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THE FIRST ORDER PDE SYSTEM FOR TYPE III OSSERMAN MANIFOLDS

Novica Blažić, Neda Bokan and **Zoran Rakić**

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Abstract. Timelike and spacelike Osserman manifolds of signature $(2,2)$ are defined in terms of the characteristic and minimal polynomials of the Jacobi operator (for details see [BBR]). Osserman manifolds with the diagonalizable Jacobi operator are characterized as rank-one symmetric spaces or flat. Geometry of Osserman manifolds with nondiagonalizable Jacobi operator is not yet completely clarified. Some partial answers can be found in [BBR], [BBRa], [BBRb]. In the most general case the Osserman type condition can be expressed in terms of the second order PDE system. In this paper we derive the first order PDE system characterizing Osserman manifolds when the minimal polynomial has a triple zero.

1. Introduction. Let M be a smooth manifold with a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ (sometimes we denote it also by g). If X is not a null vector, and $X \in T_pM$, $p \in M$, then one can split a tangent space $T_pM = X \oplus X^\perp$. The Jacobi operator $R_X : Y \mapsto R(Y,X)X$ is a symmetric endomorphism of T_pM and $\mathcal{K}_X$ is its restriction to $X^\perp$. An endomorphism $\mathcal{K}_X$ is diagonalizable if $X^\perp$ has the definite induced metric, in other case may be also undiagonalizable.

Locally rank-one symmetric and flat Riemannian spaces have constant eigenvalues of $\mathcal{K}_X$. Osserman have conjectured in [O] that the converse is also true. The positive answer for all dimensions except 2^kp, p is odd and $k > 1$, has been obtained by Chi [C], [Ca], [Cb]. For other problems related to Osserman condition see Gilkey [G], Gilkey, Swann and Vanhecke [GSV], and others (for more details see for example [BBGR]).

In pseudo-Riemannian setting the situation is rather complicated. Firstly, we denote by $S_p^\epsilon := \{X \in T_pM \mid \langle X,X\rangle = \epsilon 1\}$. the set of all unit spacelike ($\epsilon = +$) and timelike ($\epsilon = -$) tangent vectors $X \in T_pM$ at $p \in M$. We define a spacelike (resp. timelike) Osserman manifold M at p if the Jordan form of $\mathcal{K}_X$ is independent

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of $X \in S_p^\epsilon$ (resp. $X \in S_p^\epsilon$). Pointwise timelike (resp. spacelike) Osserman manifold is timelike (resp. spacelike) Osserman at each $p \in M$. If Jordan form of $\mathcal{K}_X$ is independent of $p \in M$ we say M is spacelike (resp. timelike) Osserman.

Garcia-Rio, Kupeli and Vásquez-Abal in [GKV], and the first two authors with Gilkey in [BBG] have studied Osserman Lorentzian manifolds.

Osserman 4-dimensional manifolds of signature $(2,2)$ have been studied in [BBR], [BBRa], [BBRb]. In this case the Osserman condition is equivalent with independence of the minimal polynomial of $\mathcal{K}_X$ of $X \in S_p^\epsilon$. To study these manifolds it is necessary to consider four different types of Jordan forms of $\mathcal{K}_X$. One can find a suitable pseudoorthonormal basis such that the matrix of $\mathcal{K}_X$ is one of the following

$$\text{Type I-a} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \ \text{Type I-b} \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix} \beta \neq 0,$$

$$\text{Type II} \begin{pmatrix} \alpha - 1/2 & 1/2 & 0 \\ -1/2 & \alpha + 1/2 & 0 \\ 0 & 0 & \beta \end{pmatrix} \text{or} \begin{pmatrix} -\alpha + 1/2 & -1/2 & 0 \\ 1/2 & -\alpha - 1/2 & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

$$\text{Type III} \begin{pmatrix} \alpha & 0 & \sqrt{2}/2 \\ 0 & \alpha & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 & \alpha \end{pmatrix}.$$

for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$.

It is easy to see: $\mathcal{K}_X$ is of type I-a if it is diagonalizable; type I-b corresponds to $\mathcal{K}_X$ if its characteristic polynomial has a complex root; $\mathcal{K}_X$ is of type II (resp. III) if its minimal polynomial has a double root α (resp. triple root α).

Studying examples of types II and III Osserman manifolds one can see their geometry is very reach and interesting, but their classification is not yet known (see [BBR], [Ra], GVV]). Henceforth it is important to investigate different types of PDE which express Osserman condition. In the most general case the Osserman type condition can be expressed in terms of the second order PDE system. The main purpose of this paper is to derive the first order PDE system characterizing Osserman manifolds when the minimal polynomial has a triple zero. Let us mention that studying of the corresponding PDE system in Theorem 2.3 (iii) in a diagonalizable case (Riemannian case [C] and the signature (2,2) [BBR]) was sufficient to establish characterization or to prove the nonexistence of the corresponding manifolds.

2. On the Osserman manifolds when minimal polynomial of the $\mathcal{K}_X$ has triple zero. Let M be a pseudo-Riemannian manifold of dimension 4, with the metric $\langle \cdot, \cdot \rangle$ of signature $(2,2)$. Let $\epsilon_i = -1$ for $i = 1, 2$ and $\epsilon_i = +1$ for $i = 3, 4$. We denote by $E_1, \cdots, E_4$ an orthonormal basis of M. It means $\langle E_i, E_j \rangle = \epsilon_i \delta_{ij}$. Let TM be the tangent bundle of M and X, Y, Z, etc. arbitrary vector fields. If ∇ is the Levi-Civita connection then $R(X,Y) : T_pM \longrightarrow T_pM$ is the pseudo-Riemann curvature operator given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.1}$$

If ω_j^i and Ω_j^i are the connection 1-forms and the curvature 2-forms respectively, then we have:

$$\begin{aligned} \nabla E_i &= \sum \omega_i^s E_s, \quad i, s = 1, \cdots, 4, \\ \Omega_j^i &= \frac{1}{2} \sum R_{klj}{}^i \theta^k \wedge \theta^l. \end{aligned} \tag{2.2}$$

If θ^i are the dual forms of E_i, i.e. $\theta^i(E_j) = \delta_j^i$ we write the Cartan structural equations in the following form

$$\begin{aligned} d\theta^i &= -\sum \omega_j^i \wedge \theta^j, \\ d\omega_j^i &= -\sum (\omega_k^i \wedge \omega_j^k + \Omega_j^i) \quad \text{(the second Bianchi identity)}. \end{aligned} \tag{2.3}$$

Let us mention also the symmetry properties of the connection forms as well as the curvature forms

$$\omega_j^i = -\epsilon_i \epsilon_j \omega_i^j, \qquad \Omega_j^i = -\epsilon_i \epsilon_j \Omega_i^j. \tag{2.4}$$

Recently, in the paper [BBR] we found the characterization of the Osserman manifold of signature (2,2). In this characterization appears the class of manifolds of type III, which Jacobi operators have a triple root α of its minimal polynomial. In this case we found all components of the curvature tensor R in an orthogonal frame $\{E_1, E_2, E_3, E_4\}$, using Osserman algebraic condition. The all nonzero components of the curvature tensor R are:

$$\begin{aligned} &R_{1221} = R_{4334} = \alpha, \qquad R_{1331} = R_{4224} = -\alpha, \\ &R_{1441} = R_{3223} = -\alpha, \quad R_{2114} = R_{2334} = R_{3224} = R_{1442} = -k, \\ &R_{3114} = R_{1223} = R_{1443} = R_{1332} = k, \quad \text{where} \quad k = \sqrt{2}/2. \end{aligned} \tag{2.5}$$

Now, we start with studying 1-forms which appear naturally in this context. Let us introduce the following notations

$$\begin{aligned} &A = \omega_1^2 + \omega_3^4, \quad B = \omega_2^4 - \omega_1^3, \quad C = \omega_1^4 + \omega_2^3, \\ &\tilde{A} = 2(A + B) \quad \tilde{B} = 2(B - A). \end{aligned} \tag{2.6}$$

These 1-forms are the following linear combinations of the basic forms θ^i, i.e.

$$C = \sum C_i \theta^i, \quad \tilde{A} = \sum \tilde{A}_i \theta^i \quad \text{and} \quad \tilde{B} = \sum \tilde{B}_i \theta^i. \tag{2.7}$$

The coefficients of these 1-forms are not independent. More precisely, we have

LEMMA 2.1. *(i) The coefficients $\tilde{A}_i$, $i = 1, \cdots, 4$ satisfy the following relations:*

$$\tilde{A}_1 = -\tilde{A}_4 = C_2 + C_3 \ \textit{and} \ -\tilde{A}_2 = \tilde{A}_3 = C_1 + C_4.$$

(ii) The coefficients $\tilde{B}_i$, $i = 3, 4$ satisfy the following relations:

$$\tilde{B}_3 = C_1 - C_4 - B_2 \ \textit{and} \ \tilde{B}_4 = C_3 - C_2 - B_1.$$

Proof. The second Bianchi identity implies

$$\begin{aligned} d\Omega_2^1 &= \Omega_3^1 \wedge \omega_2^3 + \Omega_4^1 \wedge \omega_2^4 - \Omega_3^2 \wedge \omega_2^1 - \Omega_3^4 \wedge \omega_4^1, \\ d\Omega_3^1 &= \Omega_2^1 \wedge \omega_3^2 + \Omega_4^1 \wedge \omega_3^4 - \Omega_3^2 \wedge \omega_2^1 - \Omega_3^4 \wedge \omega_4^1, \\ d\Omega_4^1 &= \Omega_2^1 \wedge \omega_4^2 + \Omega_3^1 \wedge \omega_4^3 - \Omega_4^2 \wedge \omega_2^1 - \Omega_4^3 \wedge \omega_3^1. \end{aligned} \tag{2.8}$$

We use now (2.5), (2.6), (2.7) and (2.8) to see

$$2B \wedge (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) - \frac{1}{2}\tilde{B} \wedge (\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4) - C \wedge (\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) = 0,$$

$$\frac{1}{2}\tilde{B} \wedge (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + 2A \wedge (\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4) - C \wedge (\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) = 0,$$

$$C \wedge (\theta^1 \wedge \theta^2 - \theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4 - \theta^3 \wedge \theta^4) - \tilde{A} \wedge (\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) = 0.$$

From the previous relations, after calculating the coefficients with $\theta^i \wedge \theta^j \wedge \theta^k$ for the different choice of indices i, j, k we have statement. □

From the relations between the coefficients of the forms in Lemma 2.1, one can find the folowing properties of the forms $\tilde{A}$ and C. More precisely, we have:

PROPOSITION 2.2. (i) $\mathrm{div}(\tilde{A}) = 0$. (ii) $\mathrm{div}(C) = 6\alpha$. (iii) $\|C\| = 0$.

Proof. We use Lemma 2.1 and (2.7) to see:

$$\begin{aligned} \|C\|^2 &= -C_1^2 - C_2^2 + C_3^2 + C_4^2, \\ \mathrm{div}(C) &= -C_{1;1} - C_{2;2} + C_{3;3} + C_{4;4}, \\ \mathrm{div}(\tilde{A}) &= -(C_2 + C_3)_{;1} + (C_1 + C_4)_{;2} - (C_2 + C_3)_{;4} + (C_1 + C_4)_{;3}. \end{aligned} \tag{2.9}$$

If we differentiate the forms $\tilde{A}$ and C we find

$$\begin{aligned} dC &= \frac{1}{8}(\tilde{A} \wedge \tilde{B}) + \Omega_2^3 + \Omega_1^4, \\ d\tilde{A} &= C \wedge \tilde{A} + 2(\Omega_1^2 + \Omega_3^4 + \Omega_2^4 - \Omega_1^3), \\ d\tilde{B} &= \tilde{B} \wedge C + 2(\Omega_2^4 - \Omega_1^3 - \Omega_1^2 - \Omega_3^4). \end{aligned} \tag{2.10}$$

Now, if we find differential of the form C using covariant derivatives we get:

$$\begin{aligned} -C_{2;1} + C_{1;2} &= -\frac{1}{8}(B_2(C_2 + C_3) + B_1(C_1 + C_4)) - 2k, \\ -C_{3;4} + C_{4;3} &= \frac{1}{8}(-B_2(C_2 + C_3) - B_1(C_1 + C_4) + 2(C_1C_3 - C_2C_4)) - 2k, \\ -C_{3;1} + C_{1;3} &= \frac{1}{8}(B_2(C_2 + C_3) + B_1(C_1 + C_4) - (C_1 - C_4)(C_2 + C_3) + 2k, \\ -C_{2;4} + C_{4;2} &= \frac{1}{8}(B_2(C_2 + C_3) + B_1(C_1 + C_4) - (C_1 - C_4)(C_2 + C_3) + 2k, \\ -C_{1;4} + C_{4;1} &= \frac{1}{8}(C_3^2 - C_2^3) + \alpha, \\ -C_{2;3} + C_{3;2} &= \frac{1}{8}(C_4^2 - C_1^3) + \alpha. \end{aligned} \tag{2.11}$$

If we add the left sides of the first four equations of (2.11) using (2.9) we obtain (i).

From $d\tilde{A}$ we have:

$$\begin{aligned} C_{1;1}+C_{2;2}+C_{3;2}+C_{4;1} &= C_1^2+C_2^2+C_2C_3+C_1C_4-2\alpha, \\ C_{1;4}+C_{2;3}+C_{3;3}+C_{4;4} &= C_3^2+C_4^2+C_2C_3+C_1C_4+2\alpha. \end{aligned} \tag{2.12}$$

Subtracting the first equation of the second one of (2.12) using (2.11) we find :

$$\operatorname{div}(C)=\frac{9}{8}\|C\|^2+6\alpha. \tag{2.13}$$

Similarly, from $d\tilde{B}$ follows:

$$B_{1;2}-B_{2;1}=-B_1C_2+B_2C_1+2\alpha, \tag{2.14}$$

$$\begin{aligned} B_{1;3}+B_{2;1}-C_{1;1}+C_{4;1} &= -B_1C_3-B_2C_1+(C_1^2-C_1C_4)+2\alpha, \\ -B_{2;4}-B_{1;2}+C_{3;2}-C_{2;2} &= B_2C_4+B_1C_2-(-C_2^2+C_2C_3)-2\alpha, \\ B_{2;4}-B_{1;3}+C_{4;4}-C_{1;4}+C_{3;3}-C_{2;3}& \\ &= B_1C_3-B_2C_4+C_2C_3+C_1C_4-C_3^2-C_4^2+2\alpha. \end{aligned}$$

If we add all four equations from (2.14) using (2.11) we find:

$$\operatorname{div}(C)=-\frac{9}{8}\|C\|^2+6\alpha. \tag{2.15}$$

Now, (ii) and (iii) follow from (2.13) and (2.15). □

Here is more convenient to deal with a following null basis $\{F_i\}$ of T_pM:

$$\begin{aligned} F_1 &= \frac{1}{\sqrt{2}}(E_1+E_4), \quad F_2=\frac{1}{\sqrt{2}}(E_2+E_3), \\ F_3 &= \frac{1}{\sqrt{2}}(E_2-E_3), \quad F_4=\frac{1}{\sqrt{2}}(E_1-E_4). \end{aligned} \tag{2.16}$$

Let $\{\xi^i, i=1,2,3,4\}$ and $\{\varphi_j^i\}$ be the corresponding dual basis of T_p^*M and connection 1-forms. Using formulas (2.2) and (2.4) one can find that the connection 1-forms $\{\varphi_j^i\}$ in this basis satisfy the following symmetric relations:

$$\begin{aligned} &\varphi_1^1+\varphi_4^4=0, \quad \varphi_1^2+\varphi_3^4=0, \quad \varphi_1^3+\varphi_2^4=0, \quad \varphi_4^1=\varphi_1^4=0, \\ &\varphi_2^2+\varphi_3^3=0, \quad \varphi_2^1+\varphi_4^3=0, \quad \varphi_3^1+\varphi_4^2=0, \quad \varphi_2^3=\varphi_3^2=0. \end{aligned} \tag{2.17}$$

Let $C=\bar{C}_i\xi^i$ and $A=\bar{A}_i\xi^i$. Let $\bar{C}_{i;j}$ be the components of the covariant derivative ∇C. Now, we give the following characterization of the type III Osserman manifolds of signature (2,2). More precisely, we have:

THEOREM 2.3. *Let M be an Osserman manifold of signature (2,2) which minimal polynomial of the Jacobi operator has a triple root. Then there exists a null frame $\{F_i\}$, with the corresponding dual basis $\{\xi^i\}$ and connection 1-forms $\{\varphi_j^i\}$ such that:*

(*i*) $\varphi^3_1(F_1) = \varphi^3_1(F_2) = 0.$

(*ii*) $\varphi^3_1(F_3) = -\frac{1}{2}2(\varphi^1_1 + \varphi^2_2)(F_1), \quad \varphi^3_1(F_4) = \frac{1}{2}(\varphi^1_1 + \varphi^2_2)(F_2).$

(*iii*) $\bar{C}_{2;1} = -\bar{C}_1\bar{C}_2 - 2\alpha,\ \bar{C}_{1;1} = \bar{C}_1^2,\ \bar{C}_{2;2} = -\bar{C}_2^2,\ \bar{C}_{2;4} = \bar{C}_1\bar{C}_2,$
$\bar{C}_{2;4} + \bar{C}_{4;2} = \bar{C}_1\bar{C}_3 + \bar{C}_2\bar{C}_4.$

Proof. (i) We know from (2.6) $A = \omega^2_1 + \omega^4_3, B = \omega^4_2 - \omega^3_1$ and $C = \omega^4_1 + \omega^3_2$. So, we have to find the relation between the connection 1-forms ω^i_j and φ^i_j. If we use the formula $[\omega^i_j] = G^{-1}[\varphi^i_j]G + G^{-1}dG$, where G is the matrix of transition from the basis $\{E_i\}$ into the basis $\{F_i\}$, and the symmetries (2.17) of the connection 1-forms $\{\varphi^i_j\}$, we find the relations $\tilde{A} = 4\varphi^3_1, \tilde{B} = 4\varphi^1_3$ and $C = \varphi^1_1 + \varphi^2_2$. Then we get the relation between dual basis using $[\xi^i] = G^\tau[\theta^i]$:

$$\xi^1 = \frac{1}{\sqrt{2}}(\theta^1 + \theta^4), \quad \xi^2 = \frac{1}{\sqrt{2}}(\theta^2 + \theta^3),$$
$$\xi^3 = \frac{1}{\sqrt{2}}(\theta^2 - \theta^3), \quad \xi^4 = \frac{1}{\sqrt{2}}(\theta^1 - \theta^4).$$

Now, we combine these relations with Lemma 2.1 (i) to get (i). One can also check that $\tilde{C}_1 = (C_1 + C_4)/\sqrt{2}$ and $\tilde{C}_2 = (C_2 + C_3)/\sqrt{2}$.

(ii) It follows directly from (i) and duality of the basis $\{\xi^i\}$ and $\{F_i\}$.

(iii) It is interesting to notice that the second equation of (2.10) has the following simpler form in the null frame $\{F_i\}$

$$d\tilde{A} = 4\alpha\xi^4 \wedge \xi^3 + C \wedge \tilde{A}. \tag{2.18}$$

In terms of this frame we express $\tilde{A} = 2(-\bar{C}_1\xi^3 + \bar{C}_2\xi^4)$. One can consider the equations (2.18) as the system given in (iii). □

Remark 1. One can see that the general Osserman type condition leads to some system of PDE of the second order. As a consequence of this system using the second Bianchi identity we obtain the PDE systems of the first order (2.18). The statements (i), (ii) from the above theorem enable us to reduce this problem to the more natural and simpler PDE systems of the first order obtained in Theorem 2.3 (iii) (in terms of the connection forms of some null frame) and gives the characterization of the type III Osserman manifold.

Remark 2. Chi has used the corresponding PDE system in Theorem 2.3 (iii) studying the Osserman conjecture for 4-dimensional Riemannian manifolds. The same system has been used in [BBR] to establish a characterization or to prove the nonexistence of the corresponding manifolds.

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Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

(Received 27 06 1997)

blazicn@matf.bg.ac.yu
neda@matf.bg.ac.yu
zrakic@matf.bg.ac.yu