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SYMPLECTIC AND COSYMPLECTIC FOLIATIONS ON COSYMPLECTIC MANIFOLDS*

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Abstract. We prove that a compact symplectic or cosymplectic foliation on a cosymplectic manifold is stable. This result extends to the odd-dimensional case the corresponding one for symplectic foliations on symplectic manifolds. A large family of examples is given.

1. Introduction

As it is* well-known a compact holomorphic foliation of a Kähler manifold is stable (see [**10**]). The result holds for compact almost complex (resp., symplectic) foliations of an almost Kähler (resp., symplectic) manifold (see [**6**, **7**]).

In this paper, we study the stability of foliations on cosymplectic manifolds. First, we introduce the notion of symplectic and cosymplectic foliations on a cosymplectic manifold, accordingly to the dimension of the foliation. Then we prove that a compact symplectic or cosymplectic foliation on a cosymplectic manifold is stable. To prove this, we use our previous results for the stability of invariant foliations of almost contact manifolds [**2**].

2. Algebraic preliminaries

Let E be a $(2n+1)$-dimensional vector space over R. The space E is called *cosymplectic* if there exist a 2-form Φ and a 1-form η such that $\eta \wedge \Phi^n \neq 0$. In such a case we say that the pair (Φ, η) is a *cosymplectic structure* on E and the triple (E, Φ, η) is called a *cosymplectic vector space*.

Let (E, Φ, η) be a cosymplectic vector space. Then there is a unique vector ξ such that

$$\eta(\xi) = 1, \quad \Phi(\xi, v) = 0,$$

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for all vector $v \in E$. The vector ξ is called the *canonical vector* of the cosymplectic vector space (E, Φ, η). Note that the vector ξ is characterized by the following condition :

$$\omega(\xi)\eta \wedge \Phi^n = \omega \wedge \Phi^n,$$

for all 1-forms ω on E.

Let $E_\eta^\perp$ be the annihilator space of η, i.e.,

$$E_\eta^\perp = \{v \in E \mid \eta(v) = 0\}.$$

It is clear that $E_\eta^\perp$ is a symplectic vector space with respect to the induced 2-form Φ.

A $2s$-dimensional subspace F is called *symplectic*if it is a symplectic subspace of $E_\eta^\perp$. If dim $F = 2s+1$, then F is called *cosymplectic* if the pair (Φ, η) induces a cosymplectic structure on F with canonical vector ξ.

A $(2n+1)$-dimensional vector space E over R is said to be *almost contact* if it admits a linear mapping $\phi : E \longrightarrow E$, a vector ξ and a 1-form $\eta : E \longrightarrow R$ such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

A subspace F of E is said to be *invariant* if $\phi(v) \in F$ for all $v \in F$ (see [**11**]). We easily see that only two cases occur for any invariant subspace F of E.

(1) If the vector $\xi \notin F$, then F has even dimension, ϕ induces an almost complex structure on F and $\eta_{|F} = 0$.

(2) If the vector $\xi \in F$, then F has odd dimension and it is an almost contact vector space endowed with the restrictions of ϕ and η.

These definitions may be extended fiberwise to vector bundles. Thus, let $\pi : E \longrightarrow M$ be a vector bundle over an n-dimensional manifold M and with fiber R^{2n+1}. Then $\pi : E \longrightarrow M$ is called *cosymplectic* if there exist cross-sections Φ and η of $\Lambda^2 E^*$ and $\Lambda^1 E^*$, respectively, whose restrictions to the fibers of E define a cosymplectic structure. Hence there exists a unique cross-section ξ of $\pi : E \longrightarrow M$ such that

$$\eta(\xi) = 1, \qquad \Phi(\xi, X) = 0,$$

for all sections X of E.

The section ξ is called the *canonical section* of the cosymplectic vector bundle $\pi : E \longrightarrow M$. Note that for each point ξ_x is the canonical vector of the cosymplectic structure induced in the fiber E_x.

A vector bundle $\pi : E \longrightarrow M$ is called *almost contact* if there exist a vector bundle automorphism ϕ, a cross-section ξ of E and a cross-section η of $\Lambda^1 E^*$, whose restrictions to the fibers of E define an almost contact structure.

In a similar way, we define symplectic and cosymplectic subbundles of a cosymplectic bundle, and invariant subbundles of an almost contact bundle.

Next, let (E, Φ, η) be a cosymplectic vector bundle over M with canonical section ξ, and F a symplectic or cosymplectic subbundle. Then we have

PROPOSITION 1. *There exists an almost contact structure* (ϕ, ξ, η) *and a metric* g *in* E *such that*:

(1) $g_x(\phi_x u, \phi_x v) = g_x(u, v) - \eta_x(u)\eta_x(v)$,

(2) $\Phi_x(u, v) = g_x(u, \phi_x v)$,

(3) F_x *is an invariant vector subspace of* E_x,

for all $x \in M, u, v \in E_x$.

Proof. Let $E_\eta^\perp$ be the symplectic subbundle of E whose fiber at $x \in M$ is the space

$$(E_\eta^\perp)_x = \{u \in E_x \mid \eta_x(u) = 0\}.$$

We consider two cases, say F is a symplectic or cosymplectic subbundle of E. First, suppose that F is a symplectic subbundle of E. Thus, F is a symplectic subbundle of $E_\eta^\perp$. Then, from Theorem 3.4 of [**6**], there exists an almost complex structure J on $E_\eta^\perp$ (i.e., J is an automorphism $J : E_\eta^\perp \longrightarrow E_\eta^\perp$ of the vector bundle $E_\eta^\perp$ with $J^2 = -I$) and a metric h in $E_\eta^\perp$ such that:

(i) $h_x(u, v) = h_x(J_x u, J_x v)$,

(ii) $\Phi_x(u, v) = h_x(u, J_x v)$,

(iii) F is a complex subbundle of $E_\eta^\perp$.

We set

$$\phi_x u = J_x(u - \eta_x(u)\xi_x),$$

and

$$g_x(u, v) = h_x(u - \eta_x(u)\xi_x, v - \eta_x(v)\xi_x) + \eta_x(u)\eta_x(v),$$

for all $x \in M, u, v \in E_x$. Then it is easy to prove that (ϕ, ξ, η) is an almost contact structure, g a metric on M and (1), (2) and (3) are satisfied.

Now, suppose that F is a cosymplectic subbundle of E. Then $F_\eta^\perp$ is a symplectic subbundle of $E_\eta^\perp$. Thus, by a similar device, we deduce the result. □

3. Foliations on cosymplectic manifolds

First, we recall some definitions about foliations on manifolds [**5**, **9**].

Let F be a foliation of dimension p on a n-dimensional manifold M. We denote by TF the vector subbundle of TM which consists of the tangent vectors to F, and by T_xF the fiber of TF over x. If X is a vector field tangent to F (i.e., $X(x) \in T_xF$ for all $x \in M$) then we put $X \in F$.

The foliation F is said to be *compact* if each leaf of F is compact. A leaf L of a compact foliation F is said to be *stable* if every neighborhood U of L contains an invariant neighborhood V of L, i.e., V is a collection of leaves. F is said to be *stable* if every leaf of F is stable.

Let M be a cosymplectic manifold with structure (Φ, η), i.e., $\eta \wedge \Phi^n \neq 0$, $d\eta = 0$, $d\Phi = 0$. Then (TM, Φ, η) is a cosymplectic vector bundle. A foliation F of dimension $p = 2s$ (resp. $p = 2s + 1$) is said to be *symplectic* (resp. *cosymplectic*) if the vector subbundle TF of TM is symplectic (resp. cosymplectic).

Let us recall that an almost contact metric manifold (M, ϕ, η, ξ, g) is called *almost cosymplectic* (in the sense of Blair [**1**]) if $d\Phi = 0, d\eta = 0$, where Φ is the fundamental 2-form of M, i.e., $\Phi(X, Y) = g(X, \phi Y)$.

Now, let (M, Φ, η) be a cosymplectic manifold with canonical vector field ξ, and F a symplectic or cosymplectic foliation. Then, from Proposition 1, we have.

PROPOSITION 2. *There exists on M an almost contact metric structure (ϕ, η, ξ, g) with fundamental 2-form Φ which is almost cosymplectic, and the foliation F is invariant.*

Finally, from Proposition 2 and Theorem 1 of [**2**] we easily deduce our main result.

THEOREM 1. *A compact symplectic or cosymplectic foliation F of a cosymplectic manifold (M, Φ, η) is stable.*

4. Examples

Let S_r be the $2r+1$-dimensional solvable non-nilpotent Lie group of matrices of the form

$$\begin{pmatrix} e^z & 0 & 0 & 0 & \dots & 0 & 0 & 0 & x_1 \\ 0 & e^{-z} & 0 & 0 & \dots & 0 & 0 & 0 & y_1 \\ 0 & 0 & e^z & 0 & \dots & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & e^{-z} & \dots & 0 & 0 & 0 & y_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^z & 0 & 0 & x_r \\ 0 & 0 & 0 & 0 & \dots & 0 & e^{-z} & 0 & y_r \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x_i, y_i, z \in R, 1 \leq i \leq r$. Then S_r may be identified with R^{2r+1} by assigning to each matrix in S_r its global coordinates $(x_1, y_1, ..., x_r, y_r, z)$.

There exists a canonical injective Lie group homomorphism $j_r : S_r \longrightarrow S_{r+1}$ defined by

$$j_r(x_1, y_1, ..., x_r, y_r, z) = (x_1, y_1, ..., x_r, y_r, 0, 0, z)$$

Then S_r may be considered as a Lie subgroup of S_{r+1} and we have a chain of Lie groups

$$\{e\} \subset S_1 \subset S_2 \subset \dots \subset S_r \subset S_{r+1} \subset \dots$$

Alternatively, S_r can be described as the semidirect group $S_r = R \propto_\phi R^{2r}$, where $\phi(z) : R^{2r} \longrightarrow R^{2r}$ is given by the matrix

$$\begin{pmatrix} e^z & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{-z} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & e^z & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & e^{-z} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^z & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & e^{-z} \end{pmatrix}$$

A simple computation shows that

$$\{\bar{\alpha}_i = e^{-z}dx_i,\ \bar{\beta}_i = e^z dy_i,\ \bar{\gamma} = dz\}$$

is a family of linearly independent left invariant 1 - forms on S_r. Then we have

$$d\bar{\alpha}_i = \bar{\alpha}_i \wedge \bar{\gamma},\ d\bar{\beta}_i = -\bar{\beta}_i \wedge \bar{\gamma},\ d\bar{\gamma} = 0.$$

The corresponding dual basis of left invariant vector fields on S_r is

$$\left\{\bar{X}_i = e^z \frac{\partial}{\partial x_i},\ \bar{Y}_i = e^{-z}\frac{\partial}{\partial y_i},\ \bar{Z} = \frac{\partial}{\partial z}\right\}$$

and we have

$$[\bar{X}_i, \bar{Z}] = -\bar{X}_i,\ [\bar{Y}_i, \bar{Z}] = \bar{Y}_i,$$

all the other brackets being zero.

Now, let $B \in \mathrm{Sl}(2, Z)$ be an unimodular matrix with positive real and different eigenvalues λ and λ^{-1} and $P \in \mathrm{Gl}(2, R)$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Let be $z_0 \in R$ such that $\lambda = e^{z_0}$ and consider the lattice $L_r = P_r(Z^{2r})$, where

$$P_r = \begin{pmatrix} P & 0 & \dots & 0 \\ 0 & P & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P \end{pmatrix}$$

Then L_r is invariant by $\phi(nz_0) = \phi(z_0)^n$, $\forall n \in Z$ and $\Gamma_r = (z_0)Z \propto_\phi L_r$ is a co-compact subgroup of S_r, i.e., $\mathrm{Solv}(r) = \Gamma_r \setminus S_r$ is a compact non-nilpotent solvmanifold of dimension $2r+1$. We notice that Solv(1) is the manifold considered in [**8**] and $\mathrm{Solv}(1) \times S^1$ is the manifold considered in [**3**, **4**].

Alternatively, the manifold $\mathrm{Solv}(r)$ may be seen as the total space of a T^{2r}-bundle over S^1. In fact, let $T^{2r} = R^{2r}/L_r$ the $2r$-dimensional torus and $\rho : Z \longrightarrow \mathrm{Diff}(T^{2r})$ the representation defined as follows : $\rho(n)$ represents the transformation of T^{2r} covered by the linear transformation of R^{2r} given by the matrix

$$\begin{pmatrix} e^z & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{-z} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & e^z & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & e^{-z} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^z & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & e^{-z} \end{pmatrix}$$

This representation determines an action $A : Z \times (T^{2r} \times R) \longrightarrow T^{2r} \times R$ defined by

$$A(n, [x_1, y_1, \dots, x_r, y_r], z) = (\rho(n)([x_1, y_1, \dots, x_r, y_r]), z + n).$$

Then $p : T^{2r} \times_Z R \longrightarrow S^1$ is a T^{2r}-bundle where the projection p is given by

$$p[[x_1, y_1, ..., x_r, y_r], z] = [z].$$

Then it is clear that $T^{2r} \times_Z R$ may be canonically identified to $\mathrm{Solv}(r)$.

Since $j_r(\Gamma_r) \subset \Gamma_{r+1}$ then j_r induces a canonical embedding

$$J_r : \mathrm{Solv}(r) \longrightarrow \mathrm{Solv}(r+1).$$

If $\pi_r : S_r \longrightarrow \mathrm{Solv}(r)$ is the canonical projection, then we have a global basis $\{\alpha_i, \beta_i, \gamma\}$ of 1-forms on $\mathrm{Solv}(r)$ such that

$$\pi_r^*\alpha_i = \bar{\alpha}_i, \quad \pi_r^*\beta_i = \bar{\beta}_i, \quad \pi_r^*\gamma = \bar{\gamma},$$
$$d\alpha_i = \alpha_i \wedge \gamma, \quad d\beta_i = -\beta_i \wedge \gamma, \quad d\gamma = 0,$$

and the corresponding dual basis of vector fields, denoted by $\{X_i, Y_i, Z\}$ verifies

$$[X_i, Z] = -X_i, \qquad [Y_i, Z] = Y_i,$$

the other brackets being all zero. Obviously, $\bar{X}_i$, $\bar{Y}_i$, $\bar{Z}$ and X_i, Y_i, Z are π_r-related.

Now, for any integer s, $1 \leq s < r$, let us consider the left invariant involutive distribution $\bar{\mathbf{F}}_s$ on S_r globally spanned by $\{\bar{X}_i, \bar{Y}_i, \bar{Z} \mid 1 \leq i \leq s\}$. Then $\bar{\mathbf{F}}_s$ is a subalgebra of the Lie algebra of S_r; in fact, $\bar{\mathbf{F}}_s$ is the Lie algebra of the Lie subgroup S_s. Thus, the leaves of the foliation $\bar{F}_s$ determined by $\bar{\mathbf{F}}_s$ are all diffeomorphic to S_s. Furthermore, since $\bar{\mathbf{F}}_s$ is left invariant, then it descends to a distribution $\mathbf{F}_s$ on $\mathrm{Solv}(r)$; $\mathbf{F}_s$ defines a foliation F_s on $\mathrm{Solv}(r)$ whose leaves are all diffeomorphic to $\mathrm{Solv}(s)$.

Consider the cosymplectic structure (Φ, η) on $\mathrm{Solv}(r)$ defined by

$$\Phi = \sum_{i=1}^{r} \alpha_i \wedge \beta_i, \qquad \eta = \gamma.$$

A simple computation shows that F_s *is a cosymplectic foliation on the cosymplectic manifold* $(\mathrm{Solv}(r), \Phi, \eta)$ *and, from Theorem 1, it is stable.*

Next, let $\mathbf{F}$ be the involutive distribution on $\mathrm{Solv}(r)$ globally spanned by $\{X_i, Y_i \mid 1 \leq i \leq r\}$. Then $\mathbf{F}$ determines a foliation F on $\mathrm{Solv}(r)$ whose leaves are precisely the fibres of the fibration $p : \mathrm{Solv}(r) \longrightarrow S^1$, which are $2r$-dimensional tori. Thus, F is a compact foliation. Furthermore, it is easy to prove that F *is a symplectic foliation on the cosymplectic manifold* $(\mathrm{Solv}(r), \Phi, \eta)$ *and, from Theorem 1, it is stable.* (We notice that this last result follows directly since the leaves of F are the fibres of p, which is a fibration with compact fibres [**9**]).

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