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NEW SPECIAL GEODESIC MAPPINGS OF GENERALIZED RIEMANNIAN SPACES

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Abstract. We define $\underset{\theta}{R}$-projective geodesic mappings ($\theta = 1, \ldots, 5$) of two generalized Riemannian spaces and obtain some invariant geometric objects of these mappings, generalizing Weyl's tensor. Also, we define $\underset{\theta}{R}$-projectively flat generalized Riemannian spaces $G\overline{R}_N$ and find necessary conditions for the space GR_N to be $\underset{\theta}{R}$-projectively flat.

0. Introduction

Definition 0.1. A differentiable N-dimensional manifold GR_N is a generalized Riemannian space in the sence of Eisenhart's definition [2] if it is equipped with a nonsymmetric basic tensor g_{ij}.

Consider two N-dimensional generalized Riemannian spaces GR_N and $G\overline{R}_N$. The connection coefficients of these spaces are generalized Christoffel's symbols of second kind, Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$ respectively. In the general case $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

Definition 0.2. The mapping $f : GR_N \to G\overline{R}_N$ is geodesic [6], if the geodesics of the space GR_N are mapped into to the geodesics of the space $G\overline{R}_N$.

We can consider these spaces in the common by this mapping system of local coordinates, i.e., if $f : M(x) \equiv M(x^1, \cdots, x^N) \mapsto \overline{M}$, then we have $\overline{M}(x) \equiv \overline{M}(x^1, \cdots, x^N)$. For Christoffel's symbols of second kind of the spaces GR_N and $G\overline{R}_N$ in the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$\overline{\Gamma}^i_{jk}(x) = \Gamma^i_{jk}(x) + P^i_{jk}(x) \qquad (i, j, k = 1, \ldots, N), \tag{0.1}$$

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where $P^i_{jk}(x)$ is **the deformation tensor** of the connection Γ of GR_N according to the mapping $f: GR_N \to G\overline{R}_N$.

A necessary and sufficient condition [6] for the mapping $f: GR_N \to G\overline{R}_N$ to be geodesic is that the deformation tensor P^i_{jk} from (0.1) at the mapping f has the form

$$P^i_{jk}(x) = \delta^i_j\,\psi_k(x) + \delta^i_k\,\psi_j(x) + \xi^i_{jk}(x), \tag{0.2}$$

where

$$\psi_i(x) = \frac{1}{N+1}(\overline{\Gamma}^\alpha_{i\alpha}(x) - \Gamma^\alpha_{i\alpha}(x)),\ \ \xi^i_{jk}(x) = P^i_{\underset{\vee}{jk}} = \frac{1}{2}(P^i_{jk} - P^i_{kj}). \tag{0.3}$$

In a generalized Riemannian space we can define four kinds of covariant derivative [3,4]. Denote by $\underset{\theta}{|}, \underset{\overline{\theta}}{|}$ a covariant derivative of the kind θ in GR_N and $G\overline{R}_N$ respectively.

For the torsion tensor of the space GR_N it holds (eq. (2.10) in [6])

$$\Gamma^\alpha_{\underset{\vee}{i\alpha}} = 0. \tag{0.4}$$

In the space GR_N we have five independent curvature tensors [5]:

$$\underset{1}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^\alpha_{jm}\Gamma^i_{\alpha n} - \Gamma^\alpha_{jn}\Gamma^i_{\alpha m}, \tag{0.5}$$

$$\underset{2}{R}{}^i{}_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{mj}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{m\alpha}, \tag{0.6}$$

$$\underset{3}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m} + \Gamma^\alpha_{nm}(\Gamma^i_{\alpha j} - \Gamma^i_{j\alpha}), \tag{0.7}$$

$$\underset{4}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m} + \Gamma^\alpha_{mn}(\Gamma^i_{\alpha j} - \Gamma^i_{j\alpha}), \tag{0.8}$$

$$\underset{5}{R}{}^i{}_{jmn} = \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{\alpha n} + \Gamma^\alpha_{mj}\Gamma^i_{n\alpha} - \Gamma^\alpha_{jn}\Gamma^i_{m\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m}). \tag{0.9}$$

Denote by $\underset{\theta}{\overline{R}}{}^i_{jmn}$ $(\theta = 1, \dots, 5)$ the corresponding curvature tensors of the space $G\overline{R}_N$.

In the case of a geodesic mapping $f: R_N \to \overline{R}_N$ of Riemannian spaces R_N and $\overline{R}_N$ we have an invariant geometric object ([8, p. 80] or [1, eq. (37.5)]):

$$W^i{}_{jmn} = R^i{}_{jmn} + \frac{1}{N-1}(\delta^i_m\,R_{jn} - \delta^i_n\,R_{jm}). \tag{0.10}$$

where $R^i{}_{jmn}$ is Riemann-Christoffel's curvature tensor of the space R_N, and R_{jm} is Ricci's tensor. The object $W^i{}_{jmn}$ is called *Weyl's tensor*, or the tensor of projective curvature [1], [8]. Having a geodesic mapping of two generalized Riemannian spaces, we can not find a generalization of Weyl's tensor as an invariant of geodesic mapping in the general case. For that reason in [7] we defined a special geodesic mapping $f: GR_N \to G\overline{R}_N$, that we called equitorsion geodesic mapping. Here we define and study new kinds of geodesic mappings of generalized Riemannian spaces.

1. $\underset{1}{R}$-projective mappings

DEFINITION 1.1. The geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{1}{R}$-projective if the following condition is satisfied

$$
\begin{aligned}
&\delta^i_j\,(\underset{1}{\mathcal{D}}_{mn} - \underset{1}{\mathcal{D}}_{nm}) + \delta^i_m\,\underset{1}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{1}{\mathcal{D}}_{jm} + (\delta^i_n\,\xi^\alpha_{jm} - \delta^i_m\,\xi^\alpha_{jn})\psi_\alpha \\
&+2\psi_j\xi^i_{mn} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{\alpha m} + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j \\
&+2\Gamma^\alpha_{\underset{\vee}{mn}}\xi^i_{j\alpha} = 0,
\end{aligned} \tag{1.1}
$$

where

$$
\begin{aligned}
\underset{1}{\mathcal{D}}_{jm} = {} & \frac{N-1}{N+1}\xi^\alpha_{jm}\psi_\alpha - \frac{1}{N^2-1}(4\Gamma^\alpha_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^\alpha_{\underset{\vee}{m\beta}}\xi^\beta_{j\alpha} - 2\Gamma^\alpha_{\underset{\vee}{j\beta}}\xi^\beta_{m\alpha}) \\
& -\frac{1}{N-1}(\xi^\alpha_{j\beta}\xi^\beta_{\alpha m} - 2\Gamma^\alpha_{\underset{\vee}{mj}}\psi_\alpha - 2\Gamma^\alpha_{\underset{\vee}{m\beta}}\xi^\beta_{j\alpha}),
\end{aligned} \tag{1.1'}
$$

and $\underset{\vee}{mj}$, as in (0.3), denotes an antisymmetrisation with respect to indices m and j.

DEFINITION 1.2. The space GR_N is $\underset{1}{R}$-projectively flat if there exists an $\underset{1}{R}$-projective mapping of the space GR_N to a flat space (i.e., to a space, whose connection coeffitients in special coordinates are zero).

Using (0.1), we get the relation between the first kind curvature tensors (0.5) of the spaces GR_N and $G\overline{R}_N$

$$
\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{1}{|}m} + P^\alpha_{jm}P^i_{\alpha n} - P^\alpha_{jn}P^i_{\alpha m} + 2\Gamma^\alpha_{\underset{\vee}{mn}}P^i_{j\alpha}. \tag{1.2}
$$

Denoting

$$
\underset{\theta}{\psi}_{mn} = \psi_{m\underset{\theta}{|}n} - \psi_m\psi_n \quad (\theta = 1,2) \tag{1.3}
$$

and substituting P with respect to (0.2) in (1.2), we obtain

$$
\begin{aligned}
\underset{1}{\overline{R}}{}^i_{jmn} = {} & \underset{1}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{1}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} - \delta^i_m\,\xi^\alpha_{jn}\psi_\alpha \\
& + \delta^i_n\,\xi^\alpha_{jm}\psi_\alpha + \xi^i_{jm\underset{1}{|}n} - \xi^i_{jn\underset{1}{|}m} + 2\psi_j\xi^i_{mn} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{\alpha m} \\
& + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\xi^i_{j\alpha}.
\end{aligned} \tag{1.4}
$$

Contracting (1.4) with respect to i and n and using (0.4) we get

$$
\begin{aligned}
\underset{1}{\overline{R}}_{jm} = {} & \underset{1}{R}_{jm} - 2\underset{1}{\psi}_{\underset{\vee}{jm}} + (1-N)\underset{1}{\psi}_{jm} + (N-1)\xi^\alpha_{jm}\psi_\alpha + \xi^\beta_{jm\underset{1}{|}\beta} \\
& - \xi^\alpha_{j\beta}\xi^\beta_{\alpha m} + 2\Gamma^\beta_{\underset{\vee}{m\beta}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^\alpha_{\underset{\vee}{m\beta}}\xi^\beta_{j\alpha}.
\end{aligned} \tag{1.5}
$$

Here $\underset{1}{\overline{R}}_{jm}$ and $\underset{1}{R}_{jm}$ denote the first kind Ricci tensors of the space $G\overline{R}_N$ and GR_N respectively.

From (1.5) we obtain

$$\begin{aligned}(1+N)\underset{1}{\psi}_{\underset{\vee}{jm}} = \underset{1}{R}_{\underset{\vee}{jm}} - \underset{1}{\overline{R}}_{\underset{\vee}{jm}} + (N-1)\xi^{\alpha}_{jm}\psi_\alpha + \xi^{\beta}_{jm\underset{1}{|}\beta} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha \\ + \Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\beta} - \Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}.\end{aligned} \tag{1.6}$$

Substituting (0.3) in (1.6) we get

$$\begin{aligned}\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jm}} + 2(N-1)\xi^{\alpha}_{jm}\psi_\alpha + 2\xi^{\beta}_{jm\underset{1}{|}\beta} \\ + 4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}\big] + (1-N)\underset{1}{\psi}_{jm} + (N-1)\xi^{\alpha}_{jm}\psi_\alpha \\ + \xi^{\beta}_{jm\underset{1}{|}\beta} - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha},\end{aligned}$$

i.e.

$$\begin{aligned}(N-1)\underset{1}{\psi}_{jm} = \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jm}} + 2(N-1)\xi^{\alpha}_{jm}\psi_\alpha \\ + 2\xi^{\beta}_{jm\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}\big] + (N-1)\xi^{\alpha}_{jm}\psi_\alpha + \xi^{\beta}_{jm\underset{1}{|}\beta} \\ - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha}.\end{aligned} \tag{1.7}$$

Replacing (1.7) in (1.4) we get

$$\begin{aligned}\underset{1}{\overline{R}}^{i}{}_{jmn} = \underset{1}{R}^{i}{}_{jmn} + \frac{1}{N-1}\delta^{i}_{j}\,\big\{\underset{1}{R}_{mn} - \underset{1}{\overline{R}}_{mn} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{mn}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{mn}} \\ + 2(N-1)\xi^{\alpha}_{mn}\psi_\alpha + 2\xi^{\beta}_{mn\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{nm}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{m\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{n\alpha}\big] \\ + (N-1)\xi^{\alpha}_{mn}\psi_\alpha + \xi^{\beta}_{mn\underset{1}{|}\beta} - \xi^{\alpha}_{m\beta}\xi^{\beta}_{\alpha n} + 2\Gamma^{\alpha}_{\underset{\vee}{nm}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{m\alpha} - \underset{1}{R}_{nm} + \underset{1}{\overline{R}}_{nm} \\ + \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{nm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{nm}} + 2(N-1)\xi^{\alpha}_{nm}\psi_\alpha + 2\xi^{\beta}_{nm\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{mn}}\psi_\alpha \\ + 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{n\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{m\alpha}\big] - (N-1)\xi^{\alpha}_{nm}\psi_\alpha - \xi^{\beta}_{nm\underset{1}{|}\beta} + \xi^{\alpha}_{n\beta}\xi^{\beta}_{\alpha m} - 2\Gamma^{\alpha}_{\underset{\vee}{mn}}\psi_\alpha \\ - 2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{n\alpha}\big\} + \frac{1}{N-1}\delta^{i}_{m}\,\big\{\underset{1}{R}_{jn} - \underset{1}{\overline{R}}_{jn} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jn}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jn}} + 2(n-1)\xi^{\alpha}_{jn}\psi_\alpha \\ + 2\xi^{\beta}_{jn\underset{1}{|}\beta} + 4\Gamma^{\alpha}_{\underset{\vee}{nj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{j\alpha} - 2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{n\alpha}\big] + (N-1)\xi^{\alpha}_{jn}\psi_\alpha + \xi^{\beta}_{jn\underset{1}{|}\beta} - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha n} \\ + 2\Gamma^{\alpha}_{\underset{\vee}{nj}}\psi_\alpha + 2\Gamma^{\alpha}_{\underset{\vee}{n\beta}}\xi^{\beta}_{j\alpha}\big\} - \frac{1}{N-1}\delta^{i}_{n}\,\big\{\underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} - \frac{1}{N+1}\big[2\underset{1}{R}_{\underset{\vee}{jm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jm}}\end{aligned}$$

$$+\,2(N-1)\xi^{\alpha}_{jm}\psi_{\alpha}+2\xi^{\beta}_{jm\underset{1}{|}\beta}+4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha}+2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha}-2\Gamma^{\alpha}_{\underset{\vee}{j\beta}}\xi^{\beta}_{m\alpha}]+(N-1)\xi^{\alpha}_{jm}\psi_{\alpha}$$

$$+\,\xi^{\beta}_{jm\underset{1}{|}\beta}-\xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m}+2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha}+2\Gamma^{\alpha}_{\underset{\vee}{m\beta}}\xi^{\beta}_{j\alpha}\}+(\delta^{i}_{n}\,\xi^{\alpha}_{jm}-\delta^{i}_{m}\,\xi^{\alpha}_{jn})\psi_{\alpha}+\xi^{i}_{jm\underset{1}{|}n}$$

$$-\,\xi^{i}_{jn\underset{1}{|}m}+2\psi_{j}\xi^{i}_{mn}+\xi^{\alpha}_{jm}\xi^{i}_{\alpha n}-\xi^{\alpha}_{jn}\xi^{i}_{\alpha m}+2\Gamma^{i}_{\underset{\vee}{mn}}\psi_{j}+2\Gamma^{\alpha}_{\underset{\vee}{mn}}\psi_{\alpha}\delta^{i}_{j}+2\Gamma^{\alpha}_{\underset{\vee}{mn}}\xi^{i}_{j\alpha}.$$

Taking into account (1.1) and (1.1′), we can write the last equation in the form

$$\overline{W}(\underset{1}{\overline{R}})^{i}{}_{jmn}=W(\underset{1}{R})^{i}{}_{jmn},\tag{1.8}$$

where

$$\begin{aligned}W(\underset{1}{R})^{i}{}_{jmn}&=\underset{1}{R}^{i}{}_{jmn}+\frac{2}{N+1}\delta^{i}_{j}\,\underset{1}{R}_{\underset{\vee}{mn}}+\frac{1}{N^{2}-1}\big[(N\underset{1}{R}_{jn}-\underset{1}{R}_{nj})\delta^{i}_{m}\\&\quad-(N\underset{1}{R}_{jm}-\underset{1}{R}_{mj})\delta^{i}_{n}\big]-\frac{2}{N+1}\delta^{i}_{j}\,\Gamma^{\beta}_{\underset{\vee}{mn}\underset{1}{|}\beta}-\frac{1}{N+1}\delta^{i}_{m}\,\Gamma^{\beta}_{\underset{\vee}{jn}\underset{1}{|}\beta}\\&\quad+\frac{1}{N+1}\delta^{i}_{n}\,\Gamma^{\beta}_{\underset{\vee}{jm}\underset{1}{|}\beta}-\Gamma^{i}_{\underset{\vee}{jm}\underset{1}{|}n}-\Gamma^{i}_{\underset{\vee}{jn}\underset{1}{|}m}.\end{aligned}\tag{1.9}$$

Therefore, we proved the next theorem

THEOREM 1.1. *The tensor* (1.9) *is an invariant of an* $\underset{1}{R}$*-projective mapping.*

THEOREM 1.2. *If* GR_N *is* $\underset{1}{R}$*-projectively flat, then*

$$W(\underset{1}{R})^{i}{}_{jmn}=0.\tag{1.10}$$

Proof. $G\overline{R}_N$ is a flat space. Then by (1.9) we get $\overline{W}(\underset{1}{\overline{R}})^{i}{}_{jmn}=0$, and using (1.8) we can see that (1.10) holds. □

2. $\underset{2}{R}$-projective mappings

DEFINITION 2.1. The geodesic mapping $f: GR_N \to G\overline{R}_N$ is $\underset{2}{R}$-projective if the following condition is satisfied

$$\begin{gathered}\delta^{i}_{j}\,(\underset{2}{\mathcal{D}}_{mn}-\underset{2}{\mathcal{D}}_{nm})+\delta^{i}_{m}\,\underset{2}{\mathcal{D}}_{jn}-\delta^{i}_{n}\,\underset{2}{\mathcal{D}}_{jm}+(\delta^{i}_{n}\,\xi^{\alpha}_{mj}-\delta^{i}_{m}\,\xi^{\alpha}_{nj})\psi_{\alpha}\\+2\psi_{j}\xi^{i}_{nm}+\xi^{\alpha}_{mj}\xi^{i}_{n\alpha}-\xi^{\alpha}_{nj}\xi^{i}_{m\alpha}+2\Gamma^{i}_{\underset{\vee}{nm}}\psi_{j}+2\Gamma^{\alpha}_{\underset{\vee}{nm}}\psi_{\alpha}\delta^{i}_{j}\\+2\Gamma^{\alpha}_{\underset{\vee}{nm}}\xi^{i}_{\alpha j}=0,\end{gathered}$$

where

$$\begin{aligned}\underset{2}{\mathcal{D}}_{jm}&=\frac{N-1}{N+1}\xi^{\alpha}_{mj}\psi_{\alpha}-\frac{1}{N^{2}-1}(4\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha}+2\Gamma^{\alpha}_{\underset{\vee}{\beta m}}\xi^{\beta}_{\alpha j}-2\Gamma^{\alpha}_{\underset{\vee}{\beta j}}\xi^{\beta}_{\alpha m})\\&\quad-\frac{1}{N-1}(\xi^{\alpha}_{\beta j}\xi^{\beta}_{m\alpha}-2\Gamma^{\alpha}_{\underset{\vee}{jm}}\psi_{\alpha}-2\Gamma^{\alpha}_{\underset{\vee}{\beta m}}\xi^{\beta}_{\alpha j}).\end{aligned}$$

Definition 2.2. The space GR_N is $\underset{2}{R}$-projectively flat if there exists an $\underset{2}{R}$-projective mapping of the space GR_N into a flat space.

For curvature tensors of the second kind (0.6) of the spaces GR_N and $G\overline{R}_N$ we get the relation

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{2}{|}m} + P^\alpha_{mj}P^i_{n\alpha} - P^\alpha_{nj}P^i_{m\alpha} + 2\Gamma^\alpha_{\underset{\vee}{nm}}P^i_{\alpha j},$$

i.e., using (0.2) and (1.3),

$$\begin{aligned}\underset{2}{\overline{R}}{}^i_{jmn} = {} & \underset{2}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{2}{\psi}_{mn} - \underset{2}{\psi}_{nm}) + \delta^i_m\,\underset{2}{\psi}_{jn} - \delta^i_n\,\underset{2}{\psi}_{jm}\\ & - \delta^i_m\,\xi^\alpha_{nj}\psi_\alpha + \delta^i_n\,\xi^\alpha_{mj}\psi_\alpha + \xi^i_{mj\underset{2}{|}n} - \xi^i_{nj\underset{2}{|}m} + 2\psi_j\xi^i_{nm} + \xi^\alpha_{mj}\xi^i_{n\alpha}\\ & - \xi^\alpha_{nj}\xi^i_{m\alpha} + 2\Gamma^i_{\underset{\vee}{nm}}\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{nm}}\psi_\alpha\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{nm}}\xi^i_{\alpha j}.\end{aligned}$$

Analogously to the previous case, we get an invariant of an $\underset{2}{R}$-projective mapping $f : GR_N \to G\overline{R}_N$

$$\begin{aligned}W(\underset{2}{R})^i{}_{jmn} = {} & \underset{2}{R}{}^i{}_{jmn} + \frac{2}{N+1}\delta^i_j\,\underset{2}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[(N\underset{2}{R}_{jn} - \underset{2}{R}_{nj})\delta^i_m\\ & - (N\underset{2}{R}_{jm} - \underset{2}{R}_{mj})\delta^i_n\big] - \frac{2}{N+1}\delta^i_j\,\Gamma^\beta_{\underset{\vee}{nm}\underset{2}{|}\beta} - \frac{1}{N+1}\delta^i_m\,\Gamma^\beta_{\underset{\vee}{nj}\underset{2}{|}\beta}\\ & + \frac{1}{N+1}\delta^i_n\,\Gamma^\beta_{\underset{\vee}{mj}\underset{2}{|}\beta} - \Gamma^i_{\underset{\vee}{mj}\underset{2}{|}n} - \Gamma^i_{\underset{\vee}{nj}\underset{2}{|}m}.\end{aligned} \tag{2.1}$$

Consequently, the next theorems are valid

Theorem 2.1. *The tensor* (2.1) *is an invariant of an $\underset{2}{R}$-projective mapping.*

Theorem 2.2. *If GR_N is $\underset{2}{R}$-projectively flat then we have*

$$W(\underset{2}{R})^i{}_{jmn} = 0.$$

3. $\underset{3}{R}$-projective mappings

Definition 3.1. The geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{3}{R}$-projective if the following condition holds

$$\begin{aligned}& \delta^i_j\,(\underset{3}{\mathcal{D}}_{mn} - \underset{3}{\mathcal{D}}_{nm}) + \delta^i_m\,\underset{3}{\mathcal{D}}_{jn} - \delta^i_n\,\underset{3}{\mathcal{D}}_{jm} + 2(\delta^i_j\,\Gamma^\alpha_{\underset{\vee}{mn}} + \delta^i_m\,\Gamma^\alpha_{\underset{\vee}{jn}})\psi_\alpha\\ & +(\delta^i_n\,\xi^\alpha_{jm} - \delta^i_m\,\xi^\alpha_{nj})\psi_\alpha + \xi^\alpha_{jm}\xi^i_{n\alpha} - \xi^\alpha_{jn}\xi^i_{\alpha m} + 2\psi_n(\Gamma^i_{\underset{\vee}{mj}} + \xi^i_{mj})\\ & +2\psi_m(\Gamma^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^\alpha_{nm}(\Gamma^i_{\underset{\vee}{\alpha j}} + \xi^i_{\alpha j}) = 0,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\underset{3}{\mathcal{D}}_{jm} &= \xi^\alpha_{jm}\psi_\alpha - \frac{1}{N^2-1}\big[4\psi_\beta(\Gamma^\beta_{\underset{\vee}{mj}} + \xi^\beta_{mj}) + 2\xi^\alpha_{\beta m}\Gamma^\beta_{\underset{\vee}{\alpha j}} - 2\xi^\alpha_{\beta j}\Gamma^\beta_{\underset{\vee}{\alpha m}}\big]\\ &+\frac{1}{N-1}\big[\xi^\alpha_{j\beta}\xi^\beta_{\alpha m} + 2\psi_\beta(\Gamma^\beta_{\underset{\vee}{mj}} + \xi^\beta_{mj}) + 2\xi^\alpha_{\beta m}(\Gamma^\beta_{\underset{\vee}{\alpha j}} + \xi^\beta_{\alpha j})\big].\end{aligned} \tag{3.1'}$$

DEFINITION 3.2. The space GR_N is $\underset{3}{R}$-projectively flat if there exists an $\underset{3}{R}$-projective mapping of the space GR_N into a flat space.

In the case of curvature tensors of the third kind (0.7) of the spaces GR_N and $G\overline{R}_N$ we get the relation

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^\alpha_{jm}P^i_{n\alpha} - P^\alpha_{nj}P^i_{\alpha m}\\ &+ 2P^\alpha_{nm}\Gamma^i_{\underset{\vee}{\alpha j}} + 2P^\alpha_{nm}P^i_{\underset{\vee}{\alpha j}}\end{aligned}$$

i.e., in virtue of (0.2) and (1.3)

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\underset{2}{\psi}_{jn} - \delta^i_n\underset{1}{\psi}_{jm}\\ &+ \psi_\alpha(\delta^i_n\xi^\alpha_{jm} - \delta^i_m\xi^\alpha_{nj}) + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} + \xi^\alpha_{jm}\xi^i_{n\alpha} - \xi^\alpha_{jn}\xi^i_{\alpha m}\\ &+ 2\psi_n(\Gamma^i_{\underset{\vee}{mj}} + \xi^i_{mj}) + 2\psi_m(\Gamma^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^\alpha_{nm}(\Gamma^i_{\underset{\vee}{\alpha j}} + \xi^i_{\alpha j}).\end{aligned} \tag{3.2}$$

Also, it holds

$$\underset{2}{\psi}_{mn} = \underset{1}{\psi}_{mn} + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha. \tag{3.3}$$

From (3.2) and (3.3) we get

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_j(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\underset{1}{\psi}_{jn} - \delta^i_n\underset{1}{\psi}_{jm}\\ &+ 2(\delta^i_j\Gamma^\alpha_{\underset{\vee}{mn}} + \delta^i_n\Gamma^\alpha_{\underset{\vee}{jn}})\psi_\alpha + (\delta^i_n\xi^\alpha_{jm} - \delta^i_m\xi^\alpha_{nj})\psi_\alpha\\ &+ \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} + \xi^\alpha_{jm}\xi^i_{n\alpha} - \xi^\alpha_{jn}\xi^i_{\alpha m}\\ &+ 2\psi_n(\Gamma^i_{\underset{\vee}{mj}} + \xi^i_{mj}) + 2\psi_m(\Gamma^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^\alpha_{nm}(\Gamma^i_{\underset{\vee}{\alpha j}} + \xi^i_{\alpha j}).\end{aligned} \tag{3.4}$$

Contracting (3.4) with respect to i and n we get

$$\begin{aligned}\underset{3}{\overline{R}}_{jm} &= \underset{3}{R}_{jm} - 2\underset{1}{\psi}_{\underset{\vee}{jm}} + (1-N)\underset{1}{\psi}_{jm} + (N+1)\xi^\alpha_{jm}\psi_\alpha + \xi^\beta_{jm\underset{2}{|}\beta}\\ &-\xi^\alpha_{j\beta}\xi^\beta_{\alpha m} + 2\psi_\beta(\Gamma^\beta_{\underset{\vee}{mj}} + \xi^\beta_{mj}) + 2\xi^\alpha_{\beta m}(\Gamma^\beta_{\underset{\vee}{\alpha j}} + \xi^\beta_{\alpha j}),\end{aligned} \tag{3.5}$$

hence

$$
\begin{aligned}
(1+N)\underset{1}{\psi}_{\underset{\vee}{jm}} &= \underset{3}{R}_{\underset{\vee}{jm}} - \underset{3}{\overline{R}}_{\underset{\vee}{jm}} + (N+1)\xi^{\alpha}_{jm}\psi_{\alpha} + \xi^{\beta}_{jm\underset{2}{|}\beta} \\
&\quad + 2\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + \xi^{\alpha}_{\beta m}\Gamma^{\beta}_{\underset{\vee}{\alpha j}} - \xi^{\alpha}_{\beta j}\Gamma^{\beta}_{\underset{\vee}{\alpha m}}.
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) in (3.5) we get

$$
\begin{aligned}
(N-1)\underset{1}{\psi}_{jm} &= \underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm} - \frac{1}{N+1}\big[2\underset{3}{R}_{\underset{\vee}{jm}} - 2\underset{3}{\overline{R}}_{\underset{\vee}{jm}} + 2(N+1)\xi^{\alpha}_{jm}\psi_{\alpha} + 2\xi^{\beta}_{jm\underset{2}{|}\beta} \\
&\quad + 4\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{\beta m}\Gamma^{\beta}_{\underset{\vee}{\alpha j}} - 2\xi^{\alpha}_{\beta j}\Gamma^{\beta}_{\underset{\vee}{\alpha m}}\big] + (N+1)\xi^{\alpha}_{jm}\psi_{\alpha} \\
&\quad + \xi^{\beta}_{jm\underset{2}{|}\beta} - \xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m} + 2\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}} + \xi^{\beta}_{mj}) + 2\xi^{\alpha}_{\beta m}(\Gamma^{\beta}_{\underset{\vee}{\alpha j}} + \xi^{\beta}_{\alpha j}).
\end{aligned}
\tag{3.7}
$$

Now, from (3.4,6,7) by conditions (3.1,1′) we get

$$
\overline{W}(\underset{3}{\overline{R}})^{i}{}_{jmn} = W(\underset{3}{R})^{i}{}_{jmn},
$$

where

$$
\begin{aligned}
W(\underset{3}{R})^{i}{}_{jmn} &= \underset{3}{R}^{i}{}_{jmn} + \frac{2}{N+1}\delta^{i}_{j}\,\underset{3}{R}_{\underset{\vee}{mn}} + \frac{1}{N^{2}-1}\big[(N\underset{3}{R}_{jn} - \underset{3}{R}_{nj})\delta^{i}_{m} \\
&\quad - (N\underset{3}{R}_{jm} - \underset{3}{R}_{mj})\delta^{i}_{n}\big] - \frac{2}{N+1}\delta^{i}_{j}\,\Gamma^{\beta}_{\underset{\vee}{mn}\underset{2}{|}\beta} \\
&\quad - \frac{1}{N+1}\delta^{i}_{m}\,\Gamma^{\beta}_{\underset{\vee}{jn}\underset{2}{|}\beta} + \frac{1}{N+1}\delta^{i}_{n}\,\Gamma^{\beta}_{\underset{\vee}{jm}\underset{2}{|}\beta} - \Gamma^{i}_{\underset{\vee}{jm}\underset{2}{|}n} + \Gamma^{i}_{\underset{\vee}{nj}\underset{2}{|}m}.
\end{aligned}
\tag{3.8}
$$

Consequently, the next theorems hold:

THEOREM 3.1. *The tensor* (3.8) *is an invariant of an* $\underset{3}{R}$*-projective mapping.*

THEOREM 3.3. *If* GR_N *is* $\underset{3}{R}$*-projectively flat then*

$$
W(\underset{3}{R})^{i}{}_{jmn} = 0.
$$

4. $\underset{4}{R}$-projective mappings

DEFINITION 4.1. A geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{4}{R}$-projective if the following condition is satisfied

$$
\begin{aligned}
&\delta^{i}_{j}\,(\underset{4}{\mathcal{D}}_{mn} - \underset{4}{\mathcal{D}}_{nm}) + \delta^{i}_{m}\,\underset{4}{\mathcal{D}}_{jn} - \delta^{i}_{n}\,\underset{4}{\mathcal{D}}_{jm} + 2(\delta^{i}_{j}\,\Gamma^{\alpha}_{\underset{\vee}{mn}} + \delta^{i}_{m}\,\Gamma^{\alpha}_{\underset{\vee}{jn}})\psi_{\alpha} \\
&+(\delta^{i}_{n}\,\xi^{\alpha}_{jm} - \delta^{i}_{m}\,\xi^{\alpha}_{nj})\psi_{\alpha} + \xi^{\alpha}_{jm}\xi^{i}_{n\alpha} - \xi^{\alpha}_{jn}\xi^{i}_{\alpha m} + 2\psi_{n}(\Gamma^{i}_{\underset{\vee}{mj}} + \xi^{i}_{mj}) \\
&+2\psi_{m}(\Gamma^{i}_{\underset{\vee}{nj}} + \xi^{i}_{nj}) + 2\xi^{\alpha}_{mn}(\Gamma^{i}_{\underset{\vee}{\alpha j}} + \xi^{i}_{\alpha j}) = 0,
\end{aligned}
$$

where

$$\underset{4}{\mathcal{D}}_{jm}=\xi^{\alpha}_{jm}\psi_{\alpha}-\frac{1}{N^2-1}\big[4\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}}+\xi^{\beta}_{mj})+2\xi^{\alpha}_{m\beta}\Gamma^{\beta}_{\underset{\vee}{\alpha j}}-2\xi^{\alpha}_{j\beta}\Gamma^{\beta}_{\underset{\vee}{\alpha m}}\big]$$
$$-\frac{1}{N-1}\big[\xi^{\alpha}_{j\beta}\xi^{\beta}_{\alpha m}-2\psi_{\beta}(\Gamma^{\beta}_{\underset{\vee}{mj}}+\xi^{\beta}_{mj})-2\xi^{\alpha}_{m\beta}(\Gamma^{\beta}_{\underset{\vee}{\alpha j}}+\xi^{\beta}_{\alpha j})\big].$$

DEFINITION 4.2. The space GR_N is $\underset{4}{R}$-projectively flat if there exists an $\underset{4}{R}$-projective mapping of the space GR_N into a flat space.

For curvature tensors of the fourth kind (0.8) we get

$$\underset{4}{\overline{R}}{}^i_{jmn}=\underset{4}{R}{}^i{}_{jmn}+P^i_{jm\underset{2}{|}n}-P^i_{nj\underset{1}{|}m}+P^{\alpha}_{jm}P^i_{n\alpha}-P^{\alpha}_{nj}P^i_{\alpha m}$$
$$+2P^{\alpha}_{mn}\Gamma^i_{\underset{\vee}{\alpha j}}+2P^{\alpha}_{mn}P^i_{\underset{\vee}{\alpha j}}$$

i.e.

$$\underset{4}{\overline{R}}{}^i_{jmn}=\underset{4}{R}{}^i{}_{jmn}+\delta^i_j\,(\underset{2}{\psi}_{mn}-\underset{1}{\psi}_{nm})+\delta^i_m\,\underset{2}{\psi}_{jn}-\delta^i_n\,\underset{1}{\psi}_{jm}$$
$$+\psi_{\alpha}(\delta^i_n\,\xi^{\alpha}_{jm}-\delta^i_m\,\xi^{\alpha}_{nj})+\xi^i_{jm\underset{2}{|}n}-\xi^i_{nj\underset{1}{|}m}+\xi^{\alpha}_{jm}\xi^i_{n\alpha}-\xi^{\alpha}_{jn}\xi^i_{\alpha m}$$
$$+2\psi_n(\Gamma^i_{\underset{\vee}{mj}}+\xi^i_{mj})+2\psi_m(\Gamma^i_{\underset{\vee}{nj}}+\xi^i_{nj})+2\xi^{\alpha}_{mn}(\Gamma^i_{\underset{\vee}{\alpha j}}+\xi^i_{\alpha j}).$$

Now analogously to the previous cases, we get an invariant of an $\underset{4}{R}$-projective mapping in the form

$$W(\underset{4}{R})^i{}_{jmn}=\underset{4}{R}{}^i{}_{jmn}+\frac{2}{N+1}\delta^i_j\,\underset{4}{R}_{\underset{\vee}{mn}}+\frac{1}{N^2-1}[(N\underset{4}{R}_{jn}-\underset{4}{R}_{nj})\delta^i_m$$
$$-(N\underset{4}{R}_{jm}-\underset{4}{R}_{mj})\delta^i_n]-\frac{2}{N+1}\delta^i_j\,\Gamma^{\alpha}_{\underset{\vee}{mn}\underset{2}{|}\alpha}-\frac{1}{N+1}\delta^i_m\,\Gamma^{\alpha}_{\underset{\vee}{jn}\underset{2}{|}\alpha}\tag{4.1}$$
$$+\frac{1}{N+1}\delta^i_n\,\Gamma^{\alpha}_{\underset{\vee}{jm}\underset{2}{|}\alpha}-\Gamma^i_{\underset{\vee}{jm}\underset{2}{|}n}+\Gamma^i_{\underset{\vee}{nj}\underset{1}{|}m},$$

i.e., the next theorems hold:

THEOREM 4.1. *The tensor* (4.1) *is an invariant of an* $\underset{4}{R}$*-projective mapping.*

THEOREM 4.2. *If* GR_N *is* $\underset{4}{R}$*-projectively flat then*

$$W(\underset{4}{R})^i{}_{jmn}=0.$$

5. $\underset{5}{R}$-projective mappings

DEFINITION 5.1. A geodesic mapping $f : GR_N \to G\overline{R}_N$ is $\underset{5}{R}$-projective if the following condition is satisfied

$$2\delta^i_m \xi^\alpha_{j\beta}\xi^\beta_{n\alpha} - 2\delta^i_n \xi^\alpha_{j\beta}\xi^\beta_{m\alpha} - (N-1)(\xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{m\alpha} + \xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{\alpha m}) = 0.$$

DEFINITION 5.2. The space GR_N is $\underset{5}{R}$-projectively flat if there exists an $\underset{5}{R}$-projective mapping of the space GR_N into a flat space.

For curvature tensors of the fifth kind (0.9) of the spaces GR_N and $G\overline{R}_N$ we find the relation

$$\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}(P^i_{\underset{3}{jm|n}} - P^i_{\underset{4}{jn|m}} + P^i_{\underset{4}{mj|n}} - P^i_{\underset{3}{nj|m}} + P^\alpha_{jm}P^i_{\alpha n} - P^\alpha_{jn}P^i_{m\alpha} + P^\alpha_{mj}P^i_{n\alpha} - P^\alpha_{nj}P^i_{\alpha m})$$

i.e.

$$\begin{aligned}\underset{5}{\overline{R}}{}^i_{jmn} = {} & \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}\delta^i_j(\underset{1}{\psi}_{mn} - \underset{2}{\psi}_{nm} + \underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \frac{1}{2}\delta^i_m(\underset{1}{\psi}_{jn} + \underset{2}{\psi}_{jn}) \\ & - \frac{1}{2}\delta^i_n(\underset{2}{\psi}_{jm} + \underset{1}{\psi}_{jm}) + \frac{1}{2}(\xi^i_{\underset{3}{jm|n}} - \xi^i_{\underset{4}{jn|m}} + \xi^i_{\underset{4}{mj|n}} - \xi^i_{\underset{3}{nj|m}} \\ & + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{m\alpha} + \xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{\alpha m}).\end{aligned} \tag{5.1}$$

Putting

$$\underset{12}{\psi}_{jn} = \frac{1}{2}(\underset{1}{\psi}_{jn} + \underset{2}{\psi}_{jn})$$

we get from (5.1)

$$\begin{aligned}\underset{5}{\overline{R}}{}^i_{jmn} = {} & \underset{5}{R}{}^i{}_{jmn} + \delta^i_j(\underset{12}{\psi}_{mn} - \underset{12}{\psi}_{nm}) + \delta^i_m\underset{12}{\psi}_{jn} - \delta^i_n\underset{12}{\psi}_{jm} \\ & + \frac{1}{2}(\xi^i_{\underset{3}{jm|n}} - \xi^i_{\underset{4}{jn|m}} + \xi^i_{\underset{4}{mj|n}} - \xi^i_{\underset{3}{nj|m}} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{m\alpha} + \\ & + \xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{\alpha m}).\end{aligned} \tag{5.2}$$

Eliminating $\underset{12}{\psi}_{mn}$ from (5.2), analogously to the previous cases, we get

$$\overline{W}(\underset{5}{R})^i{}_{jmn} = W(\underset{5}{R})^i{}_{jmn},$$

where

$$
\begin{aligned}
W(\underset{5}{R})^{i}{}_{jmn} &= \underset{5}{R}^{i}{}_{jmn} + \frac{2}{N+1}\delta^{i}_{j}\,\underset{5}{R}_{\underset{\vee}{mn}} + \frac{1}{N^{2}-1}\big[(N\underset{5}{R}_{jn} - \underset{5}{R}_{nj})\delta^{i}_{m} \\
&\quad - (N\underset{5}{R}_{jm} - \underset{5}{R}_{mj})\delta^{i}_{n}\big] - \frac{1}{N+1}\delta^{i}_{j}\,(\Gamma^{\alpha}_{\underset{\vee}{mn}\underset{3}{|}\alpha} + \Gamma^{\alpha}_{\underset{\vee}{nm}\underset{4}{|}\alpha}) \\
&\quad - \frac{1}{2(N+1)}\delta^{i}_{m}\,(\Gamma^{\alpha}_{\underset{\vee}{jn}\underset{3}{|}\alpha} + \Gamma^{\alpha}_{\underset{\vee}{nj}\underset{4}{|}\alpha}) + \frac{1}{2(N+1)}\delta^{i}_{n}\,(\Gamma^{\alpha}_{\underset{\vee}{jm}\underset{3}{|}\alpha} + \Gamma^{\alpha}_{\underset{\vee}{mj}\underset{4}{|}\alpha}) \\
&\quad - \frac{1}{2}(\Gamma^{i}_{\underset{\vee}{jm}\underset{3}{|}n} - \Gamma^{i}_{\underset{\vee}{jn}\underset{4}{|}m} + \Gamma^{i}_{\underset{\vee}{mj}\underset{4}{|}n} - \Gamma^{i}_{\underset{\vee}{nj}\underset{3}{|}m}).
\end{aligned}
\tag{5.3}
$$

Hence:

THEOREM 5.1. *The tensor* (5.3) *is an invariant of a* $\underset{5}{R}$*-projective mapping.*

THEOREM 5.2. *If* GR_N *is* $\underset{5}{R}$*-projectively flat, then*

$$W(\underset{5}{R})^{i}{}_{jmn} = 0.$$

Remark. If $GR_N(G\overline{R}_N)$ reduces to $R_N(\overline{R}_N)$, then the tensors $W(\underset{\theta}{R})^{i}{}_{jmn}$ $(\theta = 1,\dots,5)$ reduce to Weyl's tensor (0.10).

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