ON HYPERCYLINDERS IN CONFORMALLY
SYMMETRIC MANIFOLDS

Ryszard Deszcz

Abstract. Hypercylinders in conformally symmetric manifolds are considered. The main result is the following theorem: Let (M, g) be a hypercylinder in a parabolic essentially conformally symmetric manifold $(N, \widetilde{g})$, $\dim N \geq 5$ and let $\overline{U}$ be the subset od N consisting of all points of N at which the Ricci tensor $\widetilde{S}$ of $(N, \widetilde{g})$ is not recurrent. If $\overline{U} \cap M$ is a dense subset of M, then (M, g) is a conformally recurrent manifold.

1. Introduction. Totally umbilical submanifolds in locally symmetric, recurrent, conformally flat, conformally symmetric and conformally recurrent manifolds were investigated by many authors (e.g. [6], [10], [19], [21], [24]–[27], [29], [33]). An important part of these investigation treats problems concerneing totally umbilical hypersurfaces in these classes of manifolds (e.g. [7], [8], [20], [28], [30]). On the other hand, totally umbilical hypersurfaces, as well as hypercylinders, are special examples of quasi-umbilical hypersurfaces. Certain results on quasi-umbilical hypersurfaces in locally symmetric, recurrent and conformally flat manifolds are presented in [3], [34] and [22] respectively. Moreover, hypercylinders in locally symmetric and conformally flat manifolds were studied in [9] and [37] (see also [2]) respectively. We shall continue study in this direction considering hypercylinders in conformally symmetric manifolds.

Let $(N, \widetilde{g})$ be an n-dimensional, $n \geq 4$, semi-Riemannian manifold with the metric tensor $\widetilde{g}$ and let $\widetilde{\nabla}$ be the Levi-Civita connection of $(N, \widetilde{g})$. Let $(N, \widetilde{g})$ be covered by a system of charts $\{\widetilde{U}; x^r\}$. We denote by $\widetilde{g}_{rs}$, $\left\{ \begin{matrix} \widetilde{r} \\ s\ t \end{matrix} \right\}$, $\widetilde{\nabla}_s$, $\widetilde{R}_{rstu}$, $\widetilde{C}_{rstu}$, $\widetilde{S}_{ts}$ and $\widetilde{K}$ the local components of the metric tensor $\widetilde{g}$, the Christoffel symbols, the operator of covariant differentiation, the Riemann-Christoffel curvature tensor $\widetilde{R}$, the Weyl conformal curvature tensor $\widetilde{C}$, the Ricci tensor $\widetilde{S}$ and the scalar curvature $\widetilde{K}$ of $(N, \widetilde{g})$ respectively, where $r, s, t, u, v, w \in \{1, 2, \dots, n\}$.

We have

$$\widetilde{C}_{rstu} = \widetilde{R}_{rstu} + \frac{\widetilde{K}}{(n-1)(n-2)}(\widetilde{g}_{ru}\widetilde{g}_{st} - \widetilde{g}_{rt}\widetilde{g}_{su})$$
$$- \frac{1}{n-2}(\widetilde{g}_{ru}\widetilde{S}_{ts} + \widetilde{g}_{ts}\widetilde{S}_{ru} - \widetilde{g}_{rt}\widetilde{S}_{su} - \widetilde{g}_{su}\widetilde{S}_{rt}). \tag{1.1}$$

A $(0,k)$-tensor field T on N is said to be recurrent [36] if the condition

$$T(X_1, \dots, X_k)\widetilde{\nabla}T(Y_1, \dots, Y_k; Z) = T(Y_1, \dots, Y_k)\widetilde{\nabla}T(X_1, \dots, X_k; Z)$$

holds on N, where $X_1, \dots, X_k, Y_1, \dots, Y_k, Z \in \mathfrak{X}(N)$, $\mathfrak{X}(N)$ being the Lie algebra of vector fields on N. In particular, if $\widetilde{\nabla}T$ vanishes on N, then T is called parallel. A manifold $(N, \widetilde{g})$, $n \geq 4$, is said to be locally symmetric [31] (resp. conformally symmetric [4]) if its tensor $\widetilde{R}$ (resp. tensor $\widetilde{C}$) is parallel with respect to $\widetilde{\nabla}$. Further, a manifold $(N, \widetilde{g})$, $n \geq 4$, is said to be recurrent [38] (resp. conformally recurrent [1] or Ricci recurrent [32]) if its tensor $\widetilde{R}$ (resp. tensor $\widetilde{C}$ or tensor $\widetilde{S}$) is recurrent. A conformally symmetric manifold $(N, \widetilde{g})$ which is neither locally symmetric nor conformally flat is called essentially conformally symmetric or shortly e.c.s. manifold. Various examples of e.c.s. manifolds are given in [35], [11] and [18]. All e.c.s. metrics are indefinite ([16, Theorem 2]). Any e.c.s. manifold $(N, \widetilde{g})$ satisfies the following equation ([18], [17])

$$F\widetilde{C}(X,Y,Z,W) = \widetilde{S}(X,W)\widetilde{S}(Y,Z) - \widetilde{S}(X,Z)\widetilde{S}(Y,W)$$

for some function F, where $X, Y, Z, W \in \mathfrak{X}(M)$. F is called the fundamental function of $(N, \widetilde{g})$. All e.c.s. manifolds can be divided into the following five non-empty and mutually disjoint classes (according to the behaviour of the Ricci tensor and the fundamental function F [12]):

Class I. Ricci recurrent ones (they all satisfy $F = 0$).

Class II. Parabolic e.c.s. manifold [15] (satisfying $F = 0$ identically but not Ricci reccurent).

Class III. Elliptic ones [14] ($F = $ constant $\neq 0$, semidefinite everywhere).

Class IV. Hyperbolic ones [13] ($F = $ constant $\neq 0$, semidefinite nowhere)

Class V. Those with F non-constant.

LEMMA 1 (Theorem 7, 8, 9 and formula (6) of [17] and Theorem 7 of [18]). *Let $(N, \widetilde{g})$ be an e.c.s. manifold and let $\{\widetilde{U}; x^r\}$ be a chart on N. Then the following relations are satisfied on $\widetilde{U}$:*

$$\widetilde{\nabla}_v\widetilde{C}_{rstu} = 0, \tag{1.2}$$

$$F\widetilde{C}_{rstu} = \widetilde{S}_{ur}\widetilde{S}_{ts} - \widetilde{S}_{tr}\widetilde{S}_{us}, \tag{1.3}$$

$$\widetilde{\nabla}_r\widetilde{S}_{ts} = \widetilde{\nabla}_t\widetilde{S}_{rs}, \tag{1.4}$$

$$\widetilde{K} = 0, \tag{1.5}$$

$$\widetilde{S}^v{}_r \widetilde{C}_{vstu} = 0, \qquad \widetilde{S}^v{}_r = \widetilde{g}^{vt} \widetilde{S}_{tr}, \tag{1.6}$$

$$\widetilde{\nabla}_w \widetilde{\nabla}_v \widetilde{R}_{rstu} - \widetilde{\nabla}_v \widetilde{\nabla}_w \widetilde{R}_{rstu} = 0, \tag{1.7}$$

$$\widetilde{S}_{vw} \widetilde{C}_{rstu} + \widetilde{S}_{vr} \widetilde{C}_{swtu} + \widetilde{S}_{vs} \widetilde{C}_{wrtu} = 0. \tag{1.8}$$

2. Hypercylinders. Let M be a hypersurface in an n-dimensional, $n \geq 4$, semi-Riemannian manifold $(N, \widetilde{g})$ and let the tensor g, induced by the metric tensor $\widetilde{g}$, be the metric tensor of M. Moreover, let $x^r = x^r(y^a)$ be the local expression of M in N. Then we have $g_{ab} = \widetilde{g}_{rs} B_{ab}^{rs}$, where

$$B_{a_1 \ldots a_k}^{r_1 \ldots r_k} = B_{a_1}^{r_1} \ldots B_{a_k}^{r_k}, \qquad B_a^r = \partial_a x^r, \qquad \partial_a = \partial/(\partial y^a),$$

and g_{ab} are the local components of the tensor g. Further, we denote by $\left\{ \begin{matrix} a \\ b \ c \end{matrix} \right\}$, R_{abcd}, S_{ad}, C_{abcd} and K the local components of the Christoffel symbols, the curvature tensor R, the Ricci tensor S, the Weyl conformal curvature tensor C and the scalar curvature K of (M, g) respectively. Here and below, $a, b, c, d, e, f, h, i, j \in \{1, 2, \ldots, n-1\}$. Let N^r be the local components of a local unit vector field normal to M. Then we have the following relations

$$\widetilde{g}_{rs} N^r N^s = \varepsilon, \quad \widetilde{g}_{rs} N^r B_a^s = 0, \quad g^{ab} B_{ab}^{rs} = \widetilde{g}^{rs} - \varepsilon N^r N^s, \quad \varepsilon = \pm 1. \tag{2.1}$$

The hypersurface (M, g) is said to be a cylindrical hypersurface or shortly a hypercylinder (cf. [5, pp. 147–148], [9]) in $(N, \widetilde{g})$ if the second fundamental tensor H of (M, g) satisfies on M the condition $H = \beta u \otimes u$, where β is a function and u a 1-form on M, respectively. Let p be a point of the hypercylinder (M, g). Then the following equality

$$H_{ad} = \beta u_a u_d \tag{2.2}$$

holds on some neighbourhood $U \subset M$ of p, where H_{ad} and u_a are the local components of H and u on U, respectively. We denote by ∇ the operator of the van der Waerden-Bortolotti covariant derivative. Then, in virtue of (2.2), the Gauss and Weingarten formulas for (M, g) in $(N, \widetilde{g})$ take on U the following form

$$\nabla_a B_d^r = \varepsilon H_{ad} N^r = \varepsilon \beta u_a u_d N^r, \tag{2.3}$$

$$\nabla_a N^r = -H_{ac} g^{cd} B_d^r = -\beta u_a u^d B_d^r, \qquad u^d = g^{da} u_a, \tag{2.4}$$

respectively. Furthermore, by (2.2), the Gauss and Codazzi equations for (M, g) in $(N, \widetilde{g})$ can be expressed on U as follows:

$$R_{abcd} = \widetilde{R}_{rstu} B_{abcd}^{rstu} + \varepsilon(H_{ad} H_{bc} - H_{ac} H_{bd}) = \widetilde{R}_{rstu} B_{abcd}^{rstu}, \tag{2.5}$$

$$V_{bcd} = N^r \widetilde{R}_{rstu} B_{bcd}^{stu} = \nabla_d H_{bc} - \nabla_c H_{bd} = u_b(\beta_d u_c - \beta_c u_d)$$
$$+ \beta(u_c \nabla_d u_b - u_d \nabla_c u_b + u_b(\nabla_d u_c - \nabla_c u_d)), \quad \beta_c = \nabla_c \beta. \tag{2.6}$$

From this, by contraction with g^{bc} and making use of (2.1), we obtain

$$v_d = g^{bc} V_{bcd} = N^r \widetilde{S}_{rs} B_d^s$$
$$= g^{bc} u_b u_c \beta_d - u^h \beta_h u_d + \beta(-u^h \nabla_h u_d - (\nabla_h u^h) u_d + 2 u^h \nabla_d u_h). \tag{2.7}$$

LEMMA 2. *Let (M, g) be a hypercylinder in a semi-Riemannian manifold $(N, \widetilde{g})$, $n \geq 4$. If p is a point of M such that the relations (2.2) and $\beta \neq 0$ are satisfied at every point of some neighbourhood $U \subset M$ of p then the equality*

$$V_{bcd} = u_b u^h V_{hcd} + u_d(u^h V_{bch} + u_b v_c) - u_c(u^h V_{bdh} + u_b v_d) \qquad (2.8)$$

holds on U.

Proof. From (2.5), by making use of (2.3) and (2.6), it follows that

$$\nabla_e R_{abcd} = B^{vrstu}_{eabcd} \widetilde{\nabla}_v \widetilde{R}_{rstu} + \varepsilon\beta u_e(u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab}). \qquad (2.9)$$

This, by contraction with g^{bc} and an application of (2.1), (2.7) and the identity

$$\widetilde{g}^{vr} \widetilde{\nabla}_v \widetilde{R}_{rstu} = \widetilde{\nabla}_u \widetilde{S}_{ts} - \widetilde{\nabla}_t \widetilde{S}_{us}, \qquad (2.10)$$

yields

$$\nabla_e S_{ad} = B^{vts}_{eda} \widetilde{\nabla}_v \widetilde{S}_{ts} - \varepsilon N^s N^t B^{vru}_{ead} \widetilde{\nabla}_v \widetilde{R}_{rstu} + \varepsilon\beta u_e K_{ad}, \qquad (2.11)$$

where

$$K_{ad} = u_a v_d + u_d v_a + u^h V_{bch} + u^h V_{cbh}, \qquad (2.12)$$

$$k = g^{ad} K_{ad} = 4u^h v_h. \qquad (2.13)$$

On the other hand, contracting (2.9) with g^{ae} and using (2.10) and (2.2), we find

$$\nabla_d S_{bc} - \nabla_c S_{bd} = B^{uts}_{dcb}(\widetilde{\nabla}_u \widetilde{S}_{ts} - \widetilde{\nabla}_t \widetilde{S}_{us}) - \varepsilon N^v N^r B^{stu}_{bcd} \widetilde{\nabla}_v \widetilde{R}_{rstu}$$
$$+ \varepsilon\beta(V_{bcd} - u_b u^h V_{hcd} - u_c u^h V_{dbh} + u_d u^h V_{cbh}). \qquad (2.14)$$

The following equality follows immediately from the second Bianchi identity

$$N^v N^r B^{stu}_{bcd} \widetilde{\nabla}_v \widetilde{R}_{rstu} = N^v N^r B^{uts}_{dcb} \widetilde{\nabla}_u \widetilde{R}_{tvrs} - N^v N^r B^{tus}_{cdb} \widetilde{\nabla}_v \widetilde{R}_{uvrs},$$

which, in virtue of (2.11) and (2.12), turns into

$$\varepsilon N^v N^r B^{stu}_{bcd} \widetilde{\nabla}_v \widetilde{R}_{rstu} = B^{vru}_{dcb}(\widetilde{\nabla}_v \widetilde{S}_{ru} - \widetilde{\nabla}_r \widetilde{S}_{vu}) + \nabla_c S_{db} - \nabla_d S_{cb}$$
$$+ \varepsilon\beta u_d(u^h V_{cbh} + u^h V_{bch} + u_b v_c) - u_c(u^h V_{dbh} + u^h V_{bdh} + u_b v_d).$$

The last equation, together with (2.14), completes the proof.

LEMMA 3. *Let (M, g) be a hypercylinder in a semi-Riemannian manifold $(N, \widetilde{g})$, $n \geq 4$. If p is a point of M such that the relations (1.7) and (2.2) are fulfilled at every point of some neighbourhood $U \subset M$ of p then the equalities*

$$v^h V_{hef} = 0, \qquad (2.15)$$

$$u^h v_h \omega_{abc} = 0 \qquad (2.16)$$

hold on U, where

$$\omega_{acd} = \beta(u_a(\nabla_d u_c - \nabla_c u_d) + u_c(\nabla_a u_d - \nabla_d u_a) + u_d(\nabla_c u_a - \nabla_a u_c)). \qquad (2.17)$$

Proof. Transvecting (1.7) with B^{rstuvw}_{abcdef} and using the Ricci identity, (2,1), (2.5) and (2.6), we find

$$
\begin{aligned}
&\nabla_f \nabla_e R_{abcd} - \nabla_e \nabla_f R_{abcd} \\
&\quad + \varepsilon(-V_{bcd}V_{aef} + V_{acd}V_{bef} - V_{dab}V_{cef} + V_{cab}V_{def}) = 0.
\end{aligned}
\tag{2.18}
$$

Next, contracting the above equation with g^{ad} and g^{bc} and applying (2.7) we get (2.15). Finally, (2.16) is an immediate consequence of (2.15) and the following identity

$$
u_a V_{bcd} + u_c V_{bda} + u_d V_{bac} = u_b \omega_{acd}.
\tag{2.19}
$$

Our lemma is thus proved.

Remark. In the next sections we shall consider hypercylinders satisfying certain additional conditions. Let (M, g) be a hypercylinder in a manifold $(N, \widetilde{g})$ and let p be a point of M such that the relation (2.2) is fulfilled at every point of some neighbourhood $U \subset M$ of p. We assume that at every point of U one of the following relations is satisfied:

$$
g^{ad} u_a u_d = 1,
\tag{2.20}
$$

$$
g^{ad} u_a u_d = -1,
\tag{2.21}
$$

$$
g^{ad} u_a u_d = 0.
\tag{2.22}
$$

Thus the scalar $g^{ad} u_a u_d$ is a constant on U. This fact implies

$$
u^h \nabla_a u_h = 0.
\tag{2.23}
$$

Transvecting now (2.6) with u^b, u^c and u^d respectively and applying (2.20)–(2.23) we easily get

$$
u^h V_{hcd} = \eta(\beta_d u_c - \beta_c u_d + \beta(\nabla_d u_c - \nabla_c u_d)),
\tag{2.24}
$$

$$
u^h u^j V_{hjd} = \eta^2 \beta_d - \eta(u^h \nabla_h u_d + \beta u^h \nabla_h u_d),
\tag{2.25}
$$

$$
u^h V_{bch} = (u^h \beta_h u_c - \eta \beta_c) u_b + (u_c u^h \nabla_h u_b - \eta \nabla_c u_b + u_b u^h \nabla_h u_c)
\tag{2.26}
$$

respectively, where $\eta \in \{-1, 0, 1\}$.

3. Hypercylinders in conformally symmetric manifolds.

LEMMA 4. *Let (M, g) be a hypercylinder in a conformally symmetric manifold $(N, \widetilde{g})$, $n \geq 4$ and let p be a point of M such that the conditions (2.2) and $\beta \neq 0$ are satisfied at every point of some neighbourhood $U \subset M$ of p. Then we have:*

(i) *the equality*

$$
\begin{aligned}
\nabla_e C_{abcd} = \varepsilon \beta u_e &\left(\frac{k}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd}) \right. \\
&\left. + \left(u_b u_c - \frac{g_{bc}}{n-3}\right) K_{ad} + \left(u_a u_d - \frac{g_{ad}}{n-3}\right) K_{bc} \right.
\end{aligned}
$$

$$- \left(u_a u_c - \frac{g_{ac}}{n-3} \right) K_{bd} - \left(u_b u_d - \frac{g_{bd}}{n-3} \right) K_{ac} \bigg) \qquad (3.1)$$

holds on U.

(ii) *If at every point of U (2.21) is fulfilled then $\nabla C = 0$ holds on U.*

Proof. (i): Transvecting (1.2) with $\varepsilon N^s N^t B_{ead}^{vru}$ and using (1.1) and (2.1) we get

$$-\varepsilon N^s N^t B_{ead}^{vru} \widetilde{\nabla}_v \widetilde{R}_{rstu} = -\frac{B_{eda}^{vur} \widetilde{\nabla}_v \widetilde{S}_{ur} + \varepsilon(\widetilde{\nabla}_v \widetilde{S}_{ur}) N^r N^u B_e^v g_{ad}}{n-2} + \frac{B_e^v (\widetilde{\nabla}_v \widetilde{K}) g_{ad}}{(n-1)(n-2)}.$$

Substituting this in (2.11) we find

$$\frac{B_{eda}^{vur}(\widetilde{\nabla}_v \widetilde{S}_{ur})}{n-2} = \frac{\nabla_e S_{ad}}{n-3} - \frac{\varepsilon \beta u_e K_{ad}}{n-3}$$
$$+ \frac{\varepsilon N^r N^u B_e^v (\widetilde{\nabla}_v \widetilde{S}_{ur}) g_{ad}}{(n-1)(n-2)} - \frac{B_e^v (\widetilde{\nabla}_v \widetilde{K}) g_{ad}}{(n-1)(n-2)}. \qquad (3.2)$$

Contracting (3.2) with g^{ad} and using (2.1) and (2.13) we obtain

$$\frac{2\varepsilon N^r N^u B_e^v \widetilde{\nabla}_v \widetilde{S}_{ur}}{n-3} = -\frac{\nabla_e K}{n-3} + \frac{\varepsilon \beta k u_e}{n-3} + \frac{2 B_e^v \widetilde{\nabla}_v \widetilde{K}}{n-1}.$$

Now, by the above equation, (3.2) takes the form

$$\frac{B_{eda}^{vur} \widetilde{\nabla}_v \widetilde{S}_{ur}}{n-2} = \frac{\nabla_e S_{ad}}{n-3} - \frac{\varepsilon \beta u_e K_{ad}}{n-3}$$
$$+ \frac{B_e^v (\widetilde{\nabla}_v \widetilde{K}) g_{ad}}{(n-1)(n-2)} + \frac{\varepsilon \beta k u_e g_{ad}}{2(n-2)(n-3)} - \frac{(\nabla_e K) g_{ad}}{2(n-1)(n-2)}.$$

But this, together with (2.9), (1.1) and (1.2) gives

$$\nabla_e C_{abcd} = \varepsilon \beta u_e \bigg(u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab}$$
$$- \frac{g_{ad} K_{bc} + g_{bc} K_{ad} - g_{ac} K_{bd} - g_{bd} K_{ac}}{n-3} + \frac{k(g_{ad} g_{bc} - g_{ac} g_{bd})}{(n-2)(n-3)} \bigg). \qquad (3.3)$$

On the other hand, (2.8), (2.12) and (2.13) yield

$$u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab}$$
$$= u_a u_d K_{bc} + u_b u_c K_{ad} - u_a u_c K_{bd} - u_b u_d K_{ac}. \qquad (3.4)$$

Now (3.3) turns into (3.1), completing the proof of (i).

(ii): Identity (2.8), in virtue of (2.24), (2.26), (2.7) and (2.21), reduces to $V_{bcd} = 0$. Now (3.3) completes the proof.

From Lemma 4(i) the following proposition follows easily.

PROPOSITION 1. *Let (M, g) be a hypercylinder in a conformally symmetric manifold $(N, \widetilde{g})$, $n \geq 5$ and let p be a point of M such that the conditions (2.2), (2.20) and $\beta \neq 0$ are fulfilled at every point of some neighbourhood $U \subset M$ of p. Then the condition $\nabla C = 0$ is satisfied on U if and only if the relation*

$$K_{ad} = \frac{k}{2(n-2)}((n-3)u_a u_d + g_{ad})$$

holds on U.

LEMMA 5. *Let (M, g) be a hypercylinder in a conformally symmetric manifold $(N, \widetilde{g})$, $n \geq 5$ and let p be a point of M such that the relation $H_{ad} = 0$ holds at p. Then the tensor ∇C vanishes at p.*

Proof. We note that the equality

$$\nabla_e(\widetilde{C}_{rstu} B^{rstu}_{abcd}) = B^{vrstu}_{eabcd} \widetilde{\nabla}_v \widetilde{C}_{rstu} + \widetilde{C}_{rstu} B^{stu}_{bcd} \nabla_e B^r_a - \widetilde{C}_{srtu} B^{rtu}_{acd} \nabla_e B^r_b$$
$$+ \widetilde{C}_{turs} B^{urs}_{dab} \nabla_e B^t_c - \widetilde{C}_{utrs} B^{trs}_{cab} \nabla_e B^u_d$$

holds on some neighbourhood U of p. This, by (1.1), (2.5), (1.2) and (2.3) reduces at p to

$$\nabla_e R_{abcd} + \frac{(\nabla_e \widetilde{K})(g_{ad} g_{bc} - g_{ac} g_{bd})}{(n-1)(n-2)} \tag{3.5}$$
$$- \frac{1}{n-2} \left(g_{bc} B^{vru}_{ead} \widetilde{\nabla}_v \widetilde{S}_{ru} + g_{ad} B^{vst}_{ebc} \widetilde{\nabla}_v \widetilde{S}_{st} - g_{bd} B^{vrt}_{eac} \widetilde{\nabla}_v \widetilde{S}_{rt} - g_{ac} B^{vsu}_{ebd} \widetilde{\nabla}_v \widetilde{S}_{su} \right).$$

Next, contracting (3.5) with g^{bc} we obtain

$$\frac{B^{vru}_{ead} \widetilde{\nabla}_v \widetilde{S}_{ru}}{n-2} = \frac{\nabla_e S_{ad}}{n-3} + \frac{(\nabla_e \widetilde{K}) g_{ad}}{(n-1)(n-3)} - \frac{(\widetilde{\nabla}_v \widetilde{S}_{st}) B^{vst}_{ebc} g^{bc} g_{ad}}{(n-2)(n-3)}.$$

Substituting this into (3.5) we get our assertion.

LEMMA 6. *Let (M, g) be a hypercylinder in a conformally symmetric manifold $(N, \widetilde{g})$, $n \geq 5$ and let p be a point of M such that the relations (2.2) and $\beta \neq 0$ are satisfied at every point of some neighbourhood $U \subset M$ of p. If the equality (2.22) is fulfilled at p then the tensor ∇C vanishes at p.*

Proof. Transvecting (2.6) with u^b and u^d and using (2.22) we get

$$u^h V_{bch} = u^h \beta_h u_b u_c + \beta(u_c u^h \nabla_h u_b + u_b u^h \nabla_h u_c - u_b u^h \nabla_c u_h),$$
$$u^h V_{hcd} = \beta(u_c u^h \nabla_d u_h - u_d u^h \nabla_c u_h)$$

respectively. Moreover, from (2.7), by (2.22), we obtain

$$v_d = -u^h \beta_h u_d + \beta(2u^h \nabla_d u_h - (\nabla_h u^h) u_d - u^h \nabla_h u_d).$$

Now we can verify that the identity (2.8), by the above three relations, reduces to $V_{bcd} = 0$. Finally, (3.3) completes the proof.

4. Hypercylinders in non-Ricci-recurrent parabolic e.c.s. manifolds.

LEMMA 7 [**15**, Lemmas 1 and 4]. *Let* $(N, \widetilde{g})$, $n \geq 4$, *be a parabolic e.c.s. manifold. If p is a point of N such that*

$$(\widetilde{S}_{ur} \widetilde{\nabla}_v \widetilde{S}_{ts} - \widetilde{S}_{ts} \widetilde{\nabla}_v \widetilde{S}_{ur})(p) \neq 0, \tag{4.1}$$

then there exists a neighbourhood $\widetilde{U}$ of p with two vector fields A and B which are unique (up to a sign of A) determined by the following two conditions

$$\widetilde{S}_{rs} = e A_r A_s, \qquad e = \pm 1, \tag{4.2}$$

$$\widetilde{\nabla}_u \widetilde{S}_{rs} = B_u \widetilde{S}_{rs} + B_r \widetilde{S}_{su} + B_s \widetilde{S}_{ur}, \tag{4.3}$$

where A_r and B_r are the local components of A and B respectively. The vector fields A and B satisfy on $\widetilde{U}$ the following relations:

$$\widetilde{g}^{rs} A_r A_s = 0, \qquad \widetilde{g}^{rs} A_r B_s = 0, \tag{4.4}$$

$$\widetilde{\nabla}_s A_r = (1/2) A_r B_s + A_s B_r, \tag{4.5}$$

$$\widetilde{\nabla}_s B_r = B_r B_s + 3\lambda B_r A_s + \lambda A_r B_s + \sigma A_r A_s, \tag{4.6}$$

where λ and σ are some functions on $\widetilde{U}$. Moreover, we have

$$\widetilde{C}_{rstu} = -\Phi(A_r B_s - A_s B_r)(A_t B_u - A_u B_t) \tag{4.7}$$

for a certain (uniquely determined) function Φ.

LEMMA 8. *Let (M, g) be a hypercylinder in a parabolic e.c.s. manifold $(N, \widetilde{g})$, $n \geq 4$ and let p be a point of M such that the conditions: (2.2), (2.20), (4.1) and*

$$N^r A_r \neq 0 \tag{4.8}$$

are fulfilled at every point of some neighbourhood $U \subset M$ of p. Then the equality

$$\omega_{abc} = 0 \tag{4.9}$$

holds on U.

Proof. The equality (2.16), in virtue of (2.7), (4.2) and (4.8), give

$$u^h A_h \omega_{abc} = 0, \tag{4.10}$$

where

$$A_h = A_r B_h^r. \tag{4.11}$$

Suppose that at a point q of U we have

$$\omega_{abc}(q) \neq 0. \tag{4.12}$$

Then, by (4.10), the equality

$$u^h A_h = 0 \tag{4.13}$$

holds on some open subset $U' \subset U$. From this we obtain

$$A_h \nabla_c u^h + u^h \nabla_c A_h = 0. \tag{4.14}$$

Using (4.5) and (2.3), we can easily verify that the following equality is fulfilled on U

$$\nabla_c A_a = A_a B_c / 2 + A_c B_a + \varepsilon \beta N^r A_r u_a u_c, \tag{4.15}$$

$$B_c = B_r B_c^r. \tag{4.16}$$

Substituting (4.15) into (4.14) and applying (4.13) we get

$$A_h \nabla_c u^h + u^h B_h A_c + \varepsilon \beta N^r A_r u_c = 0. \tag{4.17}$$

The formula (2.17), because of (2.7), (4.2) and (4.8), yields

$$A^h V_{hef} = 0, \tag{4.18}$$

where $A^h = g^{ah} A_a$. Thus (4.18), by (2.6) and (4.13), gives

$$A^h (u_v \nabla_d u_h - u_d \nabla_c u_h) = 0,$$

whence, by (4.17), it follows that

$$u^h B_h (u_c A_d - u_d A_c) = 0.$$

If $(u_c A_d - u_d A_c)(q) = 0$ then also $A_d(q) = 0$. The last equation, in virtue of the relation

$$g^{ad} A_a A_d + \varepsilon (N^r A_r)^2 = 0, \tag{4.19}$$

which follows immediately from (4.4) and (2.1), gives $(N^r A_r)(q) = 0$. But this contradicts (4.8). If $(u^h B_h)(q) = 0$ then from (4.7), by transvection with $N^r B_{bcd}^{stu} u^b$ and the use of (1.1), (2.1), (4.2), (4.11), (4.16), (1.5) and (4.13), we obtain

$$(u^h V_{hcd} + (e/(n - 2)) N^r A_r (A_c u_d - A_d u_c))(q) = 0. \tag{4.20}$$

On the other hand, transvecting (2.19) with u^a we get

$$u_a u^h V_{hcd} + u_c u^h V_{hda} + u_d u^h V_{hac} = \omega_{acd}.$$

This, by (4.20), gives $\omega_{acd}(q) = 0$, a contradiction. Our lemma is thus proved.

LEMMA 9. *Let (M, g) be a hypercylinder in a parabolic e.c.s. manifold $(N, \widetilde{g})$ and let p be a point of M such that the conditions* (2.2), (2.20), (4.1) *and*

$$N^r A_r = 0 \tag{4.21}$$

are fulfilled. Then the equality (4.9) *holds at p.*

Proof. First of all we note that at p the following relation

$$A_d \neq 0 \tag{4.22}$$

is satisfied. In fact, if we had $A_d = 0$ the, by (4.21), we get $A(p) = 0$ and $\widetilde{S}(p) = 0$, which contradicts (4.1). Transvecting now (1.8) with $N^w B^{vrstu}_{eabcd}$ and making use of (4.2), (4.22), (4.21), (2.1) and (2.6), we find

$$A_a V_{bcd} = A_b V_{acd}. \qquad (4.23)$$

Multiplying (4.23) by u_f and summing the resulting equality cyclically in f, c, d and applying (2.19) we obtain $(A_a u_b - A_b u_a)\omega_{fcd} = 0$. Assume that $A_a u_b - A_b u_a$ vanishes at p. Then we have

$$A_a = u^h A_h u_a. \qquad (4.24)$$

So, (4.23) turns into $u^h A_h (u_a V_{bcd} - u_b V_{acd}) = 0$. Summing this cyclically in a, c, d and using again (2.19), we get $u^h A_h \omega_{acd} = 0$. From (4.24), in virtue of (4.22), it follows that $u^h A_h$ is non-zero at p. Thus the last equality completes the proof.

PROPOSITION 2. *Let (M, g) be a hypercylinder in a parabolic e.c.s. manifold $(N, \widetilde{g})$, $n \geq 5$, and let p be a point of M such that the conditions: (2.2), $\beta \neq 0$, (2.20) and (4.1) are fulfilled at every point of some neighbourhood $U \subset M$ of p. Then C is a recurrent tensor on U.*

Proof. The identity (2.19), in view of Lemmas 8 and 9 reduces to

$$u_a V_{bcd} + u_c V_{bda} + u_d V_{bac} = 0. \qquad (4.25)$$

This, by transvection with u^a, yields

$$V_{bcd} = u_d u^h V_{bch} - u_c u^h V_{bdh}. \qquad (4.26)$$

On the other hand, transvecting (4.7) with $N^r B^{stu}_{bcd}$ and using (1.1), (2.6), (4.2), (2.1), (1.5) and (4.7), we find

$$V_{bcd} = \frac{e N^r A_r}{n-2}(A_d g_{bc} - A_c g_{bd}) + \phi D_b(A_c B_d - A_d B_c), \qquad (4.27)$$

where $D_b = N^r B_r A_b - N^r A_r B_b$. Substituting (4.27) into (4.26) and (4.25) respectively, we obtain

$$\begin{aligned} V_{bcd} &= \Phi D_b(u_d C_c - u_c C_d) \\ &+ ((e N^r A_r)/(n-2))(u^h A_h(u_d g_{bc} - u_c g_{bd}) - u_b(A_c u_d - A_d u_c)), \quad (4.28) \end{aligned}$$

$$\Phi D_b(u_a(A_c B_d - A_d B_c) + u_c(A_d B_a - A_a B_d) + u_d(A_a B_c - A_c B_a))$$

$$\begin{aligned} \frac{e N^r A_r}{n-2}((u_a A_d - u_d A_a)g_{bc} \\ + (u_c A_a - u_a A_c)g_{bd} + (u_d A_c - u_c A_d)g_{ab}) = 0 \qquad (4.29) \end{aligned}$$

resepectively, where $C_c = u^h B_h A_c - u^h A_h B_c$. From (4.29), by transvection with $u^a u^b$, we get

$$\Phi u^h D_h(A_c B_d - A_d B_c) = u_d C_c - u_c C_d. \qquad (4.30)$$

Further, asumming (4.28) cyclically in b, c, d, we obtain

$$D_b(u_d C_c - u_c C_d) + D_c(u_b C_d - u_d C_b) + D_d(u_c C_b - u_b C_c) = 0,$$

which, by multiplication with u_f and antisymmetrization in b, f, gives

$$(u_f D_b - u_b D_f)(u_d C_c - u_c C_d) = (C_b u_f - C_f u_b)(u_d D_c - u_c D_d).$$

But this implies

$$C_c(u_f D_b - u_b D_f) = (D_c - u^h D_h u_c)(u_f C_b - u_b C_f). \tag{4.31}$$

If $C_c(p) = 0$ then (4.28) turns into

$$V_{bcd} = \frac{eN^r A_r}{n-2}(u^h A_h(u_d g_{bc} - u_c g_{bd}) - u_b(A_c u_d - A_d u_c)).$$

Using this we can rewrite (3.3) in the following form

$$\nabla_e C_{abcd} = \varepsilon \beta u_e \frac{k}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd})$$
$$+ \frac{2eN^r A_r u^h A_h}{n-2}(u_a u_d g_{bc} + u_b u_c g_{ad} - u_a u_c g_{bd} - u_b u_d g_{ac})$$
$$- \frac{1}{n-3}(g_{ad}K_{bc} + g_{bc}K_{ad} - g_{ac}K_{bd} - g_{bd}K_{ac}), \tag{4.32}$$

which reduces to $\nabla C = 0$. If $C_c(p) \neq 0$ then (4.31) and (4.30) yield

$$u_f D_b - u_b D_f = \tau(A_b B_f - A_f B_b), \qquad \tau \in \mathbf{R}. \tag{4.33}$$

Moreover, using (4.28) and (4.33) we obtain

$$u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab} = ((eN^r A_r)/(n-2))((u_a A_d + u_d A_a)g_{bc}$$
$$+ (u_c A_b + u_b A_c)g_{ad} - (u_a A_c + u_c A_a)g_{bd} - (u_b A_d + u_d A_b)g_{ac})$$
$$- 2\Phi\tau(A_a B_b - A_b B_a)(A_c B_d - A_d B_c).$$

This, by an application of $\widetilde{C}_{rstu} B^{rstu}_{abcd} = -\Phi(A_a B_b - A_b B_a)(A_c B_d - A_d B_c)$, (1.1), (2.5), (1.5) and (4.2), turns into

$$u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab} = 2\tau R_{abcd}$$
$$+ \frac{eN^r A_r}{n-2}((u_a A_d + u_d A_a)g_{bc} + (u_c A_b + u_b A_c)g_{ad}$$
$$- (u_a A_c + u_c A_a)g_{bd} - (u_b A_d + u_d A_b)g_{ac})$$
$$+ \frac{2e\tau}{n-2}(A_b A_c g_{ad} + A_a A_d g_{bc} - A_a A_c g_{bd} - A_b A_d g_{ac}). \tag{4.34}$$

We have now two possibilities: (a) $\tau(p) = 0$ and (b) $\tau(p) \neq 0$. (a) In this case (3.3), by (4.34), becomes

$$\nabla_e C_{abcd} = \varepsilon \beta u_e \big(g_{ad}L_{bc} + g_{bc}L_{ad} - g_{ac}L_{bd} - g_{bd}L_{ac}$$
$$+ \big(k(g_{ad}g_{bc} - g_{ac}g_{bd})\big)/\big((n-2)(n-3)\big)\big), \tag{4.35}$$

where $L_{bc} = (eN^r A_r)(u_b A_c + u_c A_b)/(n-2) - K_{bc}/(n-3)$. From (4.35) it folows that $\nabla C = 0$ at p. (b) Using (4.34), we can present (3.3) in the form

$$\nabla_e C_{abcd} = \varepsilon\beta u_e \left(\frac{k}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd}) + 2\tau R_{abcd} \right.$$

$$+ \frac{eN^r A_r}{n-2}(g_{ad}(u_c A_b + u_b A_c) + g_{bc}(u_a A_d + u_d A_a) - g_{ac}(u_b A_d + u_d A_b)$$

$$- g_{bd}(u_a A_c + u_c A_a) - \frac{1}{n-3}(g_{ad}K_{bc} + g_{bc}K_{ad} - g_{ac}K_{bd} - g_{bd}K_{ac})$$

$$\left. + \frac{2e\tau}{n-2}(g_{ad}A_b A_c + g_{bc}A_a A_d - g_{ac}A_b A_d - g_{bd}A_a A_c) \right),$$

which can be rewritten in the following form

$$\nabla_e C_{abcd} = \varepsilon\beta u_e \left(\frac{k - 2\tau K}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd}) \right.$$

$$\left. + 2\tau C_{abcd} + g_{ad}L_{bc} + g_{bc}L_{ad} - g_{ac}L_{bd} - g_{bd}L_{ac} \right) \qquad (4.36)$$

where

$$L_{bc} = \frac{2\tau}{n-3}S_{bc} - \frac{1}{n-3}K_{bc} + \frac{eN^r A_r}{n-2}(u_c A_b + u_b A_c) + \frac{2e\tau}{n-2}A_b A_c.$$

But from (4.36) we obtain $\nabla_e C_{abcd} = 2\tau\beta u_e C_{abcd}$, which states that C is recurrent. The last remark completes the proof.

Finally, combining Lemmas 5,6,4(ii) with Proposition 2 we immediately get the following theorem.

THEOREM 1. *Let (M, g) be a hypercylinder in a parabolic essentially conformally symmetric manifold $(N, \widetilde{g})$, $n \geq 5$ and let $\widetilde{U}$ be the subset of N consisting of all points of N at which the Ricci tensor $\widetilde{S}$ of $(N, \widetilde{g})$ is not recurrent. If $\widetilde{U} \cap M$ is a dense subset of M then (M, g) is a conformally recurrent manifold.*

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Department of Mathematics
Agricultural University of Wrocław
ul. C. Norwida 25
50-375 Wrocław, Poland