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ON $\mathcal{M}$-BLOCH FUNCTIONS

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Abstract. We define the class M, which contains eigenfunctions of the invariant Laplacian derivatives of $\mathcal{M}$-harmonic functions, etc. For $f \in M$ we define $\|f\|_{\mathcal{B}}$ and derive several quantities equivalent to $\|f\|_{\mathcal{B}}$. Particularly, if f is $\mathcal{M}$-harmonic function, then $\|f\|_{\mathcal{B}}$ is the usual Bloch norm. Higher-order derivatives characterisation of $\mathcal{M}$-harmonic Bloch space is also given.

1. Introduction. Let B be the open unit ball in C^n with (normalized) volume measure ν. Let S denote the boundary of B, and let σ be the usual rotation invariant measure defined on S.

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is, $\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B ($\varphi_z \in \mathrm{Aut}(B)$) taking 0 to z [**13**].

For $z \in B$ and r between 0 and 1 let $E_r(z) = \{\, w \in B \,:\, |\varphi_z(w)| < r\,\}$. We shall set $|E_r(z)| = \nu(E_r(z))$.

For fixed $r, 0 < r < 1, 0 < p \le \infty$ and $f \in C(B)$, we define

$$\widehat{f}(z,r) = \frac{1}{|E_r(z)|} \int\limits_{E_r(z)} f(w)\, d\nu(w)\,,$$

$$MO_p f(z,r) = \left(\frac{1}{|E_r(z)|} \int\limits_{E_r(z)} |f(w) - \widehat{f}(z,r)|^p \, d\nu(w) \right)^{1/p}, \quad 0 < p < \infty\,,$$

$$MO_\infty f(z,r) = \sup \left\{ |f(w) - \widehat{f}(z,r)| \,:\, w \in E_r(z) \right\},$$

$$MO_p^* f(z,r) = \left(\frac{1}{|E_r(z)|} \int\limits_{E_r(z)} |f(w) - f(z)|^p \, d\nu(w) \right)^{1/p}, \quad 0 < p < \infty\,,$$

$$MO_\infty^* f(z,r) = \sup \{ |f(w) - f(z)| \,:\, w \in E_r(z) \}\,.$$

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A function $f \in C^2(B)$ is said to be of class M if there is a constant K, $0 < K < \infty$, such that $\left|\tilde{\Delta} f(z)\right| \le K r^{-2} M O^*_\infty f(z,r)$, for all $z \in B$, $0 < r < 1$.

To show that M contains eigenfunctions of $\tilde{\Delta}$, we need the following lemma:

LEMMA 1.1 [**12**]. *If $f \in C^2(B)$ and $0 < r < 1$, then*

$$f(0) = \int_S f(r\xi)\, d\sigma(\xi) - \int_{rB} \tilde{\Delta} f(z) G(|z|, r)\, d\tau(z),$$

where $d\tau(z) = (1-|z|^2)^{-n-1}\, d\nu(z)$ *and* $G(t,r) = \dfrac{1}{2n}\displaystyle\int_t^r \rho^{1-2n}(1-\rho^2)^{n-1}\, d\rho,$ $0 < t < r < 1$.

PROPOSITION 1.2. *If $f \in X_\lambda, \lambda \in C$, i.e. $\tilde{\Delta} f = \lambda f$, then $f \in M$.*

PROOF. By Lemma 1.1,

$$\int_S (f(r\xi) - f(0))\, d\sigma(\xi) = \lambda \int_{rB} (f(z) - f(0)) G(|z|, r)\, d\tau(z) + \tilde{\Delta} f(0) \int_{rB} G(|z|, r)\, d\tau(z).$$

Using the definition of $G(|z|, r)$ and Fubini's Theorem, we get

$$\int_{rB} G(|z|, r)\, d\tau(z) = \frac{1}{2n} \int_0^r \rho^{1-2n}(1-\rho^2)^{n-1}\, d\rho \int_{\rho B} d\tau(z) = \frac{1}{2n}\int_0^r \rho(1-\rho^2)^{-1}\, d\rho.$$

Combining these results we obtain

$$\int_S (f(r\xi) - f(0))\, d\sigma(\xi) = \lambda \int_{rB} (f(z) - f(0)) G(|z|, r)\, d\tau(z) + \tilde{\Delta} f(0) \frac{1}{4n} \log \frac{1}{1-r^2}.$$

This implies that $|\tilde{\Delta} f(0)| \le K r^{-2} M O^*_\infty f(0,r)$. If z is arbitrary, we consider the function $f \circ \varphi_z$.

Recall that a function f is $\mathcal{M}$-harmonic, $f \in \mathcal{M}$, if $\tilde{\Delta} f = 0$. An application of representation theorems for derivatives of $\mathcal{M}$-harmonic functions obtained in [**2**], shows that if $f \in \mathcal{M}$, then all derivatives of f, which, in general, are not $\mathcal{M}$-harmonic, are in M.

For $f \in C^1(B), Df = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ denotes the complex gradient of f, $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_{2n})$; $z_k = x_{2k-1} + i x_{2k}\, , k = 1, 2, \dots, n$, denotes the real gradient of f.

For $f \in C^1(B)$, let $\tilde{D} f(z) = D(f \circ \varphi_z)(0)$, $z \in B$ and $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of f and the invariant real gradient of f respectively. We say that $f \in M$ is a M-Bloch function, and write $f \in M\mathcal{B}$, if $\|f\|_{\mathcal{B}} = \sup_{z \in B} |\tilde{\nabla} f(z)| < \infty$.

Let $\beta(\cdot,\cdot)$ be a Bergman metric on B. By definition [**11**, p. 45] β is the "integrated form" of the infinitesimal metric

$$G_z = (g_{ij}(z)) = \frac{1}{2}\left(\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z,z)\right),$$

where $K(z,w) = (1 - \langle z,w\rangle)^{-n-1}$ is the Bergman kernel for B.

For $f \in C^1(B)$, the following quantity, depending on f, will play a special role in

$$Qf(z) = \sup_{|w|=1}\left\{\frac{(|\langle Df(z),\bar{w}\rangle|^2 + |\langle D\bar{f}(z),\bar{w}\rangle|^2)^{1/2}}{\sqrt{\langle G_z w, w\rangle}}\right\}.$$

Let $\|\cdot\|_\beta$ denote the Lipschitz norm, i.e. if f is a continuous function on B, then $\|f\|_\beta$ is the smallest value $A \geq 0$ for which $|f(z) - f(w)| \leq A\beta(z,w)$, $z, w \in B$. We say that $f \in \operatorname{Lip}\beta$ if $\|f\|_\beta < \infty$.

We are now ready to assert our first result. For $f \in M$ we give several different quantities equivalent to $\|f\|_{\mathcal{B}}$.

THEOREM 1. *Let $0 < p < \infty$ and $0 < r < 1$. If $f \in M$, then the following statements are equivalent:*

(i) $f \in \operatorname{Lip}\beta$,

(ii) $\sup_{z\in B} Q_p f(z) < \infty$, *where* $Q_p f(z) = \left(\int_B |f\circ\varphi_z(w) - f(z)|^p \, d\nu(w)\right)^{1/p}$,

(iii) $\sup_{z\in B} MO^*_\infty f(z,r) < \infty$,

(iv) $\sup_{z\in B} MO^*_p f(z,r) < \infty$,

(v) $f \in M\mathcal{B}$,

(vi) $\sup_{z\in B} Qf(z) < \infty$,

(vii) $\sup_{z\in B} MO_\infty f(z,r) < \infty$,

(viii) $\sup_{z\in B} MO_p f(z,r) < \infty$.

Theorem 1 was first proved for the class H of holomorphic functions, then for $\mathcal{M}$-harmonic functions ([**4**], [**15**], [**3**], [**7**], [**8**]).

Since $H \subset \mathcal{M} \subset M$, a natural question is: What is the largest class for which Theorem 1 remains valid?

For $f \in \mathcal{M}$ let be $\partial f(z) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, \dots, \frac{\partial f}{\partial \overline{z_1}}, \dots, \frac{\partial f}{\partial \overline{z_n}}\right)$ and for any positive integer m we can write

$$\partial^m f(z) = \left(\partial^\alpha \overline{\partial}^\beta f(z)\right)_{|\alpha|+|\beta|=m} \quad \text{and} \quad |\partial^m f(z)|^2 = \sum_{|\alpha|+|\beta|=m} \left|\partial^\alpha \overline{\partial}^\beta f(z)\right|^2,$$

where

$$\partial^\alpha \overline{\partial}^\beta f(z) = \frac{\partial^{|\alpha|+|\beta|} f(z)}{\partial z_1^{\alpha_1}, \dots, \partial z_n^{\alpha_n} \partial \overline{z_1}^{\beta_1}, \dots, \partial \overline{z_n}^{\beta_n}},$$

and α and β are multi-indices.

Our second result is the following theorem which relates the Bloch norm of an $\mathcal{M}$-harmonic function with quantities involving integrals of higher order derivatives of the function. Even though $\|f\|_{\mathcal{B}}$, $f \in \mathcal{M}$, is not a norm, we refer to $\|f\|_{\mathcal{B}}$ as to the Bloch´s norm of the function f. The quantity $|f(0)| + \|f\|_{\mathcal{B}}$ defines a norm on the linear space $\mathcal{M}$ which, equipped with this norm, is a Banach space.

THEOREM 2. *Let $0 < p < \infty, 0 < r < 1$ and $m \in N$. Then, for an $\mathcal{M}$-harmonic function f, the following are equivalent*

$$\text{(i)}\quad \|f\|_{\mathcal{B}} < \infty, \qquad \text{(ii)}\quad \sup_{z\in B}(1-|z|)|\partial f(z)| < \infty,$$

$$\text{(iii)}\quad \sup_{z\in B}(1-|z|)^m|\partial^m f(z)| < \infty,$$

$$\text{(iv)}\quad \sup_{z\in B}\int\limits_{E_r(z)} |\partial^m f(w)|^p(1-|w|)^{mp-n-1}\,d\nu(w) < \infty.$$

For analytic functions this theorem was proved in [**5**] and [**14**].

In [**9**] it was established that (i) and (ii) are equivalent. More precisely, the following theorem was proved:

THEOREM 3. *Let $f \in \mathcal{M}$. Then the following statements are equivalent:*

$$\text{(i)}\quad \|f\|_{\mathcal{B}} < \infty, \qquad \text{(ii)}\quad \sup_{z\in B}(1-|z|)|\partial f(z)| < \infty,$$

$$\text{(iii)}\quad \sup_{z\in B}(1-|z|^2)(|Rf(z)| + |R\bar{f}(z)|) < \infty,$$

where, as usual, R denotes the radial derivative $R = \sum_{j=1}^{n} z_j \dfrac{\partial}{\partial z_j}$.

2. Proof of Theorem 1. From Theorem 13 [**6**, p. 329] it follows that (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii). It is trivial that (iii)$\Rightarrow$(iv). That (iv) implies (v) follows from the following lemma.

LEMMA 2.1 [**10**]. *Let $0 < r < 1$ and $0 < p < \infty$. There is a constant $C = C(p, r, n)$ such that if $f \in M$, then*

$$|\tilde{\nabla} f(w)|^p \le C \int\limits_{E_r(w)} |f(z) - f(w)|^p\,d\tau(z)\,, \quad \text{for all } w \in B\,.$$

In [**10**] it was proved that if $f \in C^1(B)$, then $Qf(\varphi(z)) = Q(f \circ \varphi)(z), z \in B$, for all $\varphi \in \mathrm{Aut}(B)$ (see also [**8**]).

Since $\alpha^2 = \inf_{|w|=1} \langle G_0 w, w\rangle > 0$, it follows from the definition of $Qf(z)$ that

$$\begin{aligned} Q(f\circ\varphi_z)(0) &\le \frac{1}{\alpha}\big(|D(f\circ\varphi_z)(0)|^2 + |D(\bar f\circ\varphi_z)(0)|^2\big)^{1/2} \\ &= \frac{1}{\alpha}\big(|\tilde D f(z)|^2 + |\tilde D \bar f(z)|^2\big)^{1/2} \\ &= \frac{1}{\alpha\sqrt{2}}|\tilde\nabla f(z)|\,, \quad (\text{see } [\mathbf{12}])\,, \end{aligned}$$

and, hence $Qf(z) = Q(f\circ\varphi_z)(0) \le C|\tilde\nabla f(z)|$. From the preceding we conclude that (v)⇒(vi).

(In this paper the constant is denoted by C, which may indicate different constants from one case to the other).

Let $z, w \in B$ and let $\gamma : [0,1] \to B$ be a geodesic (in the Bergman metric) with $\gamma(0) = z$ and $\gamma(1) = w$. Then

$$\begin{aligned} |f(z) - f(w)| &= \left|\int_0^1 \frac{d}{dt} f(\gamma(t))\,dt\right| \le \int_0^1 \left|\Big\langle Df(\gamma(t)), \overline{\gamma'(t)}\Big\rangle + \overline{\Big\langle D\bar f(\gamma(t)), \overline{\gamma'(t)}\Big\rangle}\right| dt \\ &\le \sqrt{2}\int_0^1 Qf(\gamma(t))\sqrt{\langle G_{\gamma(t)}\gamma'(t), \gamma'(t)\rangle}\,dt \\ &\le \sqrt{2}\sup_{\xi\in B} Qf(\xi)\beta(z,w)\,. \end{aligned}$$

Thus, $\|f\|_\beta \le \sqrt{2}\sup_{z\in B} Qf(z)$. So we have proved that the statements (i) through (vi) are equivalent.

The proof of Lemma 2.1 shows that if $f \in M$, then $|\tilde\nabla f(z)| \le C\, MO_p f(z,r)$. Obviously, $MO_p f(z,r) \le C\, MO_\infty f(z,r)$. Hence, (vii)⇒(viii)⇒(v).

Since

$$|E_r(z)| \cong (1-|z|^2)^{n+1} \cong (1-|u|^2)^{n+1}\,, u \in E_r(z),$$

we have

$$\begin{aligned} |f(w) - \hat f(z,r)| &\le C \int_{E_r(z)} |f(w) - f(u)|\,d\tau(u) \le \\ &\le C|f(w) - f(z)| + C\int_{E_r(z)} |f(u) - f(z)|\,d\tau(u). \end{aligned}$$

From this we conclude that $MO_\infty f(z,r) \le C\, MO^*_\infty f(z,r)$. Thus (iii)⇒(vii). This completes the proof of Theorem 1.

3. Proof of Theorem 2. We start with the following

LEMMA 3.1. *Let $k \geq m$ be positive integers, $0 < p < \infty$ and $0 < r < 1$. There exists a constant $C = C(k, m, p, r, n)$ such that if $f \in \mathcal{M}$, then*

$$|\partial^k f(w)|^p \leq C(1-|w|)^{(m-k)p} \int\limits_{E_r(w)} |\partial^m f(z)|^p \, d\tau(z), \quad \textit{for all } w \in B.$$

Proof. Let α and β are multi-indices. By equality (1.3) in [**2**] we have

$$F(-|\beta|, -|\alpha|, n; r^2)\partial^\alpha \overline{\partial}^\beta f(w) = \int\limits_S (1-\langle w, r\xi\rangle)^{-|\alpha|}(1-\langle r\xi, w\rangle)^{-|\beta|}\partial^\alpha\overline{\partial}^\beta f(r\xi)\, d\sigma(\xi),$$

where $F(a, b, c; x)$ denotes the usual hypergeometric function. Multiplying this equality by $2nr^{2n-1}(1-r^2)^{-n-1}h(r)\,dr$, where h is a radial function which belongs to $C^\infty(B)$, with compact support in B such that $\int\limits_B F(-|\beta|, -|\alpha|, n; |z|^2)h(z)\, d\tau(z) = 1$, then, by integrating from 0 to 1 and using the invariance of the measure τ, we obtain

$$\begin{aligned} \partial^\alpha\overline{\partial}^\beta f(w) &= \int\limits_B h(\varphi_w(z)) \frac{\partial^\alpha\overline{\partial}^\beta f(z)\, d\tau(z)}{(1-\langle w, \varphi_w(z)\rangle)^{|\alpha|}(1-\langle \varphi_w(z), w\rangle)^{|\beta|}} \\ &= \int\limits_B h(\varphi_w(z)) \frac{(1-\langle w, z\rangle)^{|\alpha|}(1-\langle z, w\rangle)^{|\beta|}}{(1-|w|^2)^{|\alpha|+|\beta|}} \partial^\alpha\overline{\partial}^\beta f(z)\, d\tau(z), \end{aligned} \tag{3.1}$$

by Theorem 2.2.2 of [**13**, p. 26].

Since, $|1-\langle z, w\rangle| \cong 1-|w|^2$, $z \in E_r(w)$, by a suitable choice of a function h, we obtain

$$|\partial^\alpha\overline{\partial}^\beta f(w)| \leq C \int\limits_{E_r(w)} |\partial^\alpha\overline{\partial}^\beta f(z)|\, d\tau(z).$$

Hence,

$$|\partial^m f(w)| \leq C \int\limits_{E_r(w)} |\partial^m f(z)|\, d\tau(z).$$

By Lemma 2.4 of [**12**] (see also [**1**]) we find that

$$|\partial^m f(w)|^p \leq C \int\limits_{E_r(w)} |\partial^m f(z)|^p\, d\tau(z).$$

By differentiating under the integral sign in (3.1), and by using the expression for $\varphi_z(w)$ [**13**] and using the same arguments as above, we conclude that

$$|D_j\partial^\alpha\overline{\partial}^\beta f(w)| \le \frac{C}{1-|w|}\int\limits_{E_r(w)} |\partial^\alpha\overline{\partial}^\beta f(z)|\,d\tau(z)\,,\quad w\in B\,,\ 1\le j\le n\,,$$

and

$$|\bar{D}_j\partial^\alpha\overline{\partial}^\beta f(w)| \le \frac{C}{1-|w|}\int\limits_{E_r(w)} |\partial^\alpha\overline{\partial}^\beta f(z)|\,d\tau(z)\,,\quad w\in B\,,\ 1\le j\le n\,.$$

Therefore,

$$|\partial^{m+1} f(w)| \le \frac{C}{1-|w|}\int\limits_{E_r(w)} |\partial^m f(z)|\,d\tau(z).$$

Adapting the argument given in [**12**, Lemma 2.4] we find that

$$|\partial^{m+1} f(w)|^p \le \frac{C}{(1-|w|)^p}\int\limits_{E_r(w)} |\partial^m f(z)|^p\,d\tau(z).$$

An argument by induction shows that

$$|\partial^{k} f(w)|^p \le \frac{C}{(1-|w|)^{(k-m)p}}\int\limits_{E_r(w)} |\partial^m f(z)|^p\,d\tau(z).$$

4. Proof of Theorem 2. If $z\in E_r(w)$ then $1-|w|^2\cong 1-|z|^2$. Hence, from Lemma 3.1, we have

$$(1-|z|)^m|\partial^m f(z)| \le C\int\limits_{E_r(z)} (1-|w|)|\partial f(w)|\,d\tau(w) \le C\|f\|_{\mathcal{B}}\,\tau(E_r(z))\,,$$

by Theorem 3. Since $\tau(E_r(z)) = r^{2n}(1-r^2)^{-n}$, we have that (ii)$\Rightarrow$(iii).

Conversely, assuming that $\partial^\alpha\overline{\partial}^\beta f(0)=0$, we have that

$$|\partial^\alpha\overline{\partial}^\beta f(z)| \le \int\limits_0^1 \left|\frac{d}{dr}\partial^\alpha\overline{\partial}^\beta f(rz)\,dr\right| \le C\int\limits_0^1 \left|\partial^{|\alpha|+|\beta|+1} f(rz)\right|dr\,.$$

Hence,

$$|\partial^k f(z)| \le C\int\limits_0^1 |\partial^{k+1} f(tz)|\,dt,$$

for any positive integer k. The implication (iii)$\Rightarrow$(ii) follows immediately.

Since $\tau(E_r(w))$ is bounded by a constant independent of w, we have that (iii)$\Rightarrow$(iv).

Let $k\ge m$ be a positive integer. Then by Lemma 3.1 we have

$$(1-|z|)^{kp}|\partial^k f(z)|^p \le C\int\limits_{E_r(z)} |\partial^m f(w)|^p(1-|w|)^{mp}\,d\tau(w).$$

Thus, (iv) implies that $\sup_{z\in B}(1-|z|)^k|\partial^k f(z)| < \infty$. This completes the proof of Theorem 2.

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