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REGULARLY VARYING ULTRADISTRIBUTIONS

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Abstract. Regularly varying ultradistributions are defined in such a way that they generalize Karamata's regularly varying functions. Basic properties are proved and a relation with q-strictly admissible tempered distributions is given.

1. Introduction.

Karamata's regularly varying functions are very useful in analysing the asymptotic behaviour of solutions of mathematical models of real systems. Slowly varying functions "measure" the growth of functions which behave more slowly than any power. Originally they were defined and analysed in the one-dimensional case ([**8**], [**2**], [**17**]). In the last years many authors proposed and elaborated the multi-dimensional case too ([**4**], [**7**], [**10**], [**20**], [**22**]) because of the applications, especially in the probability theory.

The quantum field theory, spectral operator's theory, integral transforms,... gave an impulse to the study of the asymptotic behaviour of elements belonging to different spaces of generalized functions (see [**1**], [**3**] and [**21**]). In the literature one can find different approaches to the asymptotic behaviour of distributions. Certainly, the greatest influence to the theory and applications of the asymptotics of distributions has been made by Vladimirov, Drozhinov and Zavyalov by many papers and by the book [**20**]. Since our aim is to generalize the regular variation of functions to ultradistributions we shall not count up all definitions of the asymptotics of generalized functions, but we shall mention generalizations of the regular variation.

In [**5**] and [**20**] one can find definitions, the theory and applications of the q-admissible and of the q-strictly admissible tempered distributions belonging to $\mathcal{S}'_\Gamma$, where Γ is a closed, convex, acute, solide and regular cone in $\mathcal{R}^n$, with vertex at zero. In case of one variable and $q = 0$, admissible functions are so-called dominated varying functions considered by Feller [**6**], and strictly admissible functions are regularly varying functions. In such a way q- strictly admissible tempered distributions generalize regularly varying functions.

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Denote by θ_Γ the characteristic function of the cone Γ. For an $\alpha \in \mathcal{R}$, θ_Γ^α is defined by $\mathcal{L}(\theta_\Gamma^\alpha)(z) = [K_C(x)]^\alpha = \mathcal{L}(\theta_\Gamma)^\alpha(z)$, where $\mathcal{L}$ denotes the Laplace transform. If $f \in \mathcal{S}'_\Gamma$, then we use the notation by $f^{(-\alpha)} = \theta_\Gamma^\alpha * f$. The definition of a q-strictly admissible tempered distribution f is based on the fact that there exists an $\alpha \in \mathcal{R}$ such that $f^{(-\alpha)}$ is a continuous tempered function with support belonging to Γ. Unfortunately, tempered ultradistributions, in general, have not such a property. An ultradistribution of this kind is

$$P(D)\delta = \sum_{i=0}^{\infty} a_i D^i \delta, \quad a_i \neq 0, \quad i \in \mathcal{N},$$

where $P(D)$ is an ultradifferential operator.

For this reason it was not possible to follow the idea which was used in the definition of the q-strictly admissible tempered distributions, to generalize regular variation for ultradistributions.

Let us remark that in [**18**] one can find defined and elaborated regularly varying distributions which generalize a class of functions which is wider than Karamata's regularly varying functions. In [**19**] there is a generalization of De Haan's class π_g of function [**7**]. The class π_g is a proper subclass of Karamata's class of regularly varying functions.

In Proposition 4 we prove that the set of regularly varying ultradistributions with supports in $[0,\infty)$ equals to the set $\mathcal{S}'^*_+$ of tempered ultradistributions defined on $[0,\infty)$ which have the S-asymptotics related to $h^\alpha L(h)$, where $\alpha \in \mathcal{R}$ and L is a slowly varying function. Since the S-asymptotics can be applied in analysing many mathematical models, regularly varying ultradistrubutions give possibilities to obtain more precise results in these applications.

2. Notation and notions

For the space of distributions and its subspaces we shall use the notation as in Schwartz [**16**].

The class of distributions θ_α, $\alpha \in \mathcal{R}$, belonging to $\mathcal{S}'_+$, is defined by:

$$\theta_\alpha(t) = \begin{cases} H(t)t^{\alpha-1}/\Gamma(\alpha), & \alpha > 0 \\ D^m\theta_{\alpha+m}(t), & \alpha \leq 0,\ \alpha + m > 0, \end{cases} \tag{1}$$

where D^m is the m-th derivative in the distributional sense and $H(t) = 0,\ t < 0$; $H(t) = 1,\ t \geq 0$.

A function $f \in \mathcal{L}_{loc}(\mathcal{R}_+)$ is said to be regularly varying [**2**], [**8**] ($f \in RV_\alpha$) if $f(x) > 0,\ x \geq a > 0$ and $\lim_{k\to\infty} f(kx)/f(k) = x^\alpha,\ x > 0,\ \alpha \in \mathcal{R}$. Then there exists an $a_0 > 0$ such that $f(x) = x^\alpha L(x),\ x \geq a_0$, where $L \in RV_0$ (L is slowly varying).

We shall take two slowly varying functions L_1 and L_2 as equal if

$$\lim_{x\to\infty} L_1(x)/L_2(x) = 1.$$

We denote by c a positive and locally integrable function defined on (a,∞), $a>0$. For ultradistributions we shall use notation and definitions as it is done in [**9**]. Let us repeat some of them: By $\{M_p,\ p\in\mathcal{N}_0\}$ we denote a sequence of positive numbers satisfying some of the following conditions: $M_0=M_1=1$ and

$$\text{(M.1)}\ \ M_p^2\le M_{p-1}M_{p+1},\ \ p\in\mathcal{N};$$

$$\text{(M.2)}'\ \ M_{p+1}/M_p\le AB^p;\qquad \text{(M.2)}\ \ M_p/(M_qM_{p-q})\le AB^p,\ 0\le q\le p;$$

$$\text{(M.3)}'\ \ \sum_{p=1}^{\infty}M_{p-1}/M_p<\infty;\qquad \text{(M.3)}\ \ \sum_{q=p+1}^{\infty}M_{q-1}/M_q\le AM_p/M_{p+1},$$

where A and B are constants independent of p. We will always assume (M.1), (M.2)' and (M.3)' to hold. If we need some stronger condition to be satisfied by $\{M_p,\ p\in\mathcal{N}_0\}$, we shall explicitly give them.

Let u be a positive number. By $\mathcal{E}_K^{u^pM_p}$ we denote the space of smooth functions φ on $\mathcal{R}$ such that

$$q_{u^pM_p}(\varphi)=\sup_{x\in K,\alpha\in\mathcal{N}_0}\frac{|\varphi^{(\alpha)}(x)|}{u^\alpha M_\alpha}<\infty;\ \ \mathcal{D}_K^{u^pM_p}=\{\varphi\in\mathcal{E}_K^{u^pM_p};\text{supp}\varphi\subset K\}.$$

Note that $q_{u^pM_p}$ is a norm on $\mathcal{D}_K^{u^pM_p}$. Then

$$\mathcal{E}^{(M_p)}=\text{proj}\lim_{K\subset\subset\mathcal{R}}\text{proj}\lim_{u\to0}\mathcal{E}_K^{u^pM_p};\ \ \mathcal{E}^{\{M_p\}}=\text{proj}\lim_{K\subset\subset\mathcal{R}}\text{ind}\lim_{u\to\infty}\mathcal{E}_K^{u^pM_p};$$

$$\mathcal{D}_K^{(M_p)}=\text{proj}\lim_{u\to0}\mathcal{D}_K^{u^pM_p},\ \mathcal{D}^{(M_p)}=\text{ind}\lim_{K\subset\subset\mathcal{R}}\mathcal{D}_K^{(M_p)};$$

$$\mathcal{D}_K^{\{M_p\}}=\text{ind}\lim_{u\to\infty}\mathcal{D}_K^{u^pM_p},\ \mathcal{D}^{\{M_p\}}=\text{ind}\lim_{K\subset\subset\mathcal{R}}\mathcal{D}_K^{\{M_p\}};$$

where $K\subset\subset\mathcal{R}$ means that K are compact sets which "grow" up to $\mathcal{R}$.

Spaces with the upper index (M_p) are Beurling type spaces of ultradifferentiable functions and with the upper index $\{M_p\}$ are Roumieu type spaces of ultradifferentiable functions. Their strong duals are spaces of Beurling and Roumieu type ultradistributions.

By $\mathcal{S}^{M_p,m}$, $m>0$, we denote the space of smooth functions φ on $\mathcal{R}$ such that

$$\sigma_m(\varphi)=\sup_{\substack{\alpha,\beta\in\mathcal{N}_0\\ x\in\mathcal{R}}}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\,|\,x^\beta\varphi^{(\alpha)}(x)\,|<\infty.$$

Then $\mathcal{S}^{(M_p)}=\text{proj}\lim_{m\to\infty}\mathcal{S}^{M_p,m}$; $\mathcal{S}^{\{M_p\}}=\text{ind}\lim_{m\to0}\mathcal{S}^{M_p,m}$.

In the sequel we will use $*$ for both (M_p) and $\{M_p\}$.

The strong dual of $\mathcal{S}^*$, $\mathcal{S}'^*$, is the space of tempered ultradistributions (of Beurling and Roumieu types). There holds $\mathcal{D}^*\hookrightarrow\mathcal{S}^*\hookrightarrow\mathcal{E}^*$ where $\hookrightarrow$ means that the space on the left is dense in the space on the right and that the inclusion mapping is continuous. Thus, $\mathcal{E}'^*\subset\mathcal{S}'^*\subset\mathcal{D}'^*$.

We denote ${\mathcal{S}'}^*_+ = \{f \in {\mathcal{S}'}^*; \mathrm{supp} f \subset [0,\infty)\}$.

Let $\mathcal{S}^*_{[0,\infty)}$ be the space of restrictions of functions $\varphi \in \mathcal{S}^*$ on $[0,\infty)$ i.e.

$$\mathcal{S}^*_{[0,\infty)} = \{\psi \in \mathcal{C}^\infty[0,\infty);\ \psi = \varphi\,|_{[0,\infty)} \quad \text{for some } \varphi \in \mathcal{S}^*\}$$

with the induced convergence structure from $\mathcal{S}^*$; its strong dual is denoted by ${\mathcal{S}'}^*_{[0,\infty)}$. We know [**14**] that ${\mathcal{S}'}^*_{[0,\infty)}$ and ${\mathcal{S}'}^*_+$ are isomorphic. If $f \in {\mathcal{S}'}^*$, then $f^{(-m)} = \theta_m * f \in {\mathcal{S}'}^*_+$ ($*$ is the sign of convolution).

An operator of the form

$$P(D) = \sum_{n=0}^{\infty} a_n D^n;\ \ a_n \in \mathcal{C},\ \ n \in \mathcal{N}_0$$

is called an ultradifferential operator of (M_p) class (of $\{M_p\}$ class) if there are constants L and C (for every $L > 0$ there is a constant C) such that $|a_n| \le CL^n/M_n$, $n \in \mathcal{N}_0$.

3. Definitions and comments

The next two definitions are from [**14**].

Definition A. An ultradistributions $f \in {\mathcal{S}'}^*_{[0,\infty)}$ has the quasi-asymptotics related to c if for every $\varphi \in \mathcal{S}^*_{[0,\infty)}$ there exists the limit

$$\lim_{k\to\infty} \langle f(kt)/c(k),\ \varphi(t)\rangle = \langle F, \varphi\rangle\,, \tag{2}$$

where $F \in {\mathcal{S}'}^*_{[0,\infty)}$ and $F \neq 0$. We write for short: $f \overset{q}{\sim} F$ related to c.

Definition B. An ultradistribution $T \in {\mathcal{D}'}^*$ has the S-asymptotics related to c if for every $\phi \in \mathcal{D}^*$ there exists the limit

$$\lim_{h\to\infty} \langle T(x+h)/c(h),\ \phi(x)\rangle = \langle \mathcal{U}, \phi\rangle\,, \tag{3}$$

where $\mathcal{U} \in {\mathcal{D}'}^*$ and $\mathcal{U} \neq 0$. We write for short: $T \overset{s}{\sim} \mathcal{U}$ related to c.

The quasi-asymptotics and S-asymptotics of ultradistributions are natural extensions of the same notions for distributions (see [**14**]).

By the property of ${\mathcal{D}'}^*$ and ${\mathcal{S}'}^*_{[0,\infty)}$ limits (2) and (3) are equivalent to $\lim_{k\to\infty} f(kt)/c(k) = F$ in ${\mathcal{S}'}^*_{[0,\infty)}$ and $\lim_{h\to\infty} T(x+h)/c(h) = \mathcal{U}$ in ${\mathcal{D}'}^*$ respectively.

Definition 1. An ultradistribution $T \in {\mathcal{D}'}^*$ is said to be regularly varying if and only if: (a) for a $\varphi_0 \in \mathcal{D}^*$, $\int \varphi_0(x)dx = 1$, there exists a $k_0 \in \mathcal{R}_+$ such that $\langle T(x+k), \varphi_0(x)\rangle > 0, k \ge k_0$; (b) there exists $\alpha \in \mathcal{R}$ such that for every $\varphi \in \mathcal{D}^*$ and $y > 0$

$$\lim_{k\to\infty}\left(\frac{\langle T(x+ky),\ \varphi(x)\rangle}{\langle T(x+k), \varphi_0(x)\rangle}\right) = \lim_{k\to\infty}\left(\frac{(T*\check{\varphi})(ky)}{(T*\check{\varphi}_0)(k)}\right) = \langle y^\alpha,\ \varphi(x)\rangle\,,$$

where $\check{\varphi}(x) = \varphi(-x)$. We write for short: $T \in RVU_\alpha$

Remark. 1) We can take that the last limit is $U(x, y) \in \mathcal{D}'^*$ instead of $y^\alpha H(x)$ and then prove that $U(x, y)$ has to be $y^\alpha H(x)$ using Theorem 1.4.1. in [**2**].

2) Because of (a), T cannot belong to $\mathcal{E}'^*$.

3) As a consequence of the Wiener Tauberian theorem for ultradistributions [**15**] it is enough to suppose that the limit in b) of Definition 1 is satisfied for only one $\varphi \in \mathcal{D}^*$ with the property $\mathcal{F}[\varphi](\xi) \neq 0,\ \xi \in \mathcal{R}$. By $\mathcal{F}$ we denote the Fourier transform.

4. Properties of regularly varying ultradistributions

PROPOSITION 1. *The property that an ultradistribution T belongs to RVU_α is a local property. Namely, if $S, T \in \mathcal{D}'^*$, $S = T$ on an interval (a, ∞), $a > 0$, and $S \in RVU_\alpha$, then $T \in RVU_\alpha$, as well.*

Proof. Suppose that

$$\langle S(x+k),\ \varphi_0(x)\rangle > 0,\ k \geq k_0,\ \operatorname{supp}\varphi_0 \subset [p, q],\ \int \varphi_0(x)dx = 1,$$

then $\langle T(x+k), \varphi_0(x)\rangle > 0,\ k \geq \max(k_0, a-p) = k_1$ and (a) in Definition 1 is satisfied by T.

For a fixed $y > 0$ and $k > k_1/y$, $T(x+ky) = S(x+ky)$. Since (b) in Definition 1 is satisfied by S, it is also satisfied by T.

Remark. To an ultradistribution T we associate the ultradistribution $T^\wedge$ in such a way that $T^\wedge = T$ on an interval (w, ∞), $w > 0$ and $\operatorname{supp}T^\wedge \subset [w, \infty)$. Proposition 1 asserts that an ultradistribution T is regularly varying if and only it $T^\wedge$ has this property.

PROPOSITION 2. *If $f \in \mathcal{L}_{loc}(a, \infty)$, $a > 0$ and $f \in RV_\alpha$, $\alpha \in \mathcal{R}$, then it defines an associated ultradistribution $f^\wedge$ which is regularly varying, $f^\wedge \in RVU_\alpha$.*

Proof. Let $f(t) = t^\alpha L(t)$, $t \geq a > 0$, and let K be a compact set in $\mathcal{R}$. For a fixed $y > 0$, $x = \ln\xi$, $ky = \ln\eta$, where $\xi, \eta \in \mathcal{R}_+$,

$$\begin{aligned} \lim_{k\to\infty} \frac{f(x+ky)}{f(ky)} &= \lim_{k\to\infty} \frac{(x+ky)^\alpha L(x+ky)}{(ky)^\alpha L(ky)},\quad x \in K \\ &= \lim_{\eta\to\infty} \frac{(\ln\xi\eta)^\alpha L(\ln\xi\eta)}{(\ln\eta)^\alpha L(\ln\eta)} = 1 \end{aligned} \tag{4}$$

uniformly in $x \in K$ (see Proposition 1.3.6 and Theorem 1.2.1. in [**2**]).

Now, using (4) we have

$$\lim_{k\to\infty} \frac{f(x+k)}{f(ky)} = \lim_{k\to\infty} \frac{f(x+k)}{f(k)}\frac{f(k)}{f(yk)} = y^{-\alpha} \tag{5}$$

uniformly in $x \in K$.

By supposition, f is locally integrable and positive on $[a,\infty)$, $a>0$. Let φ_0 be a positive ultradifferentiable function, $\operatorname{supp}\varphi_0 \subset K$ and $\int \varphi_0(x)dx = 1$. We can choose k_0 such that $x+k>a$ for $x\in K$ and $k\geq k_0$. Then

$$\langle f^{\wedge}(x+k),\varphi_0(x)\rangle = \int_K f(x+k)\varphi_0(x)dx > 0,\quad k\geq k_0$$

and (a) in Definition 1 is satisfied. We have to show that (b) is satisfied, as well:

$$\begin{aligned}&\lim_{k\to\infty}\left(\int_K f(x+ky)\varphi(x)dx\Big/\int_K f(x+k)\varphi_0(x)dx\right)\\ &=\lim_{k\to\infty}\left(\int_K \frac{f(x+ky)}{f(ky)}\varphi(x)dx\Big/\int_K \frac{f(x+k)}{f(ky)}\varphi_0(x)dx\right)\\ &=\langle y^{\alpha},\varphi(x)\rangle,\quad \varphi\in\mathcal{D}^*\end{aligned}$$

because of (4) and (5).

It is easy to construct a function $g\in\mathcal{L}_{loc}(\mathcal{R})$ such that g defines an ultradistribution, $g\in RVU_\alpha$ for an α fixed, but $g\notin RV_\alpha$ for any $\alpha\in\mathcal{R}$. Such a function is the following: Suppose that g is a positive function, $g\in\mathcal{L}^1(\mathcal{R})\cap\mathcal{C}(\mathcal{R})$ and has the property $g(n)=1$, $g(n+1/2)=n$, $n\in\mathcal{N}$. This function is not regularly varying for any $\alpha\in\mathcal{R}$ because the set $\{g(k/2)/g(k),\ k\in\mathcal{N}\}$ is not bounded in $\mathcal{R}$. The same property has also the function $1+g$. But the function $1+g$ defines an ultradistribution which belongs to RVU_0. Let us show it. With $\varphi\in\mathcal{D}^*$ and $y>0$

$$\begin{aligned}\int_{-\infty}^{\infty} g(x+ky)\varphi(x)dx &= -\int_{-\infty}^{\infty}\int_{-\infty}^{x} g(t+ky)dt\varphi'(x)dx\\ &=-\int_{-\infty}^{\infty}\int_{-\infty}^{x+ky} g(u)du\varphi'(x)dx\\ &\to\int_{-\infty}^{\infty} g(u)du\int_{-\infty}^{\infty}\varphi'(x)dx = 0,\quad k\to\infty.\end{aligned}$$

Let $\varphi_0\in\mathcal{D}^*, \varphi_0>0$ and $\int\varphi_0(x)dx=1$. Then a) in Definition 1 is satisfied and b) follows from (6) because of

$$\begin{aligned}&\lim_{k\to\infty}\left(\frac{\langle 1+g(x+ky),\varphi(x)\rangle}{\langle 1+g(x+k),\varphi_0(x)\rangle}\right)\\ &=\lim_{k\to\infty}\left(\frac{\langle 1,\varphi(x)\rangle}{\langle 1+g(x+k),\varphi_0(x)\rangle}\right)+\lim_{k\to\infty}\left(\frac{\langle g(x+ky),\varphi(x)\rangle}{(1+\langle g(x+k),\varphi_0(x)\rangle)}\right)\\ &=\langle 1,\varphi(x)\rangle,\ \varphi\in\mathcal{D}^*.\end{aligned}$$

PROPOSITION 3. *If $T\in RVU_\alpha$, then there exists an $\varphi_0\in\mathcal{D}^*$, $\int\varphi_0(x)dx=1$ such that $\langle T(x+k),\ \varphi_0(x)\rangle = (T*\check{\varphi}_0)(k)\in RV_\alpha$.*

Proof. For $y>0$ and φ_0 defined in (a) of Definition 1

$$\lim_{k\to\infty}\frac{\langle T(x+ky),\ \varphi_0(x)\rangle}{\langle T(x+k),\varphi_0(x)\rangle} = \lim_{k\to\infty}\frac{(T*\check{\varphi}_0)(ky)}{(T*\check{\varphi}_0)(k)} = y^{\alpha}.$$

By Theorem 1.4.1 in [**2**] it follows that $(T * \check{\varphi}_0)(k) \in RV_\alpha$.

PROPOSITION 4. *Suppose that (M.1), (M.2) and (M.3) are satisfied by* $\{M_p,\ p \in \mathcal{N}\}$. *A necessary and sufficient condition for* $T \in RVU_\alpha$ *is that* $T \overset{s}{\sim} 1$ *related to* $c(h) = h^\alpha L(h)$ *for an* $L \in RV_0$. *For a fixed* $T \in RVU_\alpha$, L *is unique in the sense of equivalence of slowly varying functions.*

A necessary condition for $T \in \mathcal{D}'^*_+$ *to belong to* RVU_α *is that* $T \in \mathcal{S}'^*_+ \setminus \mathcal{E}'^*$.

Proof. First we shall prove that if $T \in RVU_\alpha$, then there exists a slowly varying function L such that T has the S-asymptotics related to $c(h) = h^\alpha L(h)$ with the limit $U = 1$.

Let $y > 0$, then by Definition 1 there exists a $\varphi_0 \in \mathcal{D}^*$ such that

$$\lim_{k\to\infty} \left(\frac{\langle T(x+ky), \varphi(x)\rangle}{\langle T(x+k), \varphi_0(x)\rangle} \right) = \langle y^\alpha, \varphi(x)\rangle\,,\ \varphi \in \mathcal{D}^*.$$

By Proposition 3, $\langle T(x+k), \varphi_0(x)\rangle = c(k)$ is a regularly varying function of the form $c(k) = k^\alpha L(k),\ k \geq k_1 > 0$, and for $\varphi \in \mathcal{D}^*$

$$\lim_{h\to\infty} \left\langle \frac{T(x+h)}{c(h)}, \varphi(x) \right\rangle = \lim_{k\to\infty} \frac{c(k)}{c(ky)} \left\langle \frac{T(x+ky)}{c(k)}, \varphi(x) \right\rangle = \langle 1, \varphi(x)\rangle\,, y > 0.$$

If $T \in \mathcal{D}'^*_+$, then by Theorem 3 and Theorem 5 in [**14**] it follows that $T \in \mathcal{S}'^*_+$.

Concerning the uniqueness of L, suppose that $T \overset{s}{\sim} 1$ related to $h^\alpha L_1(h)$, as well. Then

$$\lim_{k\to\infty} \frac{L(k)}{L_1(k)} = \lim_{k\to\infty} \frac{\langle T(x+k)/L_1(k), \varphi_0(x)\rangle}{\langle T(x+k)/L(k), \varphi_0(x)\rangle} = 1.$$

Now, suppose that $T \in \mathcal{D}'^*_+$ and that $T \overset{s}{\sim} 1$ related to $c(h) = h^\alpha L(h)$ with an $L \in RV_0$. Then for every $\varphi \in \mathcal{D}^*$

$$\lim_{h\to\infty} \langle T(x+h)/(h^\alpha L(h)), \varphi(x)\rangle = \langle 1, \varphi(x)\rangle\,. \tag{7}$$

Among those $\varphi \in \mathcal{D}^*$ which satisfy (7) we can choose a $\varphi_0 \in \mathcal{D}^*$ so that $\int \varphi_0(t)dt = 1$. Since $h^\alpha L(h) > 0,\ h \geq h_0 > 0$, it follows from (7) that there exists $h_1 > 0$ such that $\langle T(x+h), \varphi_0(x)\rangle > 0,\ h \geq h_1$. Consequently, (a) in Definition 1 is satisfied. We have, now, to satisfy (b).

$$\begin{aligned} &\lim_{k\to\infty} \frac{\langle T(x+ky), \varphi(x)\rangle}{\langle T(x+k), \varphi_0(x)\rangle} \\ &= \lim_{k\to\infty} \frac{\langle T(x+ky), \varphi_0(x)\rangle}{(ky)^\alpha L(ky)} \cdot \frac{(ky)^\alpha L(ky)}{k^\alpha L(k)} \Big/ \frac{\langle T(x+k), \varphi_0(x)\rangle}{k^\alpha L(k)} \\ &= \langle y^\alpha,\ \varphi(x)\rangle\,, \quad \varphi \in \mathcal{D}^*. \end{aligned}$$

The necessary condition follows from the remark after Definition 1.

COROLLARIES 1. *The set of regularly varying ultradistributions, with supports in* $[0,\infty)$ *equals to the set* $\{T \in \mathcal{S}'^*_+;\ T \overset{s}{\sim} 1$ *related to* $h^\alpha L(h),\ \alpha \in \mathcal{R},\ L \in RV_0\}$. *This is a direct consequence of Proposition 4.*

2. *An ultradifferential operator $P(D)$ of $*$ class with $a_0 \neq 0$ maps the set RVU_α into itself. This is a consequence of Proposition 4 and of continuity of the operator $P(D)$. Namely, if $T \in RVU_\alpha$, then there exists $\lim_{h\to\infty} T(x+h)/(h^\alpha L(h)) = 1$ in $\mathcal{D}'^*$. Let $S = P(D)T$, then*

$$\lim_{h\to\infty} S(x+h)/(h^\alpha a_0 L(h)) = \lim_{h\to\infty} (P(D)T(x+h)/(a_0 h^\alpha L(h)) = 1.$$

The assumption $a_0 \neq 0$ is essential because if we know the asymptotic behaviour of a function, then there is no rule to provide the asymptotic behaviour of its derivative.

PROPOSITION 5. *Suppose that (M.1), (M.2) and (M.3) are satisfied by $\{M_p,\ p \in \mathcal{N}_0\}$. A necessary and sufficient condition that $T \in RVU_\alpha$ is that there exist:*

a positive number a;

an ultradifferential operator $P(D) = \sum_{i=0}^\infty a_i D^i$ of $*$ class;

continuous functions f_1 and f_2 on $[a,\infty)$ with the properties $a_0 f_1 + f_2 \in RV_\alpha$ and $\lim_{h\to\infty}(f_i(x+h)/(h^\alpha L(h))) = C_i$, $i = 1,2$, uniformly in $x \in [a,b]$, $b < \infty$, such that $T = P(D)f_1 + f_2$ on (a,∞).

Proof. By Proposition 4, if $T \in RVU_\alpha$, then it belongs to the set of ultradistributions which have the S-asymptotics related to $x^\alpha L(x)$, where $\alpha \in \mathcal{R}$ and L is slowly varying. By the property of such ultradistributions (see Theorem 5 in [**13**]) there exist an ultradifferential operator $P(D)$ of $*$ class and continuous functions f_1 and f_2 on $[a,\infty)$, $0 < a \leq 1$, such that $\lim_{h\to\infty}(f_i(x+h)/(h^\alpha L(h))) = U_i(x)$, uniformly on $[a,b]$, $b < \infty$, such that $T = P(D)f_1 + f_2$ on (a,∞).

By Corollary 4.1 in [**12**], if $U_i \neq 0$, then U_i has to be a constant $C_i \neq 0$. f_1 and f_2 as ultradistributions have the limit $\lim_{h\to\infty}(f_i(x+h)/c(h)) = C_i$ in $\mathcal{D}'^*_{[a,\infty)}$ (C_i can be zero too). But, since $P(D)$ is a continuous operator which maps $\mathcal{D}'^*$ into $\mathcal{D}'^*$, then $a_0 f_1 + f_2 \overset{s}{\sim} a_0 C_1 + C_2 \neq 0$ related to $h^\alpha L(h)$. We can suppose that $a_0 C_1 + C_2 > 0$. Then

$$\lim_{k\to\infty} \frac{a_0 f_1(kx) + f_2(kx)}{a_0 f_1(k) + f_2(k)} = \lim_{p\to\infty} \frac{a_0 f_1(x+px) + f_2(x+px)}{a_0 f_1(1+p) + f_2(1+p)}$$

$$\lim_{p\to\infty} \frac{c(px)}{c(p)} \frac{a_0 f_1(x+px) + f_2(x+px)}{c(px)} \frac{c(p)}{a_0 f_1(p+1) + f_2(p+1)} = x^\alpha,\ x > 0.$$

Conversely, the condition is sufficient. By Theorem 5 in [**13**] T has the S-asymptotics related to $c(h) = h^\alpha L(h)$. Proposition 4 asserts that $T \in RVU_\alpha$.

The relation between the property that an ultradistribution is regularly varying and that it has the S-asymptotics is given by Proposition 4. The relation between the regular variation and quasi-asymptotics of ultradistributions is more complicated; it can be considered only in $\mathcal{D}'^*_+$. It is easy to find an ultradistribution which has the quasi-asymptotics related to $c(h) = h^\alpha L(h)$ for an $\alpha \in \mathcal{R}$ and $L \in RV_0$ but which has no S-asymptotics and, consequently, is not regularly varying. Such an ultradistribution is $\theta_1(t)\sin t$ (see [**12**, p. 91]).

PROPOSITION 6. *Suppose that (M.1), (M.2) and (M.3) hold. Let $T \in RVU_\alpha$ with support in $[0,\infty)$, then*

a) if $\alpha > -1$, T has the quasi-asymptotics related to $k^\alpha L(k)$, $L \in RV_0$;

b) if $\alpha < -1$, $\alpha = -1-p-\epsilon$, where $p \in \mathcal{N}_0$ and $0 < \epsilon < 1$, then T has the quasi-asymptotics related to one of the following functions: k^{-1}, k^{-2},..., k^{-p}, k^α $L(k)$;

c) if $\alpha = -1$, $\int_{y_1}^{y} t^{-1}L(t)dt = L^\wedge(y) < \infty$, then T has the quasi-asymptotics related to k^{-1}, or $k^{-1}L(k)$;

d) if $\alpha = -1$ and $L^\wedge(y) \to \infty$, then T has the quasi-asymptotics related to $k^{-1}L^\wedge(k)$.

Proof. If $T \in RVU_\alpha$, then by Proposition 4, $T \overset{s}{\sim} 1$ related to $c(h) = h^\alpha L(h)$ for a slowly varying function L.

a) By Theorem 3 in [**14**], $T \overset{q}{\sim} \Gamma(\alpha+1)\theta_{\alpha+1}$ related to $k^\alpha L(k)$.

Other cases. Denote by ω a function belonging to $\mathcal{D}^*$ such that $\omega(x) = 1$, $x \in [0,a]$, $a > 0$. Then $T = \omega T + (1-\omega)T$; the support of ωT is compact. We know that $(1-\omega)T$ has the same S-asymptotics as T and $\text{supp}(1-\omega)T \subset [a,\infty)$. By the property of the S-asymptotic of ultradistributions (see [**14**])

$$(1-\omega)T = P(D)c(x)E_1(x) + c(x)E_2(x), \quad x > a > 0,$$

where E_i are continuous functions on $[a,\infty)$, $\lim_{x\to\infty} E_i(x) = C_i$, $i = 1,2$, and $P(D)$ is an ultradifferential operator of $*$ class. By the property of the operator $P(D)$ and the operation of the convolution, we have

$$\begin{aligned}((1-\omega)T)^{(m)} &= \theta_m * (t^\alpha L(t)(a_0 E_1(t) + E_2(t))) \\ &\quad + \sum_{i=1}^{\infty} a_i D^i(\theta_m * (t^\alpha L(t)E_1(t))), \ m+\alpha > 0, \quad m \in \mathcal{N}.\end{aligned} \tag{8}$$

Let us analyse the function $F = \theta_m * (t^\alpha L(t)E(t))$ where E is continuous, $\text{supp}E \subset [a,\infty)$ and $\lim_{x\to\infty} E(x) = C$. Let $\alpha = -1-p-\epsilon$, $0 \le \epsilon < 1$, $p \in \mathcal{N}_0$.

$$\begin{aligned}\Gamma(m)F(x) &= \int_a^x (x-t)^{m-1}t^\alpha L(t)E(t)dt \\ &= \sum_{i=0}^{m-1}\binom{m-1}{i}x^{m-1-i}\int_a^x t^{\alpha+1}L(t)E(t)dt.\end{aligned} \tag{9}$$

For $0 \le i \le p$

$$\int_a^x t^{\alpha+i}L(t)E(t)dt \to \int_a^\infty t^{\alpha+i}L(t)E(t)dt, \quad x \to \infty$$

and $x^{m-1-i}\int_a^x t^{\alpha+i}L(t)E(t)dt$ has the S-asymptotics related to $c(h) = h^{m-1-i}$ with the limit equal to a constant, $m-1-i \ge m+\alpha+\epsilon > 0$, $0 \le i \le p$.

For $p+1 \le i \le m-1$

$$\int_a^x t^{\alpha+i} L(t)E(t)dt \sim (\alpha+1+i)^{-1} x^{\alpha+1+i} L(x)E(x), \quad x \to \infty$$

and

$$x^{m-1-i} \int_a^x t^{\alpha+i} L(t)E(t)dt \sim (\alpha+1+i)^{-1} x^{m+\alpha} L(x)E(x), \quad x \to \infty$$

has the S-asymptotics related to $c(h) = h^\alpha L(h)$ with the limit equal to a constant. Since $m+\alpha > 0$, it has the quasi-asymptotics related to $c(k) = k^{m+\alpha}L(k)$. By the continuity of the operator $P(D)$ it follows by (8) that $((1-\omega)T)^{(-m)}$ has the S-asymptotics related to the same $c(h)$ as $\theta_m * (t^\alpha L(t)(a_0 E_1(t) + E_2(t)))$.

By Theorem 4 and Proposition 3 in [**14**] $(T\omega)^{(-m)}$ can have the S-asymptotics related to h^{m-j-1} for some $j \in \mathcal{N}$ and m large enough, such that $m+\alpha > 0$ and $m-j-1 > 0$, as well. Therefore $T = \omega T + (1-\omega)T$ has the quasi-asymptotics as it is asserted in b) and c). In b), $-\alpha \notin \mathcal{N}$ and in c) $L(t)$ does not converge to a constant.

In the case d) it is enough to take $m = 1$. Then

$$F(x) = \int_a^x t^{-1} L(t)E(t)dt = L^\wedge(x) \to \infty, \quad x \to \infty,$$

where $L^\wedge$ is slowly varying. Now, T has to have the quasi-asymptotics related to $k^{-1}L^\wedge(k)$. In [**14**] one can find an example which illustrates all these possibilities.

5. Some comments

5.1. Relation between RVU_α and strict admissibility. The strict admissibility in the one-dimensional case is given by the following definition (see [**5**] and [**20**]).

Definition C. The distribution $U \in \mathcal{S}'_+$ is called q-strictly admissible if the following conditions hold

1. $U^{(-q)} > 0,\ t > 0$ and $U^{(-q)} \in \mathcal{L}_{loc}$;

2. $U^{(-q)}(kt)/U^{(-q)}(k) \to t^\alpha,\ k \to \infty$, uniformly on every compact set belonging to $(0,\infty)$;

3. There exist a k_0 such that $U^{(-q)}(kt)/U^{(-q)}(k) \le \psi(t),\ k > k_0,\ t > 0$ and

$$\int_0^\infty \psi(t)(1+t)^{-m} dt < \infty, \qquad \text{for an} \quad m \in \mathcal{N}_0.$$

In comments of this definition (see [**20**]) one can find the following sentence: "0-strictly admissible functions are nothing else but regularly varying functions". To be precise, because of the definition of the regular variation (see [**2**], [**8**] and [**17**]) it should read: "0-strictly admissible functions are nothing else but regularly varying functions which satisfy condition 3 in Definition C".

By Proposition 2, regularly varying ultradistributions generalize regularly varying functions which belong to $\mathcal{L}(a,\infty)$, $a>0$. This class contains the class of 0-strictly admissible functions as a proper subclass. For example, the function $f(t)=H(t-1)t^{-2}$ is a regularly varying function belonging to $\mathcal{L}(1,\infty)$ but it does not satisfy condition 3; for $k>1$ and $a>1$, $f(kt)/f(k)=H(kt-1)t^{-2}=t^{-2}$ when $k\geq a/t$, $t>0$.

Also, by Proposition 1, it follows that if a distribution $U\in\mathcal{S}'_+$ is q-strictly admissible, then $U^{(-q)}\in RVU_\alpha$, where $\alpha\geq -1$.

Suppose that U is q-strictly admissible. Then by Theorem 1.4.1 in [**2**] there exist $\alpha\in\mathcal{R}$, $t_0>0$ and $L\in RV_0$ such that $U^{(-q)}(t)=t^\alpha L(t)$, $t\geq t_0$. Suppose that $\alpha<-1$. By Theorem 1.5.6. in [**2**] for any $\delta>0$ there exists $x(\delta)$ such that

$$\frac{U^{(-q)}(kx)}{U^{(-q)}(k)}=x^\alpha\frac{L(kx)}{L(k)}\geq x^{\alpha+\delta},\quad k\geq t_0,\ kx\geq\max\{t_0,x(\delta)\}.$$

We can choose δ such that $\alpha+\delta<-1$. Then condition 3 in Definition C can not be satisfied.

5.2. Regularly varying ultradistributions and the generalized S-asymptotics. In [**11**] Pilipović defined generalized S-asymptotics and applied it to obtain Wiener–Tauberian type results for non-negative distributions.

Definition D. [**11**]. Let $f\in\mathcal{D}'$ and $c\in\mathcal{C}^\infty$ be such that $c(x)>0$, $x\geq x_0$. f is said to have generalized S-asymptotics related to c if $\lim_{k\to\infty}(f(x+k)/c(x+k))=1$ in $\mathcal{D}'$. We write for short $f\overset{gs}{\sim}c$.

It is easy to extend the generalized S-asymptotics to ultradistributions.

Definition 2. Suppose that $T\in\mathcal{D}'^*$ and $c\in\mathcal{E}^*$, $c(x)>0$, $x\geq x_0$. T is said to have the generalized S-asymptotics related to c if $\lim_{k\to\infty}(T(x+k)/c(x+k))=1$ in $\mathcal{D}'^*$.

PROPOSITION 7. *Suppose that (M.1), (M.2) and (M.3) hold. A necessary and sufficient condition for $T\in RVU_\alpha$ is that $T\overset{gs}{\sim}\overline{c}$, where $\overline{c}\in\mathcal{E}^*$ and $\overline{c}(h)=h^\alpha\overline{L}(h)$, $h\geq h_0>0$, $\overline{L}\in RV_0\cap\mathcal{E}^*$.*

Proof. Suppose that $T\in RVU_\alpha$. By Proposition 4, $T\overset{s}{\sim}1$ related to $c(h)=h^\alpha L(h)$, $\alpha\in\mathcal{R}$, $L\in RV_0$. Using the same method as in [**11**] one can prove that there exists $\overline{c}\in\mathcal{E}^*$ and $\overline{L}\in RV_0\cap\mathcal{E}^*$ such that $\lim_{x\to\infty}(\overline{L}(x)/L(x))=1$ in $\mathcal{E}^*$, $\overline{c}(h)=h^\alpha\overline{L}(h)$, $h\geq h_0$ and $T\overset{gs}{\sim}\overline{c}$.

We will discuss many-dimensional case and applications in another paper.

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