

**POWER MOMENTS OF THE ERROR TERM
FOR THE APPROXIMATE FUNCTIONAL EQUATION
OF THE RIEMANN ZETA-FUNCTION**

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Abstract. Let $\zeta(s)$ be the Riemann zeta-function, $d(n)$ the number of positive divisors of the integer n , and

$$R(s; t/2\pi) = \zeta^2(s) - \sum'_{n \leq t/2\pi} d(n)n^{-s} - \chi^2(s) \sum'_{n \leq t/2\pi} d(n)n^{s-1},$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s).$$

We obtain the following power moment estimates:

$$\int_1^T |R(\frac{1}{2} + it; t/2\pi)|^A dt \ll \begin{cases} T^{1-\frac{1}{4}A+\varepsilon}, & 0 \leq A \leq 4, \\ 1, & A > 4. \end{cases}$$

1. Statement of results. Let $d(n)$ denote the number of positive divisors of n , γ the Euler constant, and let

$$\Delta(x) = \sum'_{n \leq x} d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4} \quad (1)$$

where the symbol \sum' indicates that the last term is to be halved if x is an integer. Kolesnik [7] proved the sharper estimate of (1):

$$\Delta(x) \ll x^{35/108+\varepsilon}. \quad (2)$$

Recently this was improved by Iwaniec and Mozzochi [3].

The asymptotic formula

$$\int_1^T \Delta^2(x) dx = \frac{1}{6\pi^2} \left\{ \sum_{n=1}^{\infty} d^2(n)n^{-3/2} \right\} T^{3/2} + O(T \log^5 T) \quad (3)$$

was proved by Tong [10], who improved an earlier result of Cramér (see [2, Theorem 13.5]). The error term of (3) has been improved to $O(T \log^4 T)$ by Preissmann [9]. Now we suppose that A is a fixed positive number (not necessarily an integer). Ivić [1] has shown the power moment estimates for $\Delta(x)$:

$$\int_1^T |\Delta(t)|^A dt \ll \begin{cases} T^{1+(A/4)+\varepsilon}, & 0 \leq A \leq 35/4, \\ T^{19/54+35A/108+\varepsilon}, & A \geq 35/4, \end{cases} \quad (4)$$

by using Kolesnik's result (2).

Let $s = \sigma + it$ ($0 \leq \sigma \leq 1$, $t \geq 1$) be a complex variable, and $\zeta(s)$ the Riemann zeta-function. We now define

$$R(s; t/2\pi) = \zeta^2(s) - \sum'_{n \leq t/2\pi} d(n)n^{-s} - \chi^2(s) \sum'_{n \leq t/2\pi} d(n)n^{s-1},$$

where $\chi(s) = 2^s \pi^{s-1} \sin(\frac{1}{2}\pi s) \Gamma(1-s)$. It has been shown by Motohashi [8] that

$$\chi(1-s)R(s; t/2\pi) = -\sqrt{2}(t/2\pi)^{-1/2} \Delta(t/2\pi) + O(t^{-1/4}). \quad (6)$$

We note that Jutila [4] gives another proof of Motohashi's result (6). The asymptotic formula

$$\int_1^T |R(\frac{1}{2} + it; t/2\pi)|^2 dt = \sqrt{2\pi} \left\{ \sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-1/2} \right\} T^{1/2} + O(T^{1/4} \log T)$$

was proved by Kiuchi and Matsumoto [5], and the error term has been improved to $O(\log^5 T)$ by Kiuchi [6], where

$$h(n) = (2/\pi)^{1/2} \int_0^{\infty} (y + \pi n)^{-1/2} \cos(y + \frac{1}{4}\pi) dy.$$

The purpose of this paper is to prove the power moment estimates for $|R(\frac{1}{2} + it; t/2\pi)|$. In view of the relation (6), to search analogues of (4) and (5) for $R(s; t/2\pi)$ is an interesting problem in itself and we can prove the following estimates:

THEOREM. *For $T \geq 1$, we have*

$$\int_1^T |R(\frac{1}{2} + it; t/2\pi)|^A dt \ll \begin{cases} T^{1-(A/4)+\varepsilon}, & 0 \leq A \leq 4, \\ 1, & A > 4. \end{cases}$$

2. Proof of Theorem. In case $\sigma = 1/2$ from the inequality $(a+b)^A \ll a^A + b^A$ ($a > 0$, $b > 0$), and (6), it follows that

$$|R(\frac{1}{2} + it; t/2\pi)|^A \ll (t/2\pi)^{-A/2} |\Delta(t/2\pi)|^A + t^{-A/4} \quad (7)$$

where A is a fixed positive number.

From (7) and Schwarz's inequality, it follows that

$$\int_{T/2}^T |R(\frac{1}{2} + it; t/2\pi)|^A dt \ll \left\{ \int_{T/2}^T (t/2\pi)^{-A} dt \right\}^{1/2} \left\{ \int_{T/2}^T |\Delta(t/2\pi)|^{2A} dt \right\}^{1/2} + T^{1-A/4}.$$

From (4) and (5), we have the following estimates:

$$\int_{T/2}^T |R(\frac{1}{2} + it; t/2\pi)|^A dt \ll \begin{cases} T^{1-(A/4)+\varepsilon}, & 0 \leq A \leq 35/8, \\ T^{(73-19A)/108+\varepsilon}, & A \geq 35/8. \end{cases}$$

Replacing T by $T/2$, $T/4$, and so on, and adding we have the theorem.

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