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GEOMETRIC INTERPRETATIONS OF SOME JORDAN ALGEBRAS

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Abstract. Geometric interpretations of Hermitian elliptic spaces whose groups of motions are compact simple Lie groups by means of Jordan algebras are constructed. In particular these results are applied to planes whose groups of motions are exceptional simple Lie groups E_6, E_7, and E_8, and to symmetry figures in these planes.

Brylinsky and Kostant [**1**] considered the Jordan algebras $\mathrm{Herm}(4, \mathbf{F})$ where the fields $\mathbf{F}$ are the real field $\mathbf{R}$, the complex field $\mathbf{C}$, and the quaternionic skew field $\mathbf{H}$ in connection with exceptional simple Lie groups E_6, E_7, and E_8. In this paper analogous Jordan algebras are applied for the geometric interpretation of these groups and for some geometric figures in the spaces whose groups of motions are these groups.

1. Jordan algebras and geometric interpretations of compact classical simple Lie groups. It is well known that compact classical simple Lie groups can be interpreted as groups of motions in complex Hermitian elliptic space $\mathbf{C}\bar{S}^n$, real elliptic space S^{2n}, quaternionic Hermitian elliptic space $\mathbf{H}\bar{S}^{n-1}$, and real elliptic space S^{2n-1}, respectively, [**2**, pp. 361–362]. All these spaces can be defined as projective spaces $\mathbf{C}P^n$, P^{2n}, $\mathbf{H}P^{n-1}$, and P^{2n-1} over corresponding algebras where the distances ω between points with projective coordinates x^i and y^i are determined by the formula

$$\cos\frac{\omega}{r} = \frac{(x,y)(y,x)}{(x,x)(y,y)}, \tag{1}$$

where (x,x) is a Hermitian quadratic form $(x,x) = \sum_i \bar{x}^i x^i$ (for $\mathbf{R}$ real quadratic form), (x,y) is a bilinear form obtained by polarization of (x,x), and r is the

AMS Subject Classification (1991): Primary 53C35 radius of curvature of the space. The spaces S^N are Riemannian manifolds V^N of constant sectional curvature, the spaces $\mathbf{C}\bar{S}^n$ and $\mathbf{H}\bar{S}^{n-1}$ are 4-pinched Riemannian manifolds V^{2n} and V^{4n-4}, respectively, of constant holomorphic sectional curvature equal to $4/r^2$. All these manifolds are symmetric spaces of rank 1 of Cartan [**3**] classes AIV, BII, CII, and DII, respectively.

The points in all these spaces can be represented by the Jordan algebras $\mathrm{Herm}(n+1,\mathbf{C})$, $\mathrm{Herm}(2n+1,\mathbf{R})$, $\mathrm{Herm}(n,\mathbf{H})$, and $\mathrm{Herm}(2n,\mathbf{R})$, respectively. The matrices (x^{ij}) in these algebras are connected with coordinates x^i of points in these spaces by the condition $x^{ij} = \bar{x}^i x^j$.

2. Hermitian elliptic spaces over tensor products of fields. Analogously, the Hermitian elliptic spaces $(\mathbf{C}\oplus\mathbf{C})\tilde{\bar{S}}^h$, $(\mathbf{C}\oplus\mathbf{H})\tilde{\bar{S}}^n$, $(\mathbf{H}\oplus\mathbf{H})\tilde{\bar{S}}^n$ over tensor products $\mathbf{C}\times\mathbf{C}$, $\mathbf{C}\times\mathbf{H}$, $\mathbf{H}\times\mathbf{H}$ are defined: these spaces are also projective spaces over corresponding algebras, where metric invariants $\hat{\omega}$ between points with projective coordinates x^i and y^i are determined for $\mathbf{C}\times\mathbf{C}$ by the formula (1) and for $\mathbf{C}\times\mathbf{H}$ and $\mathbf{H}\times\mathbf{H}$ by the more general formula

$$\cos\hat{\omega}/r = p(x,y)(x,x)^{-1}(y,x)(y,y)^{-1}p^{-1}, \tag{2}$$

where for all three cases (x,x) is a Hermitian quadratic form $(x,x) = \sum_i \tilde{\bar{x}}^i x^i$ ($x \to \bar{x}$ is the conjugate in the first tensor factor and $x \to \tilde{x}$ is the conjugate in the second tensor factor), (x,y) is a bilinear form obtained by polarization of the form (x,x), r is the radius of curvature of this space, and in two last cases p is an element of the algebra such that the right part of (2) is reduced to the form $x_0 + iIx_1$ or $x_0 + iIx_1 + jJx_2 + kKx_3$ (i, j, k and I, J, K are basis elements in the first and the second tensor factors, the expressions $x_0 + iIx_1$ are split complex numbers $a + be$, $e^2 = +1$, whose algebra is isomorphic to the direct sum $\mathbf{R}\oplus\mathbf{R}$, the expressions $x_0 + iIx_1 + jJx_2 + kKx_3$ are elements in the algebra isomorphic to the direct sum $\mathbf{R}\oplus\mathbf{R}\oplus\mathbf{R}\oplus\mathbf{R}$, and the invariants $\hat{\omega}$ are elements of the same algebras.

Since the algebras $\mathbf{C}\oplus\mathbf{C}$, $\mathbf{C}\oplus\mathbf{H}$ and $\mathbf{H}\oplus\mathbf{H}$ are isomorphic to the direct sum $\mathbf{C}\oplus\mathbf{C}$, to the algebra $\mathbf{C}_2$ of complex 2×2 matrices, and the algebra $\mathbf{R}_4$ of real 4×4 matrices, the spaces $(\mathbf{C}\oplus\mathbf{C})\tilde{\bar{S}}^n$, $(\mathbf{C}\oplus\mathbf{H})\tilde{\bar{S}}^n$, $(\mathbf{H}\oplus\mathbf{H})\tilde{\bar{S}}^n$ admit the interpretations as manifold of pairs of points of two spaces $\mathbf{C}\bar{S}^n$, as manifold of lines in $\mathbf{C}\bar{S}^{2n+1}$, and as manifold of 3-dimensional planes in S^{4n+3}, respectively. In these spaces the metrics with real distances ω equal to the root of $\sum_i \omega_i^2$, where ω_i are coefficients of $\bar{\omega}$ at 1, iI, jJ, kK; these spaces with these metrics are Riemannian manifolds V^{4n}, V^{8n}, and V^{16n}, respectively, these manifolds are symmetric spaces of rank 2, 2, and 4, respectively. The first of these spaces is the product of two spaces of the Cartan [**3**] class AIV, the second and third spaces are the spaces of the Cartan classes AIV and DII, respectively.

The points in these spaces can be represented by the Jordan algebras $\mathrm{Herm}(n+1,\mathbf{C}\oplus\mathbf{C})$, $\mathrm{Herm}(n+1,\mathbf{C}\oplus\mathbf{H})$, and $\mathrm{Herm}(n+1,\mathbf{H}\oplus\mathbf{H})$ respectively. The matrices (x^{ij}) in these algebras are connected with the coordinates x^i of the points of these spaces by the condition $x^{ij} = \tilde{\bar{x}}^i x^j$.

3. Hermitian elliptic spaces with exceptional groups of isometries. The compact exceptional Lie groups admit analogous geometric interpretations as groups of isometries of Hermitian plane $\mathbf{O}\bar{S}^2$ over alternative field $\mathbf{O}$ of octonions found by Freudenthal [**4**] and of Hermitian planes $(\mathbf{C}\oplus\mathbf{O})\tilde{\bar{S}}^2$, $(\mathbf{H}\oplus\mathbf{O})\tilde{\bar{S}}$, $(\mathbf{O}\oplus\mathbf{O})\tilde{\bar{S}}^2$ over tensor products $\mathbf{C}\otimes\mathbf{O}$, $\mathbf{H}\otimes\mathbf{O}$, $\mathbf{O}\otimes\mathbf{O}$ [**5**]. These planes are determined analogously to Hermitian spaces over fields $\mathbf{C}$ and $\mathbf{H}$ and over tensor products of these fields, but in these cases all three coordinates x^0, x^1, x^2 must belong to the same associative subalgebra of the algebra. Since in the field $\mathbf{O}$ there are involutive automorphisms $x \to \hat{x}$ which preserve a subfield of $\mathbf{O}$ isomorphic to $\mathbf{H}$, then in $\mathbf{O}$ the "projection" $x \to \bar{\hat{x}} = (x+\hat{x})/2$ onto this subfield can be defined, and the isometries of these planes can be written in the form ${}^{\prime}x^i = \sum_i \bar{\hat{a}}^i_j x^j$ where the projection $\bar{\wedge}$ is onto the subfield containing the coordinates x^0, x^1, x^2 and this transformation preserves distances ω between points determined by the formula (1) or metric invariants $\hat{\omega}$ between points determined by the formula (2). The plane $\mathbf{O}\bar{S}^2$ is 4-pinched Riemannian manifolds V^{16} of constant holomorphic sectional curvature equal to $4/r^2$. This manifold is a symmetric space of rank 1 of the Cartan [**3**] class FII. In the planes $(\mathbf{C}\otimes\mathbf{O})\tilde{\bar{S}}^2$, $(\mathbf{H}\otimes\mathbf{O})\tilde{\bar{S}}^2$, $(\mathbf{O}\otimes\mathbf{O})\tilde{\bar{S}}^2$ the metrics with real distances ω are equal to the root of $\sum_i \omega_i^2$, where ω_i are coefficients of $\bar{\omega}$ at 1, iI, jJ, kK, lL, pP, qQ, rR $(1, i, j, \ldots, r$ and $1, I, J, \ldots, R$ are basis elements in algebras $\mathbf{O})$, these planes with these metrics are Riemannian manifolds V^{32}, V^{64}, and V^{128}, respectively, these manifolds are symmetric spaces of rank 2, 4, and 8, respectively, of the Cartan [**3**] classes EIII, EVI, and EVIII, respectively.

The points in these planes can be represented by Jordan algebras $\mathrm{Herm}(3,\mathbf{O})$, $\mathrm{Herm}(3,\mathbf{C}\oplus\mathbf{O})$, $\mathrm{Herm}(3,\mathbf{H}\oplus\mathbf{O})$, and $\mathrm{Herm}(3,\mathbf{O}\oplus\mathbf{O})$, respectively. The matrices (x^{ij}) of these algebras are connected with coordinates x^i of points of these spaces by the condition $x^{ij} = \bar{x}^i x^j$ for the first case and $x^{ij} = \tilde{\bar{x}}^i x^j$ for the second, third, and fourth cases. Note that in [**4**] points of $\mathbf{O}\bar{S}^2$ were represented by matrices of $\mathrm{Herm}(3,\mathbf{O})$.

4. Symmetry figures of Hermitian elliptic spaces with exceptional groups of isometries. In the paper [**3**] mentioned above Cartan considered some symmetry figures (êtres de symétrie); manifolds of these figures can interpret symmetric spaces. In the spaces S^{2n} the symmetry figures are points (corresponding to the symmetric space BII) and m-planes (corresponding to the symmetric spaces BI). In the spaces S^{2n-1} symmetry figures are points (corresponding to the symmetric space DII), m-planes (corresponding to the symmetric spaces DI), and paratactic congruences of lines (corresponding to the symmetric spaces DIII). In the spaces $\mathbf{C}\bar{S}^n$ the symmetry figures are points (corresponding to the symmetric space AIV), m-planes (corresponding to the symmetric spaces AIII), paratactic congruences of lines (corresponding to the symmetric space AII), and normal n-chains which can be reduced to the form $x^i = \bar{x}^i$ (these figures correspond to the symmetric space AI). In the spaces $\mathbf{H}\bar{S}^{n-1}$ the symmetry figures are points and m-planes (corresponding to the symmetric spaces CII) and normal complex $(n-1)$-chains which can be reduced to the form $x^i = ix^i i^{-1}$ (these figures correspond to the symmetric space CI).

In the plane $\mathbf{O}\bar{S}^2$ the symmetry figures are points (corresponding to the symmetric space FII) and normal quaternionic 2-chains which can be reduced to the form $x^i = \hat{x}^i$ (these figures correspond to the symmetric space FI). In the plane $(\mathbf{C}\times\mathbf{O})\tilde{\bar{S}}^2$ the symmetry figures are points (corresponding to the symmetric space EIII), normal octonionic 2-chains which can be reduced to the form $x^i = \tilde{x}^i$ and whose stabilizers are isomorphic to the group of isometries in $\mathbf{O}\bar{S}^2$ (these figures correspond to the symmetric space EIV), normal biquaternionic 2-chains which can be reduced to the form $x^i = \hat{x}^i$ and whose stabilizers are isomorphic to the group of isometries in $(\mathbf{C}\times\mathbf{H})\tilde{\bar{S}}^2$ (these figures correspond to the symmetric space EII), and normal 2-bichains which can be reduced to the form $x^i = \tilde{\bar{x}}^i$ and whose stabilizers are isomorphic to the group of isometries in $\mathbf{H}\bar{S}^3$ (these figures correspond to the symmetric space EI). In the plane $(\mathbf{H}\times\mathbf{O})\tilde{\bar{S}}$ the symmetry figures are points (corresponding to the symmetric space EVI), normal bioctonionic 2-chains which can be reduced to the form $x^i = ix^i i^{-1}$ and whose stabilizers are isomorphic to the group of isometries in $(\mathbf{C}\times\mathbf{O})\tilde{\bar{S}}^2$ (these figures correspond to the symmetric space EVII), normal quaterquaternionic 2-chains which can be reduced to the form $x^i = \hat{x}^i$ and whose stabilizers are isomorphic to the group of isometries in $(\mathbf{H}\times\mathbf{H})\tilde{\bar{S}}^2$ (these figures correspond to the same symmetric space EVI, as points), and normal 2-bichains which can be reduced to the form $x^i = i\hat{x}^{i_{i-1}}$ and whose stabilizers are isomorphic to the group of isometries in $(\mathbf{C}\oplus\mathbf{H})\tilde{\bar{S}}^3$ (these figures correspond to the symmetric space EV). In the plane $(\mathbf{O}\times\mathbf{O})\tilde{\bar{S}}^2$ the symmetry figures are points (corresponding to the symmetric space EVIII), normal quateroctonionic 2-chains which can be reduced to the form $x^i = \hat{x}^i$ and whose stabilizers are isomorphic to the group of isometries in $(\mathbf{H}\times\mathbf{O})\tilde{\bar{S}}^2$ (these figures correspond to the symmetric space EIX), and normal 2-bichains which can be reduced to the form $x^i = \hat{x}^i$ and whose stabilizers are isomorphic to the group of isometries in $(\mathbf{H}\times\mathbf{H})\tilde{\bar{S}}^3$ (these figures correspond to the same symmetric space EVIII, as points).

The connection of normal 2-bichains of the planes $(\mathbf{C}\otimes\mathbf{O})\tilde{\bar{S}}^2$, $(\mathbf{H}\otimes\mathbf{O})\tilde{\bar{S}}$, and $(\mathbf{O}\otimes\mathbf{O})\tilde{\bar{S}}^2$ with the spaces $\mathbf{H}\bar{S}^3$, $(\mathbf{C}\oplus\mathbf{H})\tilde{\bar{S}}^3$, and $(\mathbf{H}\oplus\mathbf{H})\tilde{\bar{S}}^3$ is explained as follows. The normal 2-bichain $x^i = \tilde{\bar{x}}^i$ in $(\mathbf{C}\otimes\mathbf{O})\tilde{\bar{S}}^2$ can be regarded as bi-Hermitian plane $\mathbf{O}'\hat{\bar{S}}^2$ over the algebra of split octonions, that is the algebra with basis 1, i, j, k, iI, pI, qI, rI, where I is the imaginary unit in the field $\mathbf{C}$, defined by Zablotskikh [**6**] who proved that there is a bijection between the points of this plane and the lines of $\mathbf{H}\bar{S}^3$ and that there is an isomorphism between the groups of isometries of this plane and space. Analogously, the normal 2-bichain $x^i = i\hat{x}^{i_{i-1}}$ in $(\mathbf{H}\oplus\mathbf{O})\tilde{\bar{S}}^2$ can be regarded as bi-Hermitian plane over the algebra with basis 1, i, j, k, iI, pI, qI, rI, IJ, pJ, qJ, rJ, IK, pK, qK, rK, where I, J, K are basis elements in the skew field $\mathbf{H}$, and there is a bijection between the points in this plane and the lines in $(\mathbf{C} \oplus \mathbf{H})\tilde{\bar{S}}^3$ and that there is an isomorphism between the groups of isometries in this plane and space, and the normal 2-bichain $x^i = \hat{x}^i$ of $(\mathbf{O} \oplus \mathbf{O})\tilde{\bar{S}}^2$ can be regarded as a bi-Hermitian plane over the algebra with basis 1, i, j, k, iI, pI, qI, rI, lJ, pJ, qJ, rJ, IK, pK, qK, rK, lK, pL, qL, rL, lP, pP, qP, rP, lQ, PQ, qQ, rQ, lR, pR, qR, rR, where, I, J, K, L, P, Q, R are basis elements in the alternative field $\mathbf{O}$, and there is a bijection between the points in this plane and the lines in $(\mathbf{H} \times \mathbf{H})\tilde{\bar{S}}^3$ and there is an isomorphism between groups of isometries in this plane and space.

The points in the spaces $\mathbf{H}\bar{S}^3$, $(\mathbf{C} \oplus \mathbf{H})\tilde{\bar{S}}^3$, and $(\mathbf{H} \oplus \mathbf{H})\tilde{\bar{S}}^3$ which image normal 2-bichains in the Hermitian elliptic planes whose groups of isometries are groups E_6, E_7, and E_8, as we have seen in n^{os} 1 and 2, can be represented by 4×4 matrices in Jordan algebras $\mathrm{Herm}(4, \mathbf{H})$, $\mathrm{Herm}(4, \mathbf{C} \times \mathbf{H})$, and $\mathrm{Herm}(4, \mathbf{H} \times \mathbf{H})$, respectively.

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