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RECURRENCE RELATION FOR A CLASS OF POLYNOMIALS ASSOCIATED WITH THE GENERALIZED HERMITE POLYNOMIALS

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Abstract. The coefficients in an $(m+1)$-term recurrence relation for a class of polynomials associated by the generalized Hermite polynomials are determined explicitly.

Introduction. The sequence of polynomials $\{h_{n,m}^{\lambda}(x)\}_{n=0}^{+\infty}$, where λ is a real parameter and m is an arbitrary positive integer, was investigated in [**1**]. For $m = 2$, the polynomial $h_{n,m}^{\lambda}(x)$ reduces to $H_n(x,\lambda)/n!$, where $H_n(x,\lambda)$ is the Hermite polynomial with a parameter. For $\lambda = 1$, $h_{n,2}^{1}(x) = H_n(x)/n!$, where $H_n(x)$ is the classical Hermite polynomial.

Taking $\lambda = 1$ and $n = mN+q$, where $N = [n/m]$ and $0 \le q \le m-1$, Đorđević [**1**] introduced the polynomials $P_N^{(m,q)}(t)$ by $h_{n,m}^{1}(x) = (2x)^q P_N^{(m,q)}((2x)^m)$, and proved that they satisfy an $(m+1)$-term linear recurrence relation of the form

$$\sum_{i=0}^{m} A_N(i,q) P_{N+1-i}^{(m,q)}(t) = B_N(q) t P_N^{(m,q)}(t), \tag{1}$$

where $B_N(q)$ and $A_N(i,q)$ $(i = 0,1,\dots,m)$ are constants depending only on N, m and q. Fixing one of the coefficients, for example $B_N(q) = 1$, this relation becomes unique.

Recurrence relation. In this short note we determine the explicit expressions for the coefficients in (1) using some combinatorial identities. Defining the power of the standard backward difference operator ∇ by

$$\nabla^0 a_N = a_N, \quad \nabla a_N = a_N - a_{N-1}, \quad \nabla^i a_N = \nabla\big(\nabla^{i-1} a_N\big) \quad (i \in \mathbb{N}),$$

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and using the Pochhammer's symbol $(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1)$, we can prove the following result:

THEOREM *The polynomials $P_N^{(m,q)}(t)$ satisfy the $(m+1)$-term recurrence relation*

$$\sum_{i=0}^{m} \frac{1}{i!}\nabla^i (q+mN+1)_m P_{N+1-i}^{(m,q)}(t) = tP_N^{(m,q)}(t). \tag{2}$$

At first we prove an auxiliary result:

LEMMA *Let $m \in \mathbb{N}$, $q \in \{0,1,\dots,m-1\}$, $a_N = (q+mN+1)_m$ and $0 \le k \le N+1$. Then*

$$\sum_{i=0}^{G} \frac{(-1)^{N+1-k-i}}{(N+1-k-i)!}\cdot\frac{1}{i!}\nabla^i a_N = \frac{(-1)^{N+1-k}}{(N+1-k)!}a_{k-1}, \tag{3}$$

where $G = \min(m, N+1-k)$.

Proof. Let E be the shifting operator defined by $\mathrm{E}a_k = a_{k+1}$. Since

$$(\mathrm{I}-\nabla)^{N+1-k}a_N = \mathrm{E}^{-(N+1-k)}a_N = a_{k-1},$$

i.e.,

$$\sum_{i=0}^{N+1-k}(-1)^i\binom{N+1-k}{i}\nabla^i a_N = a_{k-1},$$

and $\nabla^i a_N \equiv 0$ for $i > m$, we obtain (3). □

Notice that

$$G = \begin{cases} m, & \text{if } \ 0 \le k \le N+1-m, \\ N+1-k, & \text{if } \ N+1-m \le k \le N+1. \end{cases}$$

Proof of Theorem. Taking $B_N(q) \equiv 1$ and $A_N(i,q) = \nabla^i a_N/i!$ and using an explicit representation of the polynomial $P_N^{(m,q)}(t)$, given by (see [**1**])

$$P_N^{(m,q)}(t) = \sum_{k=0}^{N}(-1)^{N-k}\frac{t^k}{(N-k)!(q+mk)!},$$

the left hand side of the relation (1) reduces to

$$\begin{aligned} L &= \sum_{i=0}^{m}\frac{1}{i!}\nabla^i a_N \sum_{k=0}^{N+1-i}(-1)^{N+1-k-i}\frac{t^k}{(N+1-k-i)!(q+mk)!} \\ &= \sum_{k=0}^{N+1-m}\left(\sum_{i=0}^{m}\frac{(-1)^{N+1-k-i}}{(N+1-k-i)!}\cdot\frac{1}{i!}\nabla^i a_N\right)\frac{t^k}{(q+mk)!} \\ &\qquad + \sum_{k=N+2-m}^{N+1}\left(\sum_{i=0}^{N+1-k}\frac{(-1)^{N+1-k-i}}{(N+1-k-i)!}\cdot\frac{1}{i!}\nabla^i a_N\right)\frac{t^k}{(q+mk)!}. \end{aligned}$$

According to Lemma, we have that

$$L=\sum_{k=0}^{N+1}\frac{(-1)^{N+1-k}}{(N+1-k)!}\cdot\frac{a_{k-1}t^k}{(q+mk)!}\,.$$

Since $a_{-1}=(q-m+1)_m=(q-m+1)(q-m+2)\cdots q=0$ and

$$\frac{a_k}{(q+m(k+1))!}=\frac{(q+mk+1)_m}{(q+m(k+1))!}=\frac{1}{(q+mk)!}\,,$$

we obtain that

$$L=\sum_{k=0}^{N}\frac{(-1)^{N-k}}{(N-k)!}\cdot\frac{t^{k+1}}{(q+mk)!}\equiv tP_N^{(m,q)}(t).$$

To complete this proof we mention the uniqueness of an $(m+1)$-term recurrence relation with $B_{N,q}=1$. □

Remark. The coefficients $A_N(i,q)$ in the recurrence relation (1) can be expressed in the form

$$A_N(0,q)=(q+mN+1)_m,\quad A_N(i,q)=\frac{1}{i}\nabla A_N(i-1,q)\quad(i=1,\dots,m).$$

Special cases. As an illustration of the previous result, we give two special cases ($m=2$ and $m=3$). For $m=2$ we obtain

$$A_N(0,q)=(q+2N+1)(q+2N+2),\quad A_N(1,q)=2(2q+4N+1),\quad A_N(2,q)=4,$$

where $q=0$ or $q=1$.

For $m=3$ we have

$$\begin{aligned}
&A_N(0,q)=(q+3N+1)_3=(q+3N+1)(q+3N+2)(q+3N+3),\\
&A_N(1,q)=3\big(2+3q+9N+3q^2+18Nq+27N^2\big),\\
&A_N(2,q)=27(q+3N-1),\\
&A_N(3,q)=27,
\end{aligned}$$

where $q\in\{0,1,2\}$.

REFERENCE

[1] G. Đorđević, *Generalized Hermite polynomials*, Publ. Inst. Math. (Beograd) (N.S.) **52(66)** (1993), 69-72

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