PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 60 (74), 1996, 153–164

ON THE CONVERGENCE OF A FACTORIZED VECTOR FINITE DIFFERENCE SCHEME

Boško S. Jovanović

Communicated by Gradimir Milovanović

Abstract. We consider a factorized vector finite difference scheme for solving multi-dimensional heat conduction equation. It can be treated as a vector version of Peaceman–Rachford scheme. A three-level version of this scheme is used to solve the wave equation. The stability and the convergence of these schemes are investigated.

Introduction

Different versions of alternating direction method [**16**], [**4**] are often used for numerical solution of multi-dimensional initial-boundary value problems of mathematical physics. Here we have, for example, splitting methods, composite methods, additive schemes, factorized schemes etc. (see [**14**]). All these methods directly or indirectly use the concept of vectorisation, i.e. one unknown mesh function is replaced by a vector mesh function.

Here we consider a factorized vector finite difference scheme proposed in [**18**]. We investigate the two-level and three-level versions of this scheme and prove its unconditional stability. The convergence rate estimates are obtained for initial-boundary value problems with generalized solutions from Sobolev spaces. Another vector finite difference schemes are considered in [**1**], [**2**], [**9**], [**10**] and [**11**].

Two-level scheme

As a model problem we consider the first initial-boundary value problem (IBVP) for the heat conduction equation

$$
(1)\qquad
\begin{aligned}
&\frac{\partial u}{\partial t} = \Delta u + f\,, \qquad (x,\,t) \in Q = \Omega \times (0,\,T) = (0,\,1)^n \times (0,\,T)\,,\\
&u(x,\,0) = u_0(x)\,, \qquad x \in \Omega\,,\\
&u(x,\,t) = 0, \qquad x \in \Gamma = \partial\Omega\,, \quad t \in (0,\,T)\,.
\end{aligned}
$$

AMS Subject Classification (1991): Primary 65M12

Supported by Ministry of Science and Technology of Serbia, grant number 04M03/C

We assume that the generalized solution of IBVP (1) belongs to the anisotropic Sobolev space $W_2^{s,\,s/2}(Q)$, $s \geq 1$ [**13**]. In this case there exists a trace $u|_{t=t'} \in W_2^{s-1}(\Omega) \subset L_2(\Omega)$ for $t' \in [0,\, T]$. We also assume that the solution u can be oddly extended in space variables outside the domain Ω, preserving the Sobolev class.

Let $\overline{\omega}$ be uniform mesh in $\overline{\Omega}$ with step size h. Let us set $\omega = \overline{\omega} \cap \Omega$, $\gamma = \overline{\omega} \setminus \omega$ and $\omega_i = \omega \cup \{x = (x_1,\, \dots,\, x_n) \in \gamma \,|\, x_i = 0,\ 0 < x_j < 1,\ j \neq i\}$. Let $\overline{\theta}$ be uniform mesh on $[0,\, T]$ with step size τ, $\theta = \overline{\theta} \cap (0,\, T)$, $\theta^- = \theta \cup \{0\}$ and $\theta^+ = \theta \cup \{T\}$. Let, also, $\overline{\vartheta}$ be uniform mesh with step size τ on interval $[-\tau/2,\, T-\tau/2]$ and $\vartheta = \overline{\vartheta} \cap (-\tau/2,\, T-\tau/2)$, $\vartheta^- = \vartheta \cup \{-\tau/2\}$. Finally, let $\overline{Q}_{h\tau} = \overline{\omega} \times \overline{\theta}$ and $\widehat{Q}_{h\tau} = \overline{\omega} \times \overline{\vartheta}$. For a function v defined on the mesh $\overline{Q}_{h\tau}$ or $\widehat{Q}_{h\tau}$ we introduce the divided difference operators v_{x_i}, $v_{\bar{x}_i}$, v_t and $v_{\bar{t}}$ in the usual manner [**17**]. Let us denote $v = v(x,\, t)$, $\hat{v} = v(x,\, t+\tau)$ and $\check{v} = v(x,\, t-\tau)$.

Let H_h be the set of discrete functions defined on the mesh $\overline{\omega}$, which vanish on γ. The identity operator on H_h will be denoted by I. We also denote

$$\Lambda_i\, v = \begin{cases} -v_{x_i\bar{x}_i}\,, & x \in \omega \\ 0\,, & x \in \gamma \end{cases} \qquad \text{and} \qquad \Lambda\, v = \sum_{i=1}^{n} \Lambda_i\, v\,.$$

We introduce the following discrete inner product

$$(v,\, w)_\omega = h^n \sum_{x\in\omega} v(x)\, w(x)$$

and the norms

$$\|v\|_\omega = (v,\, v)_\omega^{1/2} = \Big(h^n \sum_{x\in\omega} v^2(x)\Big)^{1/2} \qquad \text{and} \qquad \|v\|_{\omega_i} = \Big(h^n \sum_{x\in\omega_i} v^2(x)\Big)^{1/2}.$$

For a linear, selfadjoint and nonnegative operator A on H_h with we introduce so called “energy” seminorm

$$\|v\|_A = (A\, v,\, v)_\omega^{1/2}\,.$$

In particular

$$\|v\|_{\Lambda_i} = (\Lambda_i\, v,\, v)_\omega^{1/2} = \|v_{x_i}\|_{\omega_i}\,.$$

Let us denote $\mathbf{H}_h = H_h^n$. Elements of the space $\mathbf{H}_h$ will be denoted by $\mathbf{v} = (v^1,\, \dots,\, v^n)^T$, $\mathbf{w} = (w^1,\, \dots,\, w^n)^T$ etc. We introduce the matrix finite-difference operators $\mathbf{I} = \operatorname{diag}(I,\, \dots,\, I)$ and $\Lambda = \operatorname{diag}(\Lambda_1,\, \dots,\, \Lambda_n)$. We also define the inner product and the associated norm of vector mesh functions

$$(\mathbf{v},\, \mathbf{w}) = \sum_{i=1}^{n} (v^i,\, w^i)_\omega\,, \qquad \|\mathbf{v}\| = (\mathbf{v},\, \mathbf{v})^{1/2}\,.$$

With T_i, T_t and T_t^+ we denote the Steklov averaging operators in space variables x_i and time variable t (see [**8**])

$$T_i f(x,\,t) = \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} f(x_1,\,\dots\,,\,x_i',\,\dots\,,\,x_n,\,t)\,dx_i'\,,$$
$$T_t f(x,\,t) = T_t^+ f(x,\,t-\tau/2) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f(x_1,\,\dots\,,\,x_n,\,t')\,dt'\,.$$

Finally, C will stand for a positive generic constant, independent of h and τ.

We approximate the equation (1) by the following finite difference scheme (FDS) [**18**]

$$\text{(2)}\qquad \Big[\mathbf{I}+\frac{\tau}{2}\Big(\mathbf{L}+\frac{1}{2}\mathbf{I}\Big)\Lambda\Big]\Big[\mathbf{I}+\frac{\tau}{2}\Big(\mathbf{U}+\frac{1}{2}\mathbf{I}\Big)\Lambda\Big]\mathbf{v}_t+\mathbf{E}\,\Lambda\,\mathbf{v}=\widetilde{\mathbf{f}}\,,\qquad t\in\theta^-\,,$$

where $\widetilde{\mathbf{f}}=(\widetilde{f},\,\dots\,,\,\widetilde{f})^T$, $\quad \widetilde{f}=T_1^2\cdots T_n^2T_t^+\,f$,

$$\mathbf{L}=\begin{pmatrix}0&0&\dots&0&0\\ I&0&\dots&0&0\\ I&I&\dots&0&0\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ I&I&\dots&I&0\end{pmatrix},\quad \mathbf{U}=\begin{pmatrix}0&I&\dots&I&I\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&\dots&I&I\\ 0&0&\dots&0&I\\ 0&0&\dots&0&0\end{pmatrix}\quad\text{and}\quad \mathbf{E}=\mathbf{L}+\mathbf{I}+\mathbf{U}\,.$$

Notice, that for a fixed $t\in\overline{\theta}$ vectors $\mathbf{v}$ and $\widetilde{\mathbf{f}}$ belong to the space $\mathbf{H}_h$. The initial condition we approximate with

$$\text{(3)}\qquad \mathbf{v}\big|_{t=0}=(T_1^2\cdots T_n^2u_0,\,\dots\,,\,T_1^2\cdots T_n^2u_0)^T$$

The scheme (2–3) is a factorized vector FDS. Here the solution u of IBVP (1) is approximated by a vector mesh function $\mathbf{v}$. FDS (2–3) can be treated as a vector version of Peacemann–Rachford method (see [**16**], [**18**], [**15**]).

Since the solution of IBVP (1) may be discontinuous function, the components of the error-vector $\mathbf{z}=(z^1,\,\dots\,,\,z^n)^T$ we define by

$$z^i=T_1^2\cdots T_n^2\,u-v^i\,.$$

The vector $\mathbf{z}$ satisfies FDS

$$\text{(4)}\qquad \begin{gathered}\Big[\mathbf{I}+\frac{\tau}{2}\Big(\mathbf{L}+\frac{1}{2}\mathbf{I}\Big)\Lambda\Big]\Big[\mathbf{I}+\frac{\tau}{2}\Big(\mathbf{U}+\frac{1}{2}\mathbf{I}\Big)\Lambda\Big]\mathbf{z}_t+\mathbf{E}\,\Lambda\,\mathbf{z}=\Phi\,,\quad t\in\theta^-\,,\\ \mathbf{z}\big|_{t=0}=\mathbf{0}\,,\end{gathered}$$

where $\Phi = (\varphi^1, \ldots, \varphi^n)^T$ and

$$\varphi^i = \sum_{j=1}^{n} \left(\Lambda_j \eta^j + \Lambda_j \chi\right) + \frac{\tau}{2} \sum_{j=1}^{i-1} \Big(\sum_{k=1}^{j-1} \Lambda_j \Lambda_k \chi + \frac{1}{2} \Lambda_j^2 \chi\Big)$$
$$+\frac{\tau}{2} \sum_{j=i+1}^{n} \Big(\sum_{k=1}^{i-1} \Lambda_j \Lambda_k \chi + \frac{1}{2} \Lambda_i \Lambda_j \chi\Big) + \frac{\tau}{8} \Lambda_i^2 \chi,$$
$$\eta^j = \Big(\prod_{l \neq j} T_l^2\Big)\Big(T_j^2 u - T_t^+ u\Big), \qquad \chi = \frac{\tau}{2} T_1^2 \cdots T_n^2 u_t .$$

For an abstract two-level FDS

$$\mathbf{B}\,\mathbf{z}_t + \mathbf{A}\,\mathbf{z} = \Psi, \tag{5}$$

in a Hilbert space H the following propositions hold true.

LEMMA 1. *If* $\mathbf{A} = \mathbf{A}^* \geq \mathbf{0}$ *and* $\mathbf{B} - 0.5\,\tau\,\mathbf{A} \geq \mathbf{D} = \mathbf{D}^* > \mathbf{0}$ *then FDS* (5) *is stable and the a priori estimate*

$$\max_{t\in\theta^+} \|\mathbf{z}\|_{\mathbf{A}}^2 + \tau \sum_{t\in\theta^-} \|\mathbf{z}_t\|_{\mathbf{D}}^2 \leq 2\left(\|\mathbf{z}|_{t=0}\|_{\mathbf{A}}^2 + \tau \sum_{t\in\theta^-} \|\Psi\|_{\mathbf{D}^{-1}}^2\right)$$

holds.

Proof. Applying the inner product with $2\,\tau\,\mathbf{z}_t$ to the equation (5) we get

$$2\,\tau\,\big((\mathbf{B} - 0.5\,\tau\,\mathbf{A})\,\mathbf{z}_t,\, \mathbf{z}_t\big) + (\mathbf{A}\,\hat{\mathbf{z}},\, \hat{\mathbf{z}}) - (\mathbf{A}\,\mathbf{z},\, \mathbf{z}) = 2\,\tau\,(\Psi,\, \mathbf{z}_t)\,.$$

From here, using relations

$$\big((\mathbf{B} - 0.5\,\tau\,\mathbf{A})\,\mathbf{z}_t,\, \mathbf{z}_t\big) \geq \|\mathbf{z}_t\|_{\mathbf{D}}^2 \qquad \text{and} \qquad (\Psi,\, \mathbf{z}_t) \leq 0.5\,\|\mathbf{z}_t\|_{\mathbf{D}}^2 + 0.5\,\|\Psi\|_{\mathbf{D}^{-1}}^2\,,$$

after summation over the mesh θ^-, we obtain the desired a priori estimate. □

LEMMA 2. *If* $\mathbf{A} = \mathbf{A}^* > \mathbf{0}$, $\mathbf{B} = \mathbf{B}^*$ *and* $\mathbf{B} - 0.5\,\tau\,\mathbf{A} \geq \mathbf{0}$ *then FDS* (5) *is stable and the a priori estimate*

$$\max_{t\in\theta^+} \|\mathbf{z}\|_{\mathbf{B}-0.5\,\tau\,\mathbf{A}}^2 + \tau \sum_{t\in\theta^-} \left\|\frac{\hat{\mathbf{z}} + \mathbf{z}}{2}\right\|_{\mathbf{A}}^2 \leq 2\left(\|\mathbf{z}|_{t=0}\|_{\mathbf{B}-0.5\,\tau\,\mathbf{A}}^2 + \tau \sum_{t\in\theta^-} \|\Psi\|_{\mathbf{A}^{-1}}^2\right)$$

holds.

Proof. Applying the inner product with $\tau\,(\hat{\mathbf{z}} + \mathbf{z})$ to the equation (5) we get

$$\big((\mathbf{B} - 0.5\,\tau\,\mathbf{A})\,\hat{\mathbf{z}},\, \hat{\mathbf{z}}\big) - \big((\mathbf{B} - 0.5\,\tau\,\mathbf{A})\,\mathbf{z},\, \mathbf{z}\big) + 2\,\tau\left(\mathbf{A}\,\frac{\hat{\mathbf{z}} + \mathbf{z}}{2},\, \frac{\hat{\mathbf{z}} + \mathbf{z}}{2}\right) = 2\,\tau\left(\Psi,\, \frac{\hat{\mathbf{z}} + \mathbf{z}}{2}\right).$$

From here, using the inequality

$$\left(\Psi,\ \frac{\hat{\mathbf{z}}+\mathbf{z}}{2}\right) \le \frac{1}{2}\left\|\frac{\hat{\mathbf{z}}+\mathbf{z}}{2}\right\|_{\mathbf{A}}^2 + \frac{1}{2}\,\|\Psi\|_{\mathbf{A}^{-1}}^2\,,$$

after summation over the mesh θ^-, we obtain the desired a priori estimate. □

Applying Λ to (4) we obtain a FDS in the canonical form (5), where

$$\begin{aligned}&\mathbf{B} = \Lambda + 0.5\,\tau\,\Lambda\,\mathbf{E}\,\Lambda + 0.25\,\tau^2\,\Lambda\,(\mathbf{L}+0.5\,\mathbf{I})\,\Lambda\,(\mathbf{U}+0.5\,\mathbf{I})\,\Lambda = \mathbf{B}^* > \mathbf{0}\,,\\ &\mathbf{A} = \Lambda\,\mathbf{E}\,\Lambda = \mathbf{A}^* \ge \mathbf{0}\,, \quad \mathbf{B} - 0.5\,\tau\,\mathbf{A} \ge \Lambda\,, \quad \Psi = \Lambda\,\Phi \quad \text{and} \quad \mathbf{z}\big|_{t=0} = \mathbf{0}\,.\end{aligned} \tag{6}$$

According to lemma 1 FDS (5)–(6), or (2–3), is absolutely stable. The a priori estimate

$$\max_{t\in\theta^+}\|\mathbf{z}\|_{\mathbf{A}}^2 + \tau\sum_{t\in\theta^-}\|\mathbf{z}_t\|_{\Lambda}^2 \le 2\,\tau\sum_{t\in\theta^-}\|\Psi\|_{\Lambda^{-1}}^2 = 2\,\tau\sum_{t\in\theta^-}\|\Phi\|_{\Lambda}^2\,,$$

or, in expanded form

$$\|\mathbf{z}\|_3^2 \equiv \max_{t\in\theta^+}\left\|\sum_{i=1}^n \varLambda_i\,z^i\right\|_{\omega}^2 + \sum_{i=1}^n \tau\sum_{t\in\theta^-}\|z_t^i\|_{\varLambda_i}^2 \le 2\sum_{i=1}^n\tau\sum_{t\in\theta^-}\|\varphi^i\|_{\varLambda_i}^2 \tag{7}$$

holds.

Further

$$\|\varphi^i\|_{\varLambda_i} \le \sum_{j=1}^n \left(\|\eta^j_{x_j\bar{x}_jx_i}\|_{\omega_i} + \|\chi_{x_j\bar{x}_jx_i}\|_{\omega_i}\right) + \frac{\tau}{2}\sum_{j=1}^n\sum_{k=1}^{\min\{i,\,j\}}\|\chi_{x_j\bar{x}_jx_k\bar{x}_kx_i}\|_{\omega_i}\,.$$

The value $\eta^j_{x_j\bar{x}_jx_i}$ in the node $(x,\,t)\in\omega_i\times\theta^-$ is a bounded linear functional of $u\in W_2^{s,\,s/2}(e)$, where $e=\prod_{l=1}^n(x_l-3h,\ x_l+3h)\times(t,\ t+\tau)$ and $s\ge 1$. Moreover, $\eta^j_{x_j\bar{x}_jx_i}$ vanishes on the functions of the form $u=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\,t^{\beta}$, $\alpha_1+\ldots+\alpha_n+2\,\beta\le 4$. Using the Bramble–Hilbert lemma [**3**], [**5**], and the methodology proposed in [**12**] and developed in [**6–8**], for $\tau\asymp h^2$ (i.e. $C_1\,h^2\le\tau\le C_2\,h^2$), we obtain

$$|\eta^j_{x_j\bar{x}_jx_i}| \le C\,h^{s-4-n/2}\,|u|_{W_2^{s,\,s/2}(e)}\,, \qquad 3\le s\le 5\,.$$

From here, by summation over the mesh $\omega_i\times\theta^-$, it follows that

$$\left(\tau\sum_{t\in\theta^-}\|\eta^j_{x_j\bar{x}_jx_i}\|_{\omega_i}^2\right)^{1/2} \le C\,h^{s-3}\,\|u\|_{W_2^{s,\,s/2}(Q)}\,, \qquad 3\le s\le 5\,.$$

In the same manner we can estimate $\chi_{x_j\bar{x}_jx_i}$ and $\tau\,\chi_{x_j\bar{x}_jx_k\bar{x}_kx_i}$. From these estimates and the inequality (7) we get the following convergence rate estimate for FDS (2–3):

$$\|\mathbf{z}\|_3 \le C\,h^{s-3}\,\|u\|_{W_2^{s,\,s/2}(Q)}\,, \qquad 3\le s\le 5\,. \tag{8}$$

The estimate (8) is consistent with the smoothness of the solution of IBVP (1). Estimates of such type for IBVPs with variable coefficients are obtained in [**6**–**8**].

Another group of convergence rate estimates can be obtained in the following way. From (4) it follows that

$$\mathbf{z}_t = \Big[\mathbf{I} + \frac{\tau}{2}\Big(\mathbf{U} + \frac{1}{2}\mathbf{I}\Big)\Lambda\Big]^{-1}\Big[\mathbf{I} + \frac{\tau}{2}\Big(\mathbf{L} + \frac{1}{2}\mathbf{I}\Big)\Lambda\Big]^{-1}(\mathbf{\Phi} - \mathbf{E}\,\Lambda\,\mathbf{z})$$

and

$$\Lambda\,\mathbf{z}_t = \Lambda\Big[\mathbf{I} + \frac{\tau}{2}\Big(\mathbf{U} + \frac{1}{2}\mathbf{I}\Big)\Lambda\Big]^{-1}\Big[\mathbf{I} + \frac{\tau}{2}\Big(\mathbf{L} + \frac{1}{2}\mathbf{I}\Big)\Lambda\Big]^{-1}(\mathbf{\Phi} - \mathbf{E}\,\Lambda\,\mathbf{z})\,. \tag{9}$$

Introducing a new "error"

$$z = \varLambda^{-1}\sum_{i=1}^{n}\varLambda_i\,z^i$$

and considering that

$$\mathbf{E}\,\Lambda\,\mathbf{z} = (\varLambda\,z,\,\dots\,,\,\varLambda\,z)^T\,,$$

from (9), by summation of the components of vectors from the left and the right side of equation, we get

$$\varLambda\,z_t = \varLambda\,(\psi - A\,z)\,,$$

and

$$z_t + A\,z = \psi\,,\qquad t\in\theta^-\,;\qquad z|_{t=0} = 0\,, \tag{10}$$

where

$$\begin{aligned}
&A = \sum_{i=1}^{n} A_i\,,\qquad \psi = \varLambda^{-1}\sum_{i=1}^{n}\widetilde{A}_i\,\varphi^i\,,\\
&A_i = \varLambda_i\Big(I + \frac{\tau}{4}\varLambda_i\Big)^{-2}\prod_{j=1}^{i-1}\Big(I + \frac{\tau}{4}\varLambda_j\Big)^{-2}\Big(I - \frac{\tau}{4}\varLambda_j\Big)^{2}\,,\\
&\widetilde{A}_i = \Big\{\hat{A}_i - \frac{\tau}{2}\varLambda_i\sum_{j=i+1}^{n}\hat{A}_j\Big(I + \frac{\tau}{4}\varLambda_j\Big)^{-1}\times\\
&\qquad\times\prod_{k=i+1}^{j-1}\Big(I + \frac{\tau}{4}\varLambda_k\Big)^{-1}\Big(I - \frac{\tau}{4}\varLambda_k\Big)\Big\}\Big(I + \frac{\tau}{4}\varLambda_i\Big)^{-1}\,,\\
&\hat{A}_i = \varLambda_i\Big(I + \frac{\tau}{4}\varLambda_i\Big)^{-1}\prod_{j=1}^{i-1}\Big(I + \frac{\tau}{4}\varLambda_j\Big)^{-1}\Big(I - \frac{\tau}{4}\varLambda_j\Big)\,.
\end{aligned} \tag{11}$$

Operators $\hat{A}_i$, A_i and A are selfadjoint and satisfy the following relations

$$0 \le A_i \le \Lambda_i \,, \qquad -\Lambda_i \le \hat{A}_i \le \Lambda_i \,,$$

$$I - 0.5\,\tau\, A = \frac{1}{2}\left[I + \prod_{j=1}^{n} \Big(I - \frac{\tau}{4}\,\Lambda_j \Big)^2 \Big(I + \frac{\tau}{4}\,\Lambda_j \Big)^{-2} \right] \ge 0.5\, I \,,$$

$$0 < \Lambda \prod_{j=1}^{n} \Big(I + \frac{\tau}{4}\,\Lambda_j \Big)^{-2} \le A \le \Lambda \,.$$

For $\tau \asymp h^2$ we also have

$$A \ge \alpha\, \Lambda \,, \qquad \alpha = \text{const} > 0 \,.$$

Applying lemmas 1 and 2 to equation (10) and its consequences

$$A^{-1} z_t + z = A^{-1}\psi \qquad \text{and} \qquad A\, z_t + A^2 z = A\,\psi$$

we obtain the following a priori estimates

$$\text{(12)} \qquad \|z\|_0^2 \equiv \tau \sum_{t\in\theta^-} \left\| \frac{\hat{z}+z}{2} \right\|_\omega^2 \le C\,\tau \sum_{t\in\theta^-} \|\Lambda^{-1}\psi\|_\omega^2 \,,$$

$$\text{(13)} \qquad \|z\|_1^2 \equiv \max_{t\in\theta^+} \|z\|_\omega^2 + \tau \sum_{t\in\theta^-} \left\| \frac{\hat{z}+z}{2} \right\|_\Lambda^2 \le C\,\tau \sum_{t\in\theta^-} \|\psi\|_{\Lambda^{-1}}^2 \,,$$

$$\text{(14)} \qquad \|z\|_2^2 \equiv \max_{t\in\theta^+} \|z\|_\Lambda^2 + \tau \sum_{t\in\theta^-} \left\| \Lambda \frac{\hat{z}+z}{2} \right\|_\omega^2 + \tau \sum_{t\in\theta^-} \|z_t\|_\omega^2 \le C\,\tau \sum_{t\in\theta^-} \|\psi\|_\omega^2 \,.$$

Further

$$\text{(15)} \qquad \|\Lambda^{-1}\psi\|_\omega \le C \Big(\sum_{j=1}^{n} \|\eta^j\|_\omega + \|\chi\|_\omega \Big) \,,$$

$$\text{(16)} \qquad \|\psi\|_{\Lambda^{-1}} \le C \sum_{j=1}^{n} \Big(\|\eta^j_{x_j}\|_{\omega_j} + \|\chi_{x_j}\|_{\omega_j} \Big) \,,$$

$$\text{(17)} \qquad \|\psi\|_\omega \le C \sum_{j=1}^{n} \Big(\|\eta^j_{x_j\bar{x}_j}\|_\omega + \|\chi_{x_j\bar{x}_j}\|_\omega \Big) \,.$$

In such a way, the problem of deriving the convergence rate estimates for FDS (10)–(11), or (2–3), is reduced to estimation of terms η^j, χ, $\eta^j_{x_j}$, χ_{x_j}, $\eta^j_{x_j\bar{x}_j}$ and $\chi_{x_j\bar{x}_j}$. Using the Bramble–Hilbert lemma, in the same manner as in the previous case, from (12–17) we get

$$\text{(18)} \qquad \|z\|_0 \le C\, h^s\, \|u\|_{W_2^{s,\,s/2}(Q)} \,, \qquad 1 \le s \le 2 \,,$$

$$\text{(19)} \qquad \|z\|_1 \le C\, h^{s-1}\, \|u\|_{W_2^{s,\,s/2}(Q)} \,, \qquad 1 \le s \le 3 \,,$$

$$\text{(20)} \qquad \|z\|_2 \le C\, h^{s-2}\, \|u\|_{W_2^{s,\,s/2}(Q)} \,, \qquad 2 \le s \le 4 \,.$$

The estimates (18–20) are also consistent with the smoothness of data.

Three-level scheme

Three-level version of the previous FDS we will consider for the case of the first IBVP for the wave equation

$$
(21)\qquad
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \Delta u + f\,, \qquad (x,\,t) \in Q = \Omega \times (0,\,T) = (0,\,1)^n \times (0,\,T)\,,\\
&u(x,\,0) = u_0(x)\,, \qquad \frac{\partial u(x,\,0)}{\partial t} = u_1(x)\,, \qquad x \in \Omega\,,\\
&u(x,\,t) = 0, \qquad x \in \Gamma = \partial\Omega\,, \quad t \in (0,\,T)\,.
\end{aligned}
$$

We assume that the generalized solution of IBVP (21) belongs to the Sobolev space $W_2^s(Q)$, $s \geq 2$ [**13**]. In this case there exists a trace $u|_{t=t'} \in W_2^{s-1/2}(\Omega) \subset L_2(\Omega)$ for $t' \in [0,\,T]$. We also assume that the solution u can be oddly extended in space variables outside the domain Ω, preserving the Sobolev class.

On the mesh $\widehat{Q}_{h\tau}$ we approximate IBVP (21) by the following three-level factorized vector FDS

$$
(22)\qquad \Big[\mathbf{I} + \frac{\tau^2}{4}\Big(\mathbf{L} + \frac{1}{2}\,\mathbf{I}\Big)\,\Lambda\Big]\,\Big[\mathbf{I} + \frac{\tau^2}{4}\Big(\mathbf{U} + \frac{1}{2}\,\mathbf{I}\Big)\,\Lambda\Big]\,\mathbf{v}_{t\bar{t}} + \mathbf{E}\,\Lambda\,\mathbf{v} = \widetilde{\mathbf{f}}\,, \qquad t \in \vartheta\,,
$$

where $\widetilde{\mathbf{f}} = (\widetilde{f},\,\ldots\,,\,\widetilde{f})^T$, $\widetilde{f} = T_1 \cdots T_n T_t\, f$. The initial conditions we approximate by

$$
(23)\qquad \mathbf{v}|_{t=\mp\tau/2} = \big(T_1 \cdots T_n\,(u_0 \mp 0.5\,\tau\,u_1),\,\ldots\,,\,T_1 \cdots T_n\,(u_0 \mp 0.5\,\tau\,u_1)\big)^T\,.
$$

Similarly as in the previous case, we define the errors

$$
z^i = T_1 \cdots T_n\, u - v^i \qquad \text{and set} \qquad \mathbf{z} = (z^1,\,\ldots\,,\,z^n)^T\,.
$$

Vector $\mathbf{z}$ satisfies FDS

$$
(24)\qquad
\begin{gathered}
\Big[\mathbf{I} + \frac{\tau^2}{4}\Big(\mathbf{L} + \frac{1}{2}\,\mathbf{I}\Big)\,\Lambda\Big]\,\Big[\mathbf{I} + \frac{\tau^2}{4}\Big(\mathbf{U} + \frac{1}{2}\,\mathbf{I}\Big)\,\Lambda\Big]\,\mathbf{z}_{t\bar{t}} + \mathbf{E}\,\Lambda\,\mathbf{z} = \Phi\,, \quad t \in \vartheta\,,\\
\mathbf{z}_t|_{t=-\tau/2} = \mathbf{b}\,, \qquad 0.5\,(\mathbf{z} + \hat{\mathbf{z}})|_{t=-\tau/2} = \mathbf{d}\,,
\end{gathered}
$$

where $\Phi = (\varphi^1,\,\ldots\,,\,\varphi^n)^T$, $\mathbf{b} = (\beta,\,\ldots\,,\,\beta)^T$, $\mathbf{d} = (\delta,\,\ldots\,,\,\delta)^T$ and

$$
\begin{aligned}
\varphi^i &= \xi + \sum_{j=1}^{n} (\eta^j + \Lambda_j \zeta) + \frac{\tau^2}{4} \Big[\sum_{j=1}^{i-1} \Big(\sum_{k=1}^{j-1} \Lambda_k \Lambda_j \zeta + \frac{1}{2} \Lambda_j^2 \zeta \Big) \\
&\quad + \sum_{j=i+1}^{n} \Big(\sum_{k=1}^{i-1} \Lambda_k \Lambda_j \zeta + \frac{1}{2} \Lambda_i \Lambda_j \zeta \Big) + \frac{1}{4} \Lambda_i^2 \zeta \Big] , \\
\xi &= T_1 \cdots T_n \Big(u_{t\bar{t}} - T_t \frac{\partial^2 u}{\partial t^2} \Big) , \\
\eta^j &= T_1 \cdots T_n \Big(T_t \frac{\partial^2 u}{\partial x_j^2} - u_{x_j \bar{x}_j} \Big) , \\
\zeta &= \frac{\tau^2}{4} T_1 \cdots T_n \, u_{t\bar{t}} , \\
\beta &= T_1 \cdots T_n \Big(T_t \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \Big) \Big|_{t=0} , \\
\delta &= 0.5 \, T_1 \cdots T_n \, \big(u|_{t=-\tau/2} - 2 \, u|_{t=0} + u|_{t=\tau/2} \big) .
\end{aligned}
$$

For a three-level FDS

$$
\mathbf{C} \, \mathbf{z}_{t\bar{t}} + \mathbf{A} \, \mathbf{z} = \Psi , \tag{25}
$$

in a Hilbert space H the following assertion holds true.

LEMMA 3. *If* $\mathbf{A} = \mathbf{A}^* \geq \mathbf{0}$ *and* $\mathbf{C} - 0.25 \, \tau^2 \, \mathbf{A} \geq D = D^* > \mathbf{0}$ *then FDS* (25) *is stable and the a priori estimate*

$$
\max_{t \in \vartheta} N(\mathbf{z}) \leq N(\mathbf{z}) \big|_{t=-\tau/2} + \tau \sum_{t \in \vartheta} \|\Psi\|_{D^{-1}} ,
$$

holds, where

$$
N^2(\mathbf{z}) = \big\| \mathbf{z}_t \big\|^2_{\mathbf{C} - 0.25 \, \tau^2 \, \mathbf{A}} + \Big\| \frac{\mathbf{z} + \hat{\mathbf{z}}}{2} \Big\|^2_{\mathbf{A}} .
$$

Proof. Applying the inner product with $\hat{\mathbf{z}} - \check{\mathbf{z}} = \tau \, (\mathbf{z}_t + \mathbf{z}_{\bar{t}})$ to the equation (25) we get

$$
N^2(\mathbf{z}) - N^2(\check{\mathbf{z}}) = \tau \, (\Psi, \mathbf{z}_t + \mathbf{z}_{\bar{t}}) .
$$

From here, using the inequalities

$$
(\Psi, \mathbf{z}_t + \mathbf{z}_{\bar{t}}) \leq \|\Psi\|_{\mathbf{D}^{-1}} \big(\|\mathbf{z}_t\|_{\mathbf{D}} + \|\mathbf{z}_{\bar{t}}\|_{\mathbf{D}} \big) \leq \|\Psi\|_{\mathbf{D}^{-1}} \big[N(\mathbf{z}) + N(\check{\mathbf{z}}) \big] ,
$$

dividing by $N(\mathbf{z}) + N(\check{\mathbf{z}})$ and summing over the mesh ϑ, we obtain the desired a priori estimate. □

Applying Λ to (24) we obtain a FDS in the canonical form (25), where

$$\mathbf{C} = \Lambda + \frac{\tau^2}{4}\,\Lambda\,\mathbf{E}\,\Lambda + \frac{\tau^4}{16}\,\Lambda\,(\mathbf{L}+0.5\,\mathbf{I})\,\Lambda\,(\mathbf{U}+0.5\,\mathbf{I})\,\Lambda = \mathbf{C}^* > \mathbf{0}\,,$$
$$\mathbf{A} = \Lambda\,\mathbf{E}\,\Lambda = \mathbf{A}^* \ge \mathbf{0}\,, \qquad \mathbf{C} - 0.25\,\tau^2\,\mathbf{A} \ge \Lambda \qquad \text{and} \qquad \Psi = \Lambda\,\Phi\,.$$

Applying lemma 3 we obtain the a priori estimate

$$\max_{t\in\vartheta} N(\mathbf{z}) \le N(\mathbf{z})\big|_{t=-\tau/2} + \tau\sum_{t\in\vartheta}\|\Psi\|_{\Lambda^{-1}}\,. \tag{26}$$

Further

$$N^2(\mathbf{z}) = \big\|\mathbf{z}_t\big\|^2_{\mathbf{C}-0.25\,\tau^2\,\mathbf{A}} + \Big\|\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\Big\|^2_{\mathbf{A}} \ge \big\|\mathbf{z}_t\big\|^2_{\Lambda} + \Big\|\Lambda\,\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\Big\|^2_{\mathbf{E}}$$
$$= \sum_{i=1}^n \big\|z_t^i\big\|^2_{\varLambda_i} + \Big\|\sum_{i=1}^n \varLambda_i\,\frac{z^i+\hat{z}^i}{2}\Big\|^2_{\omega} \equiv \big\|\mathbf{z}\big\|^2_2\,,$$
$$N^2(\mathbf{z})\big|_{t=-\tau/2} = \|\mathbf{b}\|^2_{\mathbf{C}-0.25\,\tau^2\,\mathbf{A}} + \|\mathbf{d}\|^2_{\mathbf{A}}$$
$$= \sum_{i=1}^n \Big(\|\beta\|^2_{\varLambda_i} + \frac{\tau^4}{16}\Big\|\frac{1}{2}\,\varLambda_i\,\beta + \sum_{j=i+1}^n \varLambda_j\,\beta\Big\|^2_{\varLambda_i}\Big) + \Big\|\sum_{i=1}^n \varLambda_i\,\delta\Big\|^2_{\omega}\,,$$
$$\big\|\Psi\big\|_{\Lambda^{-1}} = \big\|\Phi\big\|_{\Lambda} = \Big(\sum_{i=1}^n \big\|\varphi^i\big\|^2_{\varLambda_i}\Big)^{1/2}.$$

Replacing these in (26), in the case when h and τ are of the same order ($\tau \asymp h$), we get

$$\max_{t\in\vartheta}\|\mathbf{z}\|_2 \le C\sum_{i=1}^n \Big(\big\|\beta_{x_i}\big\|_{\omega_i} + \big\|\delta_{x_i\bar{x}_i}\big\|_{\omega} + \tau\sum_{t\in\vartheta}\big\|\varphi^i_{x_i}\big\|_{\omega_i}\Big)\,. \tag{27}$$

To derive the convergence rate estimate we need to estimate the terms $\varphi^i_{x_i}$, β_{x_i} i $\delta_{x_i\bar{x}_i}$. Using the Bramble–Hilbert lemma, similarly as in the previous cases, for $\tau \asymp h$, we get from (27)

$$\max_{t\in\vartheta}\|\mathbf{z}\|_2 \le C\,h^{s-3}\,\|u\|_{W_2^s(Q)}\,, \qquad 3 \le s \le 5\,. \tag{28}$$

Similarly as in the case of two-level FDS, denoting

$$z = \varLambda^{-1}\sum_{i=1}^n \varLambda_i\,z^i\,,$$

from (24) we get a FDS in the form

$$\begin{gathered} z_{t\bar{t}} + A\,z = \psi\,, \qquad t\in\vartheta\,, \\ z_t\big|_{t=-\tau/2} = \beta\,, \qquad 0.5\,(z+\hat{z})\big|_{t=-\tau/2} = \delta\,, \end{gathered} \tag{29}$$

where

$$A = \sum_{i=1}^{n} A_i\,, \qquad \psi = \Lambda^{-1} \sum_{i=1}^{n} \widetilde{A}_i\, \varphi^i\,,$$

$$A_i = \Lambda_i \left(I + \frac{\tau^2}{8}\,\Lambda_i\right)^{-2} \prod_{j=1}^{i-1} \left(I + \frac{\tau^2}{8}\,\Lambda_j\right)^{-2} \left(I - \frac{\tau^2}{8}\,\Lambda_j\right)^{2},$$

$$\widetilde{A}_i = \left\{ \hat{A}_i - \frac{\tau}{2}\,\Lambda_i \sum_{j=i+1}^{n} \hat{A}_j \left(I + \frac{\tau^2}{8}\,\Lambda_j\right)^{-1} \times \right.$$

$$\left. \times \prod_{k=i+1}^{j-1} \left(I + \frac{\tau^2}{8}\,\Lambda_k\right)^{-1} \left(I - \frac{\tau^2}{8}\,\Lambda_k\right) \right\} \left(I + \frac{\tau^2}{8}\,\Lambda_i\right)^{-1},$$

$$\hat{A}_i = \Lambda_i \left(I + \frac{\tau^2}{8}\,\Lambda_i\right)^{-1} \prod_{j=1}^{i-1} \left(I + \frac{\tau^2}{8}\,\Lambda_j\right)^{-1} \left(I - \frac{\tau^2}{8}\,\Lambda_j\right).$$

Operators $\hat{A}_i$, A_i and A are selfadjoint and satisfy the relations

$$0 \le A_i \le \Lambda_i\,, \qquad -\Lambda_i \le \hat{A}_i \le \Lambda_i\,,$$

$$I - 0.25\,\tau^2 A = \frac{1}{2}\left[I + \prod_{j=1}^{n} \left(I - \frac{\tau^2}{8}\,\Lambda_j\right)^{2} \left(I + \frac{\tau^2}{8}\,\Lambda_j\right)^{-2} \right] \ge 0.5\, I\,,$$

$$0 < \Lambda \prod_{j=1}^{n} \left(I + \frac{\tau^2}{8}\,\Lambda_j\right)^{-2} \le A \le \Lambda\,.$$

For $\tau \asymp h$ we have

$$A \ge \alpha\, \Lambda\,, \qquad \alpha = \text{const} > 0\,.$$

Using these relations and lemma 3 one obtains the a priori estimate

(30)
$$\max_{t\in\vartheta} \|z\|_1 \equiv \max_{t\in\vartheta} \left(\left\| z_t \right\|_\omega^2 + \left\| \frac{z+\hat{z}}{2} \right\|_\Lambda^2 \right)^{1/2}$$
$$\le C \left(\|\beta\|_\omega + \sum_{i=1}^{n} \left\| \delta_{x_i} \right\|_{\omega_i} + \tau \sum_{t\in\vartheta} \sum_{i=1}^{n} \|\varphi^i\|_\omega \right).$$

Similarly, applying operator A^{k-1} $(k = 2,\, 3,\, \dots)$ to equation (29) and repeating the same procedure, we get

(31)
$$\max_{t\in\vartheta} \|z\|_k \equiv \max_{t\in\vartheta} \left(\left\| z_t \right\|_{\Lambda^{k-1}}^2 + \left\| \frac{z+\hat{z}}{2} \right\|_{\Lambda^k}^2 \right)^{1/2}$$
$$\le C \left(\|\beta\|_{\Lambda^{k-1}} + \left\| \delta \right\|_{\Lambda^k} + \tau \sum_{t\in\vartheta} \sum_{i=1}^{n} \|\varphi^i\|_{\Lambda^{k-1}} \right).$$

In such a way, estimating the right-hand side terms in (30) and (31) using Bramble–Hilbert lemma, in the same manner as in the previous cases, we obtain the following convergence rate estimate for the FDS (29), or (22–23)

$$\max_{t\in\vartheta} \|z\|_k \le C\, h^{s-k-1}\, \|u\|_{W_2^s(Q)}\,, \qquad k+1 \le s \le k+3\,; \qquad k = 1,\, 2,\, \dots\,.$$

References

1 В.Н. Абрашин, *Об одном варианте метода переменных направлений решения многомерных задач математической физики,* Дифференциальные уравнения **26** (1990), 314–323.

2 В.Н. Абрашин, В.А. Муха, *Об одном классе экономичных разностных схем решения многомерных задач математической физики,* Дифференциальные уравнения **28** (1992), 1786–1799.

3 J.H. Bramble, S.R. Hilbert, *Bounds for a class of linear functionals with application to Hermite interpolation,* Numer. Math. **16** (1971), 362–369.

4 J. Douglas, *On numerical integration of* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \partial u/\partial t$ *by implicit methods,* J. Soc. Industr. Appl. Math. **3** (1955), 42–65.

5 T. Dupont, R. Scott, *Polynomial approximation of functions in Sobolev spaces,* Math. Comput. **34** (1980), 441–463.

6 B.S. Jovanović, *On the convergence of finite–difference schemes for parabolic equations with variable coefficients,* Numer. Math. **54** (1989), 395–404.

7 B.S. Jovanović, *Convergence of finite–difference schemes for parabolic equations with variable coefficients,* Z. Angew. Math. Mech. **71** (1991), 647–650.

8 B.S. Jovanović, *The finite–difference method for boundary–value problems with weak solutions,* Posebna izdan. Mat. Inst. **16**, Belgrade 1993.

9 B.S. Jovanović, *On the convergence of a multicomponent alternating direction difference scheme,* Publ. Inst. Math. **55 (69)** (1994).

10 B.S. Jovanović, *On the convergence of a multicomponent three level alternating direction difference scheme,* Mat. vesnik **46** (1994), 99–103.

11 B.S. Jovanović, *On a class of multicomponent alternating direction methods.* In: D. Bainov, A. Dishliev (eds.), Third Int. Coll. Numer. Anal. held in Plovdiv 1994, SCTP, Singapore 1995, 97–106.

12 Р.Д. Лазаров, *К вопросу о сходимости разностных схем для обобщенных решений уравнения Пуассона,* Дифференциальные уравнения **17** (1981), 1287–1294.

13 J.L. Lions, E. Magenes, *Boundary–value problems and its applications,* Springer, Berlin 1972.

14 Г.И. Марчук, *Методы расщепления,* Наука, Москва (1988).

15 J.M. Ortega, W.C. Rheinboldt, *Iterative solution of nonlinear equations of several variables,* Academic Press, New York 1970.

16 D.W. Peaceman, H.H. Rachford, *The numerical solution of parabolic and elliptic differential equations,* J. Soc. Industr. Appl. Math. **3** (1955), 28–42.

17 А.А. Самарский, *Теория разностных схем,* Наука, Москва 1983.

18 П.Н. Вабищевич, *Векторные аддитивные разностные схемы,* Москва, Инст. Математ. Моделир. РАН, 1994, препр. № **2**.

Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

(Received 27 10 1995)