:(0,+\infty)\to(0,+\infty)$ with $\lim_{x\to\infty} g(x)=+\infty$. Let $\sum_{n=1}^{\infty} 1/(ng(n))=+\infty$. Then for each set $M=\{m_1<m_2<\dots\}\subset N$ with $\sum_{n=1}^{\infty} m_n^{-1}<+\infty$ we have $L(M,g)=0$.*

Proof. Suppose that $L(M,g)>0$. Then there exists a $\delta>0$ and $n_0\in N$ such that

$$M(n)g(n)/n\ge\delta>0$$

for each $n>n_0$. From this we get

$$\frac{\delta}{ng(n)}\le\frac{M(n)}{n^2}\qquad(n>n_0). \tag{16}$$

Let i_0 be the first positive integer with $n_0<m_{i_0}$. Then the set of all positive integers $n>m_{i_0}$ can be partitioned into the intervals $(m_r,m_{r+1}]$, $(r=i_0,i_0+1,\dots)$.

Let $m_r<n\le m_{r+1}$. Then $M(n)\le r+1$ and so $M(n)/n^2\le(r+1)/n^2$. By a simple estimation we get

$$\begin{aligned}\sum_{m_r<n\le m_{r+1}}\frac{M(n)}{n^2}&\le(r+1)\cdot\sum_{m_r<n\le m_{r+1}}\frac{1}{n^2}\\&<(r+1)\int_{m_r}^{m_{r+1}}\frac{dt}{t^2}=(r+1)\left(\frac{1}{m_r}-\frac{1}{m_{r+1}}\right).\end{aligned} \tag{17}$$

We shall show that

$$\sum_{n=1}^{\infty}\frac{M(n)}{n^2}<+\infty. \tag{18}$$

For this it suffices to show by Cauchy's condition for convergence of series that for each $\varepsilon>0$ there is a $j_0\ge i_0$ such that for any two numbers $j\ge j_0$ and $k\in N$ we have

$$\sum_{n=m_j+1}^{m_{j+k}}\frac{M(n)}{n^2}<\varepsilon. \tag{19}$$

Using (17) we get

$$\sum_{n=m_{j+1}}^{m_{j+k}} \frac{M(n)}{n^2} < \sum_{r=j}^{j+k-1} (r+1)\left(\frac{1}{m_r} - \frac{1}{m_{r+1}}\right)$$
$$= (j+1)\left(\frac{1}{m_j} - \frac{1}{m_{j+1}}\right) + (j+2)\left(\frac{1}{m_{j+1}} - \frac{1}{m_{j+2}}\right)$$
$$+ \cdots + (j+k-1)\left(\frac{1}{m_{j+k-2}} - \frac{1}{m_{j+k-1}}\right) + (j+k)\left(\frac{1}{m_{j+k-1}} - \frac{1}{m_{j+k}}\right)$$
$$= \frac{j+1}{m_j} + \frac{1}{m_{j+1}} + \cdots + \frac{1}{m_{j+k-1}} - \frac{j+k}{m_{j+k}}$$
$$< \frac{j+1}{m_j} + \frac{1}{m_{j+1}} + \cdots + \frac{1}{m_{j+k-1}}.$$

Hence we get

$$\sum_{n=m_{j+1}}^{m_{j+k}} \frac{M(n)}{n^2} < \frac{j+1}{m_j} + \frac{1}{m_{j+1}} + \cdots + \frac{1}{m_{j+k-1}}. \tag{20}$$

Choose a j_0 such that for each $j \geq j_0$ we have

$$(j+1)/m_j < \varepsilon/2 \tag{21}$$

(see Theorem A) and

$$\sum_{n=j+1}^{\infty} \frac{1}{m_n} < \frac{\varepsilon}{2}. \tag{22}$$

Then (19) follows from (20) because of (21), (22). Hence (18) holds and from (16) we get $\sum_{n=1}^{\infty} 1/(ng(n)) < +\infty$ — a contradiction. Q.E.D.

THEOREM 8. *Let $M = \{m_1 < m_2 < \ldots\} \subset N$. Then $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$ If and only if $\sum_{n=1}^{\infty} M(n)/n^2 < +\infty$.*

Proof. (1) Let $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$. The convergence of the series $\sum_{n=1}^{\infty} M(n)/n^2$ is already proved in the proof of Theorem 7.

(2) Let $\sum_{n=1}^{\infty} M(n)/n^2 < +\infty$. We shall prove that $\sum_{n=1}^{\infty} m_n^{-1} < +\infty$. Put $C_k = \sum_{m_k \leq n < m_{k+1}} M(n)/n^2$ $(k = 1, 2, \ldots)$. Then $C = \sum_{n=1}^{\infty} M(n)/n^2 = \sum_{k=1}^{\infty} C_k$. By a simple estimation we get

$$C_k = k \cdot \sum_{m_k \leq n < m_{k+1}} n^{-2} \geq k \int_{m_k}^{m_{k+1}} \frac{dt}{t^2} = k\left(\frac{1}{m_k} - \frac{1}{m_{k+1}}\right).$$

But then we have for each $n = 1, 2, \ldots$

$$C \geq \sum_{k=1}^{n} C_k \geq 1\left(\frac{1}{m_1} - \frac{1}{m_2}\right) + 2\left(\frac{1}{m_2} - \frac{1}{m_3}\right) + \cdots + n\left(\frac{1}{m_n} - \frac{1}{m_{n+1}}\right)$$
$$= \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_n} - \frac{n}{m_{n+1}},$$

hence

$$\sum_{k=1}^{n} \frac{1}{m_k} \le C + \frac{n}{m_{n+1}} \le C + 1 \tag{23}$$

since $n/m_{n+1} \le 1$. As (23) holds for each $n = 1, 2, \dots$, we get by $n \to \infty$

$$\sum_{k=1}^{\infty} m_k^{-1} \le C + 1 < +\infty. \qquad \square$$

Another proof of Theorem 8 is given by Krzyś [**2**] and is also noted by Šalát [**7**].

Remark 5. (to Theorem B and previous theorems) For each set $M = \{m_1 < m_2 < \dots m_n < \dots\} \subset N$ satisfying $M(x) = O\big(x/(\log x)^{1+\varepsilon}\big)$, $\sum_{m \in M} m^{-1} < +\infty$, then $c_M = 0$, where $c_M = \lim_{x\to\infty} M(x) \log x / x$.

Proof. We have

$$\frac{M(x)\log x}{x} \le \frac{Kx}{(\log x)^{1+\varepsilon}} \cdot \frac{\log x}{x} = \frac{K}{(\log x)^{\varepsilon}}$$

for some constants $K, \varepsilon > 0$. Hence

$$\lim_{x\to\infty} \frac{K}{(\log x)^{\varepsilon}} = 0 \quad \text{and thus} \quad \lim_{x\to\infty} \frac{M(x)\log x}{x} = 0.$$

Thus Theorem B with stronger hypothesis is true.

Example 5. Let $M = \{1^2, 2^2, 3^2, \dots, n^2, \dots\}$. Then

$$M(x) = \sqrt{x} = O\big(x/(\log x)^{1+\varepsilon}\big), \qquad c_M = \lim_{x\to\infty} \frac{\sqrt{x}\log x}{x} = 0.$$

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