PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 65 (79), 1999, 109–111

A NOTE ON TWO INDICES OF SEMIGROUP ELEMENTS

Vladimir Rakočević

Communicated by Žarko Mijajlović

Abstract. We answer some questions from a recent paper by Kečkić [**3**].

Let us recall that Drazin [**1**] has introduced and investigated a generalized inverse (he called it *pseudoinverse*) in associative rings and semigroups, i.e., if S is an algebraic semigroup (or associative ring), then an element $a \in S$ is said to have a *Drazin* inverse if there exists an $x \in S$ such that

$$a^m = a^{m+1}x \quad \text{for some non-negative integer } m, \tag{1}$$

$$x = ax^2 \quad \text{and} \quad ax = xa. \tag{2}$$

If a has a Drazin inverse, then the smallest non-negative integer m in (1) is called the *index* (*Drazin index*) $i(a)$ of a. It is well known that there is at most one x such that equations (1) and (2) hold. The unique x is denoted by a^D and called the *Drazin* inverse of a.

Following [**3**], for some integer $k > 1$, $a \in S$ let us consider the following systems:

$$a^{k+1}x = a^k, \qquad ax = xa, \tag{S_k}$$

and

$$axa = a, \qquad a^kx = xa^k. \tag{Σ_k}$$

If k is the smallest positive integer such that (S_k) is consistent, we say that the S-index of a is k and we write $i_S(a) = k$. Similarly, if k is the smallest positive

AMS Subject Classification (1991): Primary 20M05

Supported by the Science Fund of Serbia, grant 04M03, through Matematički institut

integer such that (Σ_k) is consistent, we say that the Σ-index of a is k and we write $i_\Sigma(a) = k$. Kečkić [**3**, A.2 and E.2] has proved that if the system (Σ_k) is consistent, then the system (S_k) is also consistent, and the converse is not true in general.

For the sake of completness let us recall that Kečkić [**3**], among other things, asked the following questions:

Question 1. Is there a semigroup in which, for some $k > 2$, both systems (Σ_k) and (S_{k-1}) are consistent, while (Σ_{k-1}) is inconsistent?

Question 2. Describe the semigroups in which $i_S(a) = k \Leftrightarrow i_\Sigma(a) = k$.

To give the answers to these questions, we start with the following well-known auxiliary result (see e.g. [**2**, p. 190, Proof of (A)]).

Lemma 1. Let S be a semigroup, and $a \in S$. Then the system (S_k) is consistent if and only if the the system $(S_k) \wedge ax^2 = x$ is consistent. In that case a is Drazin invertible and $i_S(a) = i(a)$.

Now we shall prove that the answer to the Question 1 is negative.

THEOREM 1. *Let S be a semigroup, $a \in S$, and $k > 2$. If both systems (Σ_k) and (S_{k-1}) are consistent, then the system (Σ_{k-1}) is also consistent.*

Proof. Suppose that both systems (Σ_k) and (S_{k-1}) are consistent. As the system (Σ_k) is consistent, there is an $x \in S$ such that

$$axa = a, \qquad a^k x = xa^k. \tag{3}$$

Furthermore, as the system (S_{k-1}) is consistent, by Lemma 1, a is Drazin invertible and $i_S(a) = i(a) \le k-1$. Now (see [**1**]) a^k is Drazin invertible, $(a^k)^D = (a^D)^k$, and

$$(a^k)^D(a^k x) = (a^k x)(a^k)^D = (xa^k)(a^k)^D. \tag{4}$$

Taking into account that a and a^D are commuting and $a^D a$ is an idempotent, we obtain

$$a^D ax = xaa^D. \tag{5}$$

Multiplying (5) by a and taking into account (3) we get

$$a^D a = a^D(axa) = (a^D ax)a = (xaa^D)a = xa^2a^D, \tag{6}$$

i.e.,

$$a^D a = xa^2 a^D. \tag{7}$$

Multiplying (7) by a^{k-1} we get

$$a^D a^k = xa^2 a^D a^{k-1}, \tag{8}$$

i.e.,

$$a^{k-1} = xa^k. \tag{9}$$

Finally by (3) we have

$$a^{k-1}x = (xa^k)x = x(a^kx) = x(xa^k) = xa^{k-1}, \tag{10}$$

i.e., we prove that the system (Σ_{k-1}) is consistent. ■

Concerning Question 2, by Theorem 1 and [**3**, A.3] we get the following corollaries

COROLLARY 1. *Let S be a semigroup and $a \in S$. If a is Drazin invertible with the Drazin index $i(a) = n$, and if the system (Σ_k), $k \geq n$ is consistent, then $i_\Sigma(a) = i(a)$.*

COROLLARY 2. *Let S be a semigroup and $a \in S$. If the system (Σ_k) is consistent, then we know that by* [**3**, A.2] *the system (S_k) is consistent, and we have $i_\Sigma(a) = i_S(a) = i(a)$.*

Proof. By Lemma 1 and Corollary 1. ■

Hence Corollary 2 is an answer to Question 2, under the restriction that for each $a \in S$ the system (Σ_k) is consistent.

REFERENCES

1. M.P. Drazin, *Pseudoinvese in associative rings and semigroups*, Amer. Math. Monthly **65** (1958), 506–514.
2. M.P. Drazin, *Extremal Definitions of Generalized Inverses*, Linear Algebra Appl. **165** (1992), 185–196.
3. J. Kečkić, *Some remarks on possible generalized inverses in semigroups*, Publ. Inst. Math. (Beograd) **61(75)** (1997), 33–40.

Filozofski fakultet
18000 Niš
Serbia, Yugoslavia
vrakoc@bankerinter.net
vrakoc@archimed.filfak.ni.ac.yu

(Received 18 05 1998)