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THE GEOMETRY OF SELF-ADJUNCTION

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ABSTRACT. This paper is a companion to another paper where it is shown that the multiplicative monoids of Temperley-Lieb algebras are isomorphic to monoids of endomorphisms in categories where an endofunctor is adjoint to itself. Such a self-adjunction underlies the orthogonal group case of Brauer's representation of the Brauer centralizer algebras. The present paper provides detailed proofs of results on the presentation of various monoids of diagrams by generators and relations, on which the other paper depends.

1. Introduction

As an offshoot of Jones' polynomial approach to knot and link invariants, Temperley-Lieb algebras have played in the 1990s a prominent role in knot theory and low-dimensional topology (see [**14**], [**17**] and [**20**]). In [**6**] it is shown that the multiplicative monoids of Temperley-Lieb algebras are closely related to the general notion of adjunction, one of the fundamental notions of category theory, and of mathematics in general (see [**18**]). More precisely, it is shown that these monoids are isomorphic to monoids of endomorphisms in categories involved in one kind of self-adjoint situation, where an endofunctor is adjoint to itself.

As shown in [**6**], such a self-adjunction may be found in categories whose arrows are matrices, where the functor adjoint to itself is based on the Kronecker product of matrices. This self-adjunction underlies the orthogonal group case of Brauer's representation of the Brauer algebras, which can be restricted to the Temperley-Lieb subalgebras of the Brauer algebras (see [**3**], [**21**, Section 3], and [**10**, Section 3]). This leads in [**6**] to a representation of braid groups in Temperley-Lieb algebras similar to the standard one that stems from Jones, but not the same. The present paper is devoted to proving precisely results on which [**6**] relies.

A self-adjunction, which we will also call *$\mathcal{L}$-adjunction*, is an adjunction in which an endofunctor is adjoint to itself, which means that it is both left and right adjoint to itself (for the general notion of adjunction see [**18**, Chapter IV]). In an

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$\mathcal{L}$-adjunction we have a category $\langle \mathcal{A}, \circ, \mathbf{1} \rangle$ and a functor F from $\mathcal{A}$ to $\mathcal{A}$ such that φ is the counit of the adjunction with components $\varphi_a : FFa \rightarrow a$ and γ is the unit of the adjunction with components $\gamma_a : a \rightarrow FFa$. Let κ_a be an abbreviation for $\varphi_a \circ \gamma_a : a \rightarrow a$. A *$\mathcal{K}$-adjunction* is an $\mathcal{L}$-adjunction that satisfies the additional equation $F\kappa_a = \kappa_{Fa}$. A *$\mathcal{J}$-adjunction* is an $\mathcal{L}$-adjunction that satisfies the additional equation $\kappa_a = \mathbf{1}_a$. Every $\mathcal{J}$-adjunction is a $\mathcal{K}$-adjunction, but not vice versa.

In the paper we proceed as follows. We first present by generators and relations monoids for which it is shown in [**6**] that they are engendered by categories involved in self-adjoint situations. These categories engender monoids, whose names will be indexed by ω, when we consider a total binary operation on all arrows defined with the help of composition. Our categories engender monoids of a different kind, with names indexed by n, when we consider just composition, restricting ourselves to endomorphisms in the category. We deal first with the monoids related to $\mathcal{L}$-adjunction, and next with those related $\mathcal{K}$-adjunction, which is the adjunction encountered in connection with Temperley-Lieb algebras.

Next we prove in detail that our monoids are isomorphic to monoids made of equivalence classes of diagrams which in knot theory would be called planar tangles, without crossings, and which we call *friezes*. In these representations, there are two different notions of equivalence of friezes: the $\mathcal{L}$ notion is based purely on planar ambient isotopies, whereas the $\mathcal{K}$ notion allows circles to cross lines, which is forbidden in the $\mathcal{L}$ notion. So the mathematical content of the most general notion of self-adjunction is caught by the notion of planar ambient isotopy. The diagrammatic representation of the $\mathcal{K}_n$ monoids is not a new result, but $\mathcal{L}_n$ and its diagrammatic representation don't seem to have been treated so far.

In the diagrammatic representation of the third notion of self-adjunction, $\mathcal{J}$-adjunction, we don't take account of circles at all. This notion is more strict than $\mathcal{K}$-adjunction. With the help of friezes, we show for this third notion that it is maximal in the sense that we could not extend it with any further assumption without trivializing it. This maximality is an essential ingredient in the proof given in [**6**] that we have in matrices an isomorphic representation of the monoids of Temperley-Lieb algebras. However, this maximality need not serve only for that particular goal, which can be reached by other means, as mentioned in [**6**]. Maximality can serve to establish the isomorphism of other nontrivial representations of the monoids of Temperley-Lieb algebras.

We consider in Section 11 monoids interpreted in friezes with points labelled by all integers, and not only positive integers, and also monoids interpreted in cylindrical friezes. This matter, though related to other matters in the paper, is independent of its main thrust, and is presented with less details.

In the main body of the paper, however, we strive, as we said above, to give detailed proofs of isomorphisms of monoids, and so our style of exposition will be occasionally rather formal. It will be such at the beginning, and it might help the reader while going through Sections 2–4 to take a look at Sections 5–7, to get some motivation.

2. The monoids $\mathcal{L}_\omega$ and $\mathcal{K}_\omega$

The monoid $\mathcal{L}_\omega$ has for every $k \in \mathbf{N}^+ = \mathbf{N} - \{0\}$ a generator $\lfloor k \rfloor$, called a *cup*, and a generator $\lceil k \rceil$, called a *cap*. The *terms* of $\mathcal{L}_\omega$ are defined inductively by stipulating that the generators and $\mathbf{1}$ are terms, and that if t and u are terms, then (tu) is a term. As usual, we will omit the outermost parentheses of terms. In the presence of associativity we will omit all parentheses, since they can be restored as we please.

The monoid $\mathcal{L}_\omega$ is freely generated from the generators above so that the following equations hold between terms of $\mathcal{L}_\omega$ for $l \leqslant k$:

$$
\begin{array}{rl}
(1) & \mathbf{1}t = t, \quad t\mathbf{1} = t, \\
(2) & t(uv) = (tu)v, \\
(cup) & \lfloor k \rfloor \lfloor l \rfloor = \lfloor l \rfloor \lfloor k+2 \rfloor, \\
(cap) & \lceil l \rceil \lceil k \rceil = \lceil k+2 \rceil \lceil l \rceil, \\
(cup\text{-}cap\ 1) & \lfloor l \rfloor \lceil k+2 \rceil = \lceil k \rceil \lfloor l \rfloor, \\
(cup\text{-}cap\ 2) & \lfloor k+2 \rfloor \lceil l \rceil = \lceil l \rceil \lfloor k \rfloor, \\
(cup\text{-}cap\ 3) & \lfloor k \rfloor \lceil k \pm 1 \rceil = \mathbf{1}.
\end{array}
$$

The monoid $\mathcal{K}_\omega$ is defined as the monoid $\mathcal{L}_\omega$ save that we have the additional equation

$$(cup\text{-}cap\ 4) \quad \lfloor k \rfloor \lceil k \rceil = \lfloor k+1 \rfloor \lceil k+1 \rceil,$$

which, of course, implies $\lfloor k \rfloor \lceil k \rceil = \lfloor l \rfloor \lceil l \rceil$. To understand the equations of $\mathcal{L}_\omega$ and $\mathcal{K}_\omega$ it helps to have in mind their diagrammatic interpretation of Sections 5-7 (see in particular the diagrams corresponding to $\lfloor k \rfloor$ and $\lceil k \rceil$ at the beginning of Section 6).

Let $[k]$ be an abbreviation for $\lfloor k \rfloor \lceil k \rceil$, and let us call such terms *circles*. Then (*cup-cap* 4) says that we have only one circle, which we designate by c. We have the following equations in $\mathcal{L}_\omega$ for $l \leqslant k$:

$$
\begin{array}{l}
\lfloor k \rfloor [l] = [l] \lfloor k \rfloor, \\
\lfloor l \rfloor [k+2] = [k] \lfloor l \rfloor.
\end{array}
$$

For the first equation we have

$$
\begin{array}{rll}
\lfloor k \rfloor \lfloor l \rfloor \lceil l \rceil & = \lfloor l \rfloor \lfloor k+2 \rfloor \lceil l \rceil, & \text{by } (cup) \\
& = \lfloor l \rfloor \lceil l \rceil \lfloor k \rfloor, & \text{by } (cup\text{-}cap\ 2),
\end{array}
$$

and for the second

$$
\begin{array}{rll}
\lfloor l \rfloor \lfloor k+2 \rfloor \lceil k+2 \rceil & = \lfloor k \rfloor \lfloor l \rfloor \lceil k+2 \rceil, & \text{by } (cup) \\
& = \lfloor k \rfloor \lceil k \rceil \lfloor l \rfloor, & \text{by } (cup\text{-}cap\ 1).
\end{array}
$$

We derive analogously the following dual equations of $\mathcal{L}_\omega$ for $l \leqslant k$:

$$
\begin{array}{r}
[l] \lceil k \rceil = \lceil k \rceil [l], \\
[k+2] \lceil l \rceil = \lceil l \rceil [k].
\end{array}
$$

So in $\mathcal{K}_\omega$ we have the equations

$$\lfloor k \rfloor c = c \lfloor k \rfloor,$$
$$\lceil k \rceil c = c \lceil k \rceil,$$

which yield the equation $tc = ct$ for any term t.

3. Finite multisets, circular forms and ordinals

Let an *o-monoid* be a monoid with an arbitrary unary operation o, and consider the free *commutative* o-monoid $\mathcal{F}$ generated by the empty set of generators. In $\mathcal{F}$ the operation o is a one-one function.

The elements of $\mathcal{F}$ may be designated by parenthetical words, i.e., well-formed words in the alphabet $\{(,)\}$, which will be precisely defined in a moment, where the empty word stands for the unit of the monoid, concatenation is monoid multiplication, and $o(a)$ is written simply (a). Parenthetical words are defined inductively as follows:

(0) the empty word is a parenthetical word;
(1) if a is a parenthetical word, then (a) is a parenthetical word;
(2) if a and b are parenthetical words, then ab is a parenthetical word.

We consider next several isomorphic representations of $\mathcal{F}$, via finite multisets, circular forms in the plane and ordinals.

If we take that $()$ is the empty multiset, then the elements of $\mathcal{F}$ of the form (a) may be identified with finite multisets, i.e., the hierarchy of finite multisets obtained by starting from the empty multiset $\emptyset$ as the only *urelement*. To obtain a more conventional notation for these multisets, just replace $()$ everywhere by $\emptyset$, replace the remaining left parentheses (by left braces $\{$ and the remaining right parentheses) by right braces $\}$, and put in commas where concatenation occurs.

The elements of $\mathcal{F}$ may also be identified with nonintersecting finite collections of circles in the plane factored through homeomorphisms of the plane mapping one collection into another (cf. [**12**, Section II]). For this interpretation, just replace (a) by $\textcircled{a}$. Since we will be interested in particular in this plane interpretation, we call the elements of $\mathcal{F}$ *circular forms*. The *empty circular form* is the unit of $\mathcal{F}$. When we need to refer to it we use e. We refer to other circular forms with parenthetical words.

The free commutative o-monoid $\mathcal{F}$ has another isomorphic representation in the ordinals contained in the ordinal $\varepsilon_0 = \min\{\xi \mid \omega^\xi = \xi\}$, i.e., in the ordinals lesser than ε_0. By Cantor's Normal Form Theorem (see, for example, [**15**, VII.7, Theorem 2, p. 248], or [**16**, IV.2, Theorem 2.14, p. 127],), for every ordinal $\alpha > 0$ in ε_0 there is a unique finite ordinal $n \geqslant 1$ and a unique sequence of ordinals $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ contained in α, i.e., lesser than α, such that $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. The *natural sum* $\alpha \sharp \beta$ of

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}, \quad \alpha_1 \geqslant \cdots \geqslant \alpha_n,$$
$$\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}, \quad \beta_1 \geqslant \cdots \geqslant \beta_m,$$

is defined as $\omega^{\gamma_1} + \cdots + \omega^{\gamma_{n+m}}$ where $\gamma_1, \dots, \gamma_{n+m}$ is obtained by permuting the sequence $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$ so that $\gamma_1 \geqslant \cdots \geqslant \gamma_{n+m}$ (this operation was introduced by Hessenberg; see [**15**, p. 252] or [**16**, p. 130]). We also have $\alpha \sharp 0 = 0 \sharp \alpha = \alpha$. The natural sum $\sharp$ and the ordinal sum $+$ don't coincide in general: $\sharp$ is commutative, but $+$ is not (for example, $\omega = 1 + \omega \neq \omega + 1$, but $1 \sharp \omega = \omega \sharp 1 = \omega + 1 = \omega^{\omega^0} + \omega^0$). However, if $\alpha_1 \geqslant \cdots \geqslant \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} = \omega^{\alpha_1} \sharp \dots \sharp \omega^{\alpha_n}$.

Let $\omega^{\cdots}$ be the unary operation that assigns to every $\alpha \in \varepsilon_0$ the ordinal $\omega^\alpha \in \varepsilon_0$. Then it can be shown that the commutative o-monoid $\langle \varepsilon_0, \sharp, 0, \omega^{\cdots} \rangle$ is isomorphic to $\mathcal{F}$ by the isomorphism $\iota : \varepsilon_0 \to \mathcal{F}$ such that $\iota(0)$ is the empty word and

$$\iota(\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}) = \iota(\omega^{\alpha_1} \sharp \dots \sharp \omega^{\alpha_n}) = (\iota(\alpha_1)) \dots (\iota(\alpha_n)).$$

That the function $\iota^{-1} : \mathcal{F} \to \varepsilon_0$ defined inductively by

$$\begin{aligned}
&\iota^{-1}(e) = 0,\\
&\iota^{-1}(ab) = \iota^{-1}(a) \sharp \iota^{-1}(b),\\
&\iota^{-1}((a)) = \omega^{\iota^{-1}(a)}
\end{aligned}$$

is the inverse of ι is established by easy inductions relying on the fact that

$$\iota(\alpha \sharp \beta) = \iota(\alpha)\iota(\beta).$$

It is well known in proof theory that the ordinal ε_0 and natural sums play an important role in Gentzen's proof of the consistency of formal Peano arithmetic PA (see [**7**, Paper 8, §4]). Induction up to any ordinal lesser than ε_0 is derivable in PA; induction up to ε_0, which is not derivable in PA, is not only sufficient, but also necessary, for proving the consistency of PA (see [**7**, Paper 9]).

From the isomorphism of $\mathcal{F}$ with $\langle \varepsilon_0, \sharp, 0, \omega^{\cdots} \rangle$ we obtain immediately a normal form for the elements of $\mathcal{F}$. Circular forms inherit a well-ordering from the ordinals, and we have the following inductive definition. The empty word is in normal form, and if $a_1, \dots, a_n$, $n \geqslant 1$, are parenthetical words in normal form such that $a_1 \geqslant \cdots \geqslant a_n$, then $(a_1) \dots (a_n)$ is in normal form. We call this normal form of parenthetical words the *Cantor normal form*.

fLet a commutative o-monoid be called *solid* iff it satisfies

$$(\textit{solid}) \quad o(a) = o(1)a,$$

where 1 is the unit of the monoid. The free solid commutative o-monoid $\mathcal{F}'$ generated by the empty set of generators is isomorphic to the structure $\langle \mathbf{N}, +, 0, \dots +1 \rangle$ by the isomorphism that assigns to n the sequence of n pairs ( ). So (*solid*) makes $\langle \varepsilon_0, \sharp, \emptyset, \omega^{\cdots} \rangle$ collapse into $\langle \omega, \sharp, \emptyset, \dots +1 \rangle$. For $k \in \mathbf{N}$, let $k\mathbf{N} = \{kn \mid n \in \mathbf{N}\}$ and $k^{\mathbf{N}} = \{k^n \mid n \in \mathbf{N}\}$. If $k \geqslant 1$, then $\langle \mathbf{N}, +, 0, \dots +1 \rangle$ is isomorphic to $\langle k\mathbf{N}, +, 0, \dots +k \rangle$, which for $k \geqslant 2$ is isomorphic to $\langle k^{\mathbf{N}}, \cdot, 1, \dots \cdot k \rangle$.

The equation (*solid*) is what a unary function $o : \mathcal{M} \to \mathcal{M}$, for a monoid $\mathcal{M}$, has to satisfy to be in the image of the Cayley monomorphic representation of $\mathcal{M}$ in $\mathcal{M}^{\mathcal{M}}$, which assigns to every $a \in \mathcal{M}$ the function $f_a \in \mathcal{M}^{\mathcal{M}}$ such that $f_a(b) = ab$. In the presence of (*solid*), the function $f_{o(a)}$ will be equal to $o \circ f_a$. The equation (*solid*) can be replaced by $o(ab) = o(a)b$, and in commutative o-monoids it could, of course, as well be written $o(a) = ao(1)$.

4. Normal forms in $\mathcal{L}_\omega$ and $\mathcal{K}_\omega$

For $k \in \mathbf{N}^+$ let c_k^0 be the term $\mathbf{1}$ of $\mathcal{L}_\omega$. For $\alpha > 0$ an ordinal in ε_0 whose Cantor normal form is $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ let the term c_k^α of $\mathcal{L}_\omega$ be defined inductively as

$$\lfloor k \rfloor c_{k+1}^{\alpha_1} \lceil k \rceil \ldots \lfloor k \rfloor c_{k+1}^{\alpha_n} \lceil k \rceil .$$

Next, let a_k^0 be the term $\lfloor k \rfloor$, and let a_k^α be the term $\lfloor k \rfloor c_{k+1}^\alpha$. Similarly, let b_k^0 be the term $\lceil k \rceil$, and let b_k^α be the term $c_{k+1}^\alpha \lceil k \rceil$.

Consider terms of $\mathcal{L}_\omega$ of the form

$$b_{j_1}^{\beta_1} \ldots b_{j_m}^{\beta_m} c_{k_1}^{\gamma_1} \ldots c_{k_l}^{\gamma_l} a_{i_1}^{\alpha_1} \ldots a_{i_n}^{\alpha_n}$$

where $n, m, l \geqslant 0$, $n + m + l \geqslant 1$, $j_1 > \cdots > j_m$, $k_1 < \cdots < k_l$, $i_1 < \cdots < i_n$, and for every $p \in \{1, \ldots, l\}$ we have $\gamma_p \neq 0$. If n is 0, the sequence $a_{i_1}^{\alpha_1} \ldots a_{i_n}^{\alpha_n}$ is empty, and analogously if m or l is 0. Terms of $\mathcal{L}_\omega$ of this form and the term $\mathbf{1}$ will be said to be in *normal form.*

In the definition of normal form we could have required that $k_1 > \cdots > k_n$, or, as a matter of fact, we could have imposed any other order on these particular indices, with the same effect. We have chosen the order above for the sake of definiteness. (Putting aside complications involving the terms c_k^α and the ordinals, the idea of our normal form may be found in [**1**, p. 106].

To reduce terms of $\mathcal{L}_\omega$ to normal form we use an alternative formulation of $\mathcal{L}_\omega$, which is obtained as follows. Now the generators are the *a terms* a_k^α, the *b terms* b_k^α and the *c terms* c_k^α for $k \in \mathbf{N}^+$ and $\alpha \in \varepsilon_0$. These terms are now primitive, and not defined. We generate terms with these generators, $\mathbf{1}$ and multiplication, and we stipulate the following equations for $l \leqslant k$:

$$
\begin{array}{ll}
(1) & \mathbf{1}t = t, \quad t\mathbf{1} = t, \\
(2) & t(uv) = (tu)v, \\
(aa) & a_k^\alpha a_l^\beta = a_l^\beta a_{k+2}^\alpha, \\
(bb) & b_l^\alpha b_k^\beta = b_{k+2}^\beta b_l^\alpha, \\
(c1) & c_k^0 = \mathbf{1}, \\
(c2) & c_k^\alpha c_k^\beta = c_k^{\alpha \sharp \beta}, \\
(cc) & c_k^\alpha c_l^\beta = c_l^\beta c_k^\alpha, \quad \text{for } l < k,
\end{array}
$$

ab equations:

$$
\begin{array}{ll}
(ab\,1) & a_l^\alpha b_{k+2}^\beta = b_k^\beta a_l^\alpha, \\
(ab\,2) & a_{k+2}^\alpha b_l^\beta = b_l^\beta a_k^\alpha, \\
(ab\,3.1) & a_k^\alpha b_{k+1}^\beta = c_k^\beta c_{k+1}^\alpha, \\
(ab\,3.2) & a_{k+1}^\alpha b_k^\beta = c_k^\alpha c_{k+1}^\beta, \\
(ab\,3.3) & a_k^\alpha b_k^\beta = c_k^{\omega^{\alpha \sharp \beta}},
\end{array}
$$

ac equations:

$$\begin{aligned}
&(ac\,1) \quad a_k^\alpha c_l^\gamma = c_l^\gamma a_k^\alpha,\\
&(ac\,2) \quad a_l^\alpha c_{k+2}^\gamma = c_k^\gamma a_l^\alpha,\\
&(ac\,3) \quad a_k^\alpha c_{k+1}^\gamma = a_k^{\alpha\sharp\gamma},
\end{aligned}$$

bc equations:

$$\begin{aligned}
&(bc\,1) \quad c_l^\gamma b_k^\beta = b_k^\beta c_l^\gamma,\\
&(bc\,2) \quad c_{k+2}^\gamma b_l^\beta = b_l^\beta c_k^\gamma,\\
&(bc\,3) \quad c_{k+1}^\gamma b_k^\beta = b_k^{\gamma\sharp\beta}.
\end{aligned}$$

It is tiresome, but pretty straightforward, to derive all these equations in the original formulation of $\mathcal{L}_\omega$ for defined c_k^α, a_k^α and b_k^α, while with $\lfloor k \rfloor$ defined as a_k^0 and $\lceil k \rceil$ defined as b_k^0, we easily derive in the new formulation the equations of the original formulation of $\mathcal{L}_\omega$. We can, moreover, derive in the new formulation the inductive definitions of c_k^α, a_k^α and b_k^α. We can then prove the following lemma for $\mathcal{L}_\omega$.

NORMAL FORM LEMMA. *Every term is equal to a term in normal form.*

PROOF. We will give a reduction procedure that transforms every term t of $\mathcal{L}_\omega$ into a term t' in normal form such that $t = t'$ in $\mathcal{L}_\omega$. (In logical jargon, we establish that this procedure is strongly normalizing—namely, that any sequence of reduction steps terminates in a term in normal form.)

Take a term t in the new alternative formulation of $\mathcal{L}_\omega$, and let subterms of this term of the forms on the left-hand sides of the equations of the alternative formulation except (2) be called *redexes*. A reduction consists in replacing a redex of t by the term on the right-hand side of the corresponding equation. Note that the terms on the left-hand sides of these equations cover all possible cases for terms of the forms $a_k^\alpha b_l^\beta$, $a_k^\alpha c_k^\gamma$ and $c_l^\gamma b_k^\beta$, and all these cases exclude each other.

A subterm of t which is an a term will be called an *a subterm* of t, and analogously with b and c. For a particular subterm a_k^α of t let $\sigma(a_k^\alpha)$ be the number of b subterms of t on the right-hand side of a_k^α in t. Let n_1 be the sum of all the numbers $\sigma(a_k^\alpha)$ for every a subterm a_k^α of t. If there are no a subterms of t, then n_1 is zero.

For a particular subterm a_k^α of t let $\sigma_a(a_k^\alpha)$ be the number of a subterms a_l^β of t on the right-hand side of a_k^α in t such that $l \leqslant k$. Let σ_a be the sum of all the numbers $\sigma_a(a_k^\alpha)$ for every a subterm a_k^α of t. For a particular subterm b_k^β of t let $\sigma_b(b_k^\beta)$ be the number of b subterms b_l^α of t on the left-hand side of b_k^β in t such that $l \leqslant k$. Let σ_b be the sum of all the numbers $\sigma_b(b_k^\beta)$ for every b subterm b_k^β of t. For a particular subterm c_l^γ of t let $\tau(c_l^\gamma)$ be the number of a subterms on the left-hand side c_l^γ in t plus the number of b subterms on the right-hand side of c_l^γ in t. Let τ be the sum of all the numbers $\tau(c_l^\gamma)$ for every c subterm c_l^γ of t. Let ν_c be the number of c subterms of t, and let $\nu_{\mathbf{1}}$ be the number of subterms $\mathbf{1}$ of t. Then let n_2 be $\sigma_a + \sigma_b + \tau + 2\nu_c + \nu_{\mathbf{1}}$.

Let σ_c be defined as σ_a save that a is everywhere replaced by c, and let n_3 be σ_c. With reductions based on $(c2)$ and (cc), the number n_3 decreases, while n_1 and n_2 don't increase. With reductions based on (1), $(c1)$, (aa), (bb) and the ac and bc equations, n_2 decreases, n_1 doesn't change, and n_3 may even increase in case we apply $(ac\,2)$ or $(bc\,2)$. With reductions based on the ab equations, n_1 decreases, while n_2 and n_3 may increase. Then we take as the *complexity measure* of t the ordered triple (n_1, n_2, n_3). These triples are well-ordered lexicographically, and with every reduction the complexity measure decreases. So by induction on the complexity measure, we obtain that every term in the new formulation is equal to a term without redexes, and it is easy to check that such a term stands for a term in normal form of the original formulation of $\mathcal{L}_\omega$. □

Let c stand for $[k]$, where $k \in \mathbf{N}^+$. Let c^0 be the empty sequence, and let c^{n+1} be $c^n c$. Consider terms of $\mathcal{L}_\omega$ of the form

$$\lceil j_1 \rceil \ldots \lceil j_m \rceil c^l \lfloor i_1 \rfloor \ldots \lfloor i_n \rfloor$$

where $n, m, l \geqslant 0$, $n + m + l \geqslant 1$, $j_1 > \cdots > j_m$ and $i_1 < \cdots < i_n$. Terms of this form and the term $\mathbf{1}$ will be said to be in $\mathcal{K}$-*normal form*. (We could as well put c^l on the extreme left, or on the extreme right, or, actually, anywhere, but for the sake of definiteness, and, by analogy with the normal form of $\mathcal{L}_\omega$, we put c^l in the middle.)

We can easily derive from the Normal Form Lemma for $\mathcal{L}_\omega$ the Normal Form Lemma for $\mathcal{K}_\omega$, which says that every term is equal in $\mathcal{K}_\omega$ to a term in $\mathcal{K}$-normal form. For that it is enough to use the uniqueness of c and $tc = ct$. However, the Normal Form Lemma for $\mathcal{K}_\omega$ has a much simpler direct proof, which does not require the introduction of an alternative formulation of $\mathcal{K}_\omega$. This proof is obtained by simplifying the proof of the Normal Form Lemma for $\mathcal{L}_\omega$. The complications of the previous proof were all due to distinguishing $[k]$ from $[k+1]$ and to the absence of $tc = ct$. In $\mathcal{K}_\omega$ we have in fact assumed (*solid*), and the ordinals in ε_0 have collapsed into natural numbers.

5. Friezes

By a one-manifold with boundary we understand a Hausdorff topological space whose points have open neighbourhoods homeomorphic to the real intervals $(-1, 1)$ or $[0, 1)$, the boundary points having the latter kind of neighbourhoods. For $a > 0$ a real number, let R_a be $[0, \infty) \times [0, a]$. Let $\{(x, a) \mid x \geqslant 0\}$ be the *top* of R_a and $\{(x, 0) \mid x \geqslant 0\}$ the *bottom* of R_a.

An ω-*diagram* D in R_a is a compact one-manifold with boundary with denumerably many connected components embedded in R_a

such that the intersection of D with the top of R_a is $t(D) = \{(i, a) \mid i \in \mathbf{N}^+\}$ the intersection of D with the bottom of R_a is $b(D) = \{(i, 0) \mid i \in \mathbf{N}^+\}$ and $t(D) \cup b(D)$ is the set of boundary points of D.

It follows from this definition that every ω-diagram has denumerably many components homeomorphic to $[0, 1]$, which are called *threads*, and at most a denumerable number of components homeomorphic to S^1, which are called *circular* *components*. The threads and the circular components make all the connected components of an ω-diagram. All these components are mutually disjoint. Every thread has two end points that belong to the boundary $t(D) \cup b(D)$. When one of these end points is in $t(D)$ and the other in $b(D)$, the thread is *transversal*. A transversal thread is *vertical* when the first coordinates of its end points are equal. A thread that is not transversal is a *cup* when both of its end points are in $t(D)$, and it is a *cap* when they are both in $b(D)$.

A *frieze* is an ω-diagram with a finite number of cups, caps and circular components. Although many, but not all, of the definitions that follow can be formulated for all ω-diagrams, and not only for friezes, we will be interested here only in friezes, and we will formulate our definitions only with respect to them. The notion of frieze corresponds to a special kind of *tangle* of knot theory, in which there are no crossings (see [**4**, p. 99], [**19**, Chapter 9], [**11**, Chapter 12]).

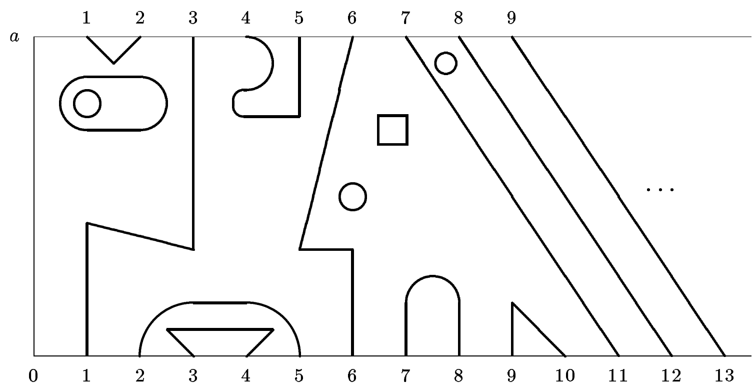
For D_1 a frieze in R_a and D_2 frieze in R_b, we say that D_1 is $\mathcal{L}$*-equivalent* to D_2, and write $D_1 \cong_{\mathcal{L}} D_2$, iff there is a homeomorphism $h : R_a \to R_b$ such that $h[D_1] = D_2$ and for every $i \in \mathbf{N}^+$ we have $h(i, 0) = (i, 0)$ and $h(i, a) = (i, b)$. It is straightforward to check that $\mathcal{L}$-equivalence between friezes is indeed an equivalence relation.

This definition is equivalent to a definition of $\mathcal{L}$-equivalence in terms of ambient isotopies. The situation is analogous to what one finds in knot theory, where one can define equivalence of knots either in terms of ambient isotopies or in a simpler manner, analogous to what we have in the preceding paragraph. The equivalence of these two definitions is proved with the help of Alexander's trick (see [**4**, Chapter 1B]), an adaptation of which also works in the case of $\mathcal{L}$-equivalence.

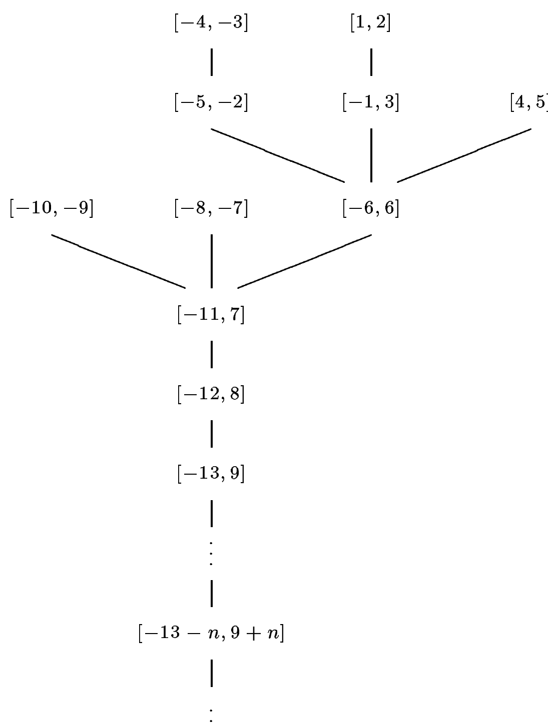
For D_1 a frieze in R_a and D_2 a frieze in R_b, we say that D_1 is $\mathcal{K}$*-equivalent* to D_2, and write $D_1 \cong_{\mathcal{K}} D_2$, iff there is a homeomorphism $h : D_1 \to D_2$ such that for every $i \in \mathbf{N}^+$ we have $h(i, 0) = (i, 0)$ and $h(i, a) = (i, b)$. It is clear that this defines an equivalence relation on friezes, which is wider than $\mathcal{L}$-equivalence: namely, if $D_1 \cong_{\mathcal{L}} D_2$, then $D_1 \cong_{\mathcal{K}} D_2$, but the converse need not hold. If D_1 and D_2 are without circular components, then $D_1 \cong_{\mathcal{L}} D_2$ iff $D_1 \cong_{\mathcal{K}} D_2$. The relation of $\mathcal{K}$-equivalence takes account only of the number of circular components, whereas $\mathcal{L}$-equivalence takes also account of whether circular components are one in another, and, in general, in which region of the diagram they are located.

If i stands for (i, a) and $-i$ stands for $(i, 0)$, we may identify the end points of each thread in a frieze in R_a by a pair of integers in $\mathbf{Z} - \{0\}$. For M an ordered set and for $a, b \in M$ such that $a < b$, let a *segment* $[a, b]$ in M be $\{z \in M \mid a \leqslant z \leqslant b\}$. The numbers a and b are the end points of $[a, b]$. We say that $[a, b]$ *encloses* $[c, d]$ iff $a < c$ and $d < b$. A set of segments is *nonoverlapping* iff every two distinct segments in it are either disjoint or one of these segments encloses the other.

We may then establish a one-to-one correspondence between the set Θ of threads of a frieze and a set S_Θ of nonoverlapping segments in $\mathbf{Z} - \{0\}$. Every element of $\mathbf{Z} - \{0\}$ is an end point of a segment in S_Θ. Since enclosure is irreflexive and transitive, S_Θ is partially ordered by enclosure. This is a tree-like ordering without root, with a finite number of branching nodes. For example, in the frieze



the set Θ of threads corresponds to the following tree in S_Θ:



The branching points of this tree are $[-6, 6]$ and $[-11, 7]$. This tree-like ordering of S_Θ induces an isomorphic ordering of Θ.

If from a frieze D in R_a we omit all the threads, we obtain a disjoint family of connected sets in R_a, which are called the *regions* of D. Every circular component of D is included in a unique region of D. The closure of a region of D has a border that includes a nonempty set of threads. In the tree-like ordering, this set must have a lowest thread, and all the other threads in the set, if any, are its immediate successors. Every thread is the lowest thread for some region. In our example, in the region in which one finds as circular components a circle and a square, the lowest thread is the one corresponding to $[-11, 7]$, and its immediate successors correspond to $[-10, -9]$, $[-8, -7]$ and $[-6, 6]$. Assigning to every region of a frieze the corresponding lowest thread in the border establishes a one-to-one correspondence between regions and threads.

The collection (possibly empty) of circular components in a single region of a frieze corresponds to a circular form (see Section 3), which can then be coded by an ordinal in ε_0. In every frieze we can assign to every thread the ordinal that corresponds to the collection of circular components in the region for which this is the lowest thread. This describes all the circular components of a frieze. (In an ω-diagram that is not a frieze it is possible that one collection of circular components, which is in a region without lowest thread, is not covered.)

Then it is easy to establish the following.

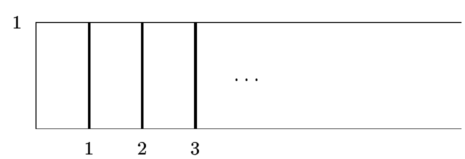
REMARK 1$\mathcal{L}$. The friezes D_1 and D_2 are $\mathcal{L}$-equivalent iff
(i) the end points of the threads in D_1 are identified with the same pairs of integers as the end points of the threads in D_2,
(ii) the same ordinals in ε_0 are assigned to the threads of D_1 and D_2 that are identified with the same pairs of integers.

This means that the $\mathcal{L}$-equivalence class of a frieze may be identified with a function $f : S_\Theta \to \varepsilon_0$, where the domain S_Θ of f is a set of nonoverlapping segments in $\mathbf{Z} - \{0\}$.

REMARK 1$\mathcal{K}$. The friezes D_1 and D_2 are $\mathcal{K}$-equivalent iff
(i) the end points of the threads in D_1 are identified with the same pairs of integers as the end points of the threads in D_2,
(ii) D_1 and D_2 have the same number of circular components.

This means that the $\mathcal{K}$-equivalence class of a frieze may be identified with a pair (S_Θ, l) where S_Θ is a set of nonoverlapping segments in $\mathbf{Z} - \{0\}$, and l is a natural number, which is the number of circular components.

The set of $\mathcal{L}$-equivalence classes of friezes is endowed with the structure of a monoid in the following manner. Let the *unit frieze* I be $\{(i, y) \mid i \in \mathbf{N}^+ \textit{ and } y \in [0, 1]\}$ in R_1. So I has no circular components and all of its threads are vertical threads. We draw I as follows:



For two friezes D_1 in R_a and D_2 in R_b let the *composition* of D_1 and D_2 be defined as follows:

$$D_2 \circ D_1 = \{(x, y + b) \mid (x, y) \in D_1\} \cup D_2.$$

It is easy to see that $D_2 \circ D_1$ is a frieze in R_{a+b}.

For $1 \leqslant i \leqslant 4$, let D_i be a frieze in R_{a_i} and suppose $D_1 \cong_{\mathcal{L}} D_3$ with the homeomorphism $h_1 : R_{a_1} \to R_{a_3}$ and $D_2 \cong_{\mathcal{L}} D_4$ with the homeomorphism $h_2 : R_{a_2} \to R_{a_4}$. Then $D_2 \circ D_1 \cong_{\mathcal{L}} D_4 \circ D_3$ with the homeomorphism $h : R_{a_1+a_2} \to R_{a_2+a_4}$ defined as follows. For p^1 the first and p^2 the second projection, let

$$h(x,y) = \begin{cases} (p^1(h_1(x,y-a_2)), p^2(h_1(x,y-a_2)) + a_4), & \text{if } y > a_2 \\ h_2(x,y), & \text{if } y \leqslant a_2 \end{cases}$$

So the composition $\circ$ defines an operation on $\mathcal{L}$-equivalence classes of friezes.

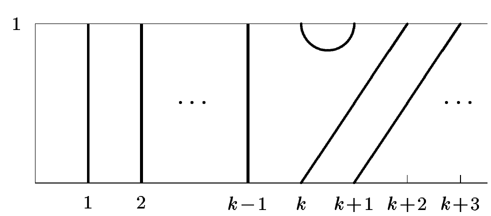
We can then establish that

$$\begin{array}{ll} (1) & I \circ D \cong_{\mathcal{L}} D, \quad D \circ I \cong_{\mathcal{L}} D, \\ (2) & D_3 \circ (D_2 \circ D_1) \cong_{\mathcal{L}} (D_3 \circ D_2) \circ D_1. \end{array}$$

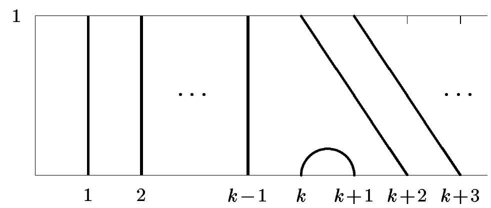
The equivalences of (1) follow from the fact that the threads of $I \circ D$, $D \circ I$ and D are identified with the same pairs of integers, because all the threads of I are vertical transversal threads, and from the fact that I has no circular component. Then we apply Remark 1$\mathcal{L}$. For the equivalence (2), it is clear that $D_3 \circ (D_2 \circ D_1)$ is actually identical to $(D_3 \circ D_2) \circ D_1$. So the set of $\mathcal{L}$-equivalence classes of friezes has the structure of a monoid, and the monoid structure of the set of $\mathcal{K}$-equivalence classes of friezes is defined quite analogously. We will show for these monoids that they are isomorphic to $\mathcal{L}_\omega$ and $\mathcal{K}_\omega$ respectively.

6. Generating friezes

For $k \in \mathbf{N}^+$ let the *cup frieze* V_k be the frieze in R_1 without circular components, with a single semicircular cup with the end points $(k,1)$ and $(k+1,1)$; all the other threads are straight line segments connecting $(i,0)$ and $(i,1)$ for $i < k$ and $(i,0)$ and $(i+2,1)$ for $i \geqslant k$. This frieze looks as follows:



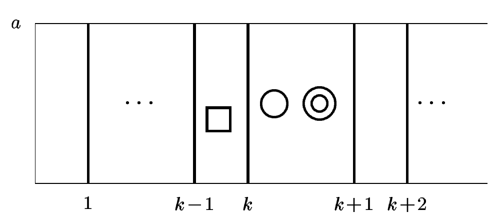
For $k \in \mathbf{N}^+$ let the *cap frieze* Λ_k be the frieze in R_1 that is defined analogously to V_k and looks as follows:



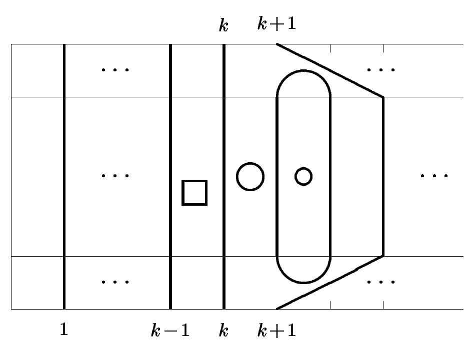
Let a frieze without cups and caps be called a *circular frieze*. Note that according to this definition the unit frieze I is a circular frieze. For circular friezes we can prove the following lemma.

GENERATING CIRCLES LEMMA. *Every circular frieze is $\mathcal{L}$-equivalent to a frieze generated from the unit frieze I and the cup and cap friezes with the operation of composition* $\circ$.

PROOF. If there are no circular components in our circular frieze, then, by Remark 1$\mathcal{L}$, this frieze is $\mathcal{L}$-equivalent to the unit frieze I. Suppose then that there are circular components in our circular frieze, and take a circular component in this frieze that is not within another circular component. For example, let that be the right outer circle in the following frieze



We replace this by

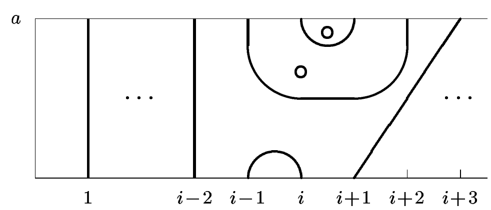


which is $\mathcal{L}$-equivalent to the original frieze. In the frieze in the middle there are less circular components than in the original frieze, and the lemma follows by induction. (By judicious choices, we can ensure that the composition of cup and cap friezes we obtain at the end corresponds to a term of $\mathcal{L}_\omega$ in normal form.) □

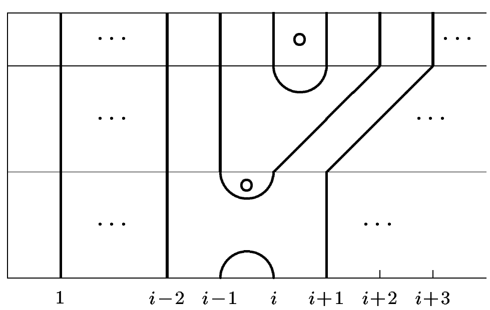
Note that the unit frieze I is $\mathcal{L}$-equivalent to $\mathrm{V}_k \circ \Lambda_{k+1}$ (or to $\mathrm{V}_{k+1} \circ \Lambda_k$), for any $k \in \mathbf{N}^+$, so that, strictly speaking, the mentioning of I is superfluous in the preceding and in the following lemma.

GENERATING LEMMA. *Every frieze is $\mathcal{L}$-equivalent to a frieze generated from the unit frieze I and the cup and cap friezes with the operation of composition $\circ$.*

PROOF. We proceed by induction on the sum of the numbers of cups and caps in the given frieze. The basis of the induction is covered by the Generating Circles Lemma. If our frieze has cups, it must have a cup whose end points are (i, a) and $(i + 1, a)$. If, for example, we have



we replace this by



which is $\mathcal{L}$-equivalent to the original frieze. In the lowest frieze there are less cups, and the same number of caps. We apply to this frieze the induction hypothesis, and we apply the Generating Circles Lemma to the highest frieze. We proceed analogously with caps. (Again, by judicious choices, we can ensure that the composition of cup and cap friezes we obtain at the end corresponds to a term of $\mathcal{L}_\omega$ in normal form.) □

Since $\mathcal{L}$-equivalence implies $\mathcal{K}$-equivalence, we have the Generating Circles Lemma and the Generating Lemma also for $\mathcal{L}$-equivalence replaced by $\mathcal{K}$-equivalence.

It follows from the Generating Lemma that there are only denumerably many $\mathcal{L}$-equivalence classes of friezes, and the same holds a fortiori for $\mathcal{K}$-equivalence classes. If we had allowed infinitely many cups or caps in friezes, then we would have a continuum of different $\mathcal{L}$ or $\mathcal{K}$-equivalence classes of friezes (which is clear from the fact that we can code 0-1 sequences with such friezes). The corresponding monoids could not then be finitely generated, as $\mathcal{L}_\omega$ and $\mathcal{K}_\omega$ are. With infinitely many circular components we would have a continuum of different $\mathcal{L}$-equivalence classes, but not so for $\mathcal{K}$-equivalence classes (see Section 9).

7. $\mathcal{L}_\omega$ and $\mathcal{K}_\omega$ are monoids of friezes

Let $\mathcal{F}$ be the set of friezes. We define as follows a map δ from the terms of $\mathcal{L}_\omega$ into $\mathcal{F}$:

$$\begin{aligned}&\delta(\lfloor k \rfloor) = \mathrm{V}_k,\\ &\delta(\lceil k \rceil) = \Lambda_k,\\ &\delta(\mathbf{1}) = I,\\ &\delta(tu) = \delta(t) \circ \delta(u).\end{aligned}$$

We can then prove the following.

Soundness Lemma. *If $t = u$ in $\mathcal{L}_\omega$, then $\delta(t) \cong_{\mathcal{L}} \delta(u)$.*

Proof. We already verified in Section 5 that we have replacement of equivalents, and that the equations (1) and (2) of the axiomatization of $\mathcal{L}_\omega$ are satisfied for I and $\circ$. It just remains to verify the remaining equations, which is quite straightforward. □

We have an analogous Soundness Lemma for $\mathcal{K}_\omega$ and $\cong_{\mathcal{K}}$, involving the additional checking of (*cup-cap* 4).

Let $[\mathcal{F}]_{\mathcal{L}}$ be the set of $\mathcal{L}$-equivalence classes $[D]_{\mathcal{L}} = \{D' : D \cong_{\mathcal{L}} D'\}$ for all friezes D (and analogously with $\mathcal{L}$ replaced by $\mathcal{K}$). This set is a monoid whose unit is $[I]_{\mathcal{L}}$ and whose multiplication is defined by taking that $[D_1]_{\mathcal{L}}[D_2]_{\mathcal{L}}$ is $[D_1 \circ D_2]_{\mathcal{L}}$. The Soundness Lemma guarantees that there is a homomorphism, defined via δ, from $\mathcal{L}_\omega$ to the monoid $[\mathcal{F}]_{\mathcal{L}}$, and the Generating Lemma guarantees that this homomorphism is onto. We have the same with $\mathcal{L}$ replaced by $\mathcal{K}$. It remains to establish that these homomorphisms from $\mathcal{L}_\omega$ onto $[\mathcal{F}]_{\mathcal{L}}$ and from $\mathcal{K}_\omega$ onto $[\mathcal{F}]_{\mathcal{K}}$ are also one-one.

We can prove the following lemmata.

Auxiliary Lemma. *If t and u are terms of $\mathcal{L}_\omega$ in normal form and $\delta(t) \cong_{\mathcal{L}} \delta(u)$, then t and u are the same term.*

Proof. Let t and u be the following two terms:

$$\begin{aligned}&b_{j_1}^{\beta_1} \dots b_{j_m}^{\beta_m} c_{k_1}^{\gamma_1} \dots c_{k_l}^{\gamma_l} a_{i_1}^{\alpha_1} \dots a_{i_n}^{\alpha_n},\\ &b_{j'_1}^{\beta'_1} \dots b_{j'_{m'}}^{\beta'_{m'}} c_{k'_1}^{\gamma'_1} \dots c_{k'_{l'}}^{\gamma'_{l'}} a_{i'_1}^{\alpha'_1} \dots a_{i'_{n'}}^{\alpha'_{n'}}.\end{aligned}$$

If $a_{i_1}^{\alpha_1} \dots a_{i_n}^{\alpha_n}$ is different from $a_{i'_1}^{\alpha'_1} \dots a_{i'_{n'}}^{\alpha'_{n'}}$, then either $n < n'$ or $n' < n$ or ($n = n'$ and for some $p \in \{1, \dots, n\}$ either $i_p \neq i'_p$ or $\alpha_p \neq \alpha'_p$). Since $i_1 < \dots < i_n$, each index i_p corresponds to the left end point of a cup. So if $n < n'$ or $n' < n$ or $i_p \neq i'_p$, then $\delta(t)$ and $\delta(u)$ don't have the same left end points of cups, and hence, they cannot be $\mathcal{L}$-equivalent by Remark $1\mathcal{L}(i)$. If, on the other hand, $\delta(t)$ and $\delta(u)$ have cups identified with the same pairs of integers, then for some $p \in \{1, \dots, n\}$

we have $\alpha_p \neq \alpha'_p$, and, since different ordinals are assigned to threads of $\delta(t)$ and $\delta(u)$ identified with the same pairs of integers, by Remark $1\mathcal{L}(ii)$, the friezes $\delta(t)$ and $\delta(u)$ cannot be $\mathcal{L}$-equivalent. We reason analogously with a replaced by b and c. $\square$

COMPLETENESS LEMMA. *If $\delta(t) \cong_{\mathcal{L}} \delta(u)$, then $t = u$ in $\mathcal{L}_\omega$.*

PROOF. By the Normal Form Lemma of Section 4, for every term t and every term u of $\mathcal{L}_\omega$ there are terms t' and u' in normal form such that $t = t'$ and $u = u'$ in $\mathcal{L}_\omega$. By the Soundness Lemma, we obtain $\delta(t) \cong_{\mathcal{L}} \delta(t')$ and $\delta(u) \cong_{\mathcal{L}} \delta(u')$, and if $\delta(t) \cong_{\mathcal{L}} \delta(u)$, it follows that $\delta(t') \cong_{\mathcal{L}} \delta(u')$. Then, by the Auxiliary Lemma, the terms t' and u' are the same term, and hence $t = u$ in $\mathcal{L}_\omega$. $\square$

The Auxiliary Lemma and the Completeness Lemma are easily obtained when $\mathcal{L}$ is replaced by $\mathcal{K}$. So we may conclude that our homomorphisms from $\mathcal{L}_\omega$ onto $[\mathcal{F}]_{\mathcal{L}}$ and from $\mathcal{K}_\omega$ onto $[\mathcal{F}]_{\mathcal{K}}$ are one-one, and hence $\mathcal{L}_\omega$ is isomorphic to $[\mathcal{F}]_{\mathcal{L}}$ and $\mathcal{K}_\omega$ is isomorphic to $[\mathcal{F}]_{\mathcal{K}}$.

We may also conclude that for every term t of $\mathcal{L}_\omega$ there is a *unique* term t' in normal form such that $t = t'$ in $\mathcal{L}_\omega$. If $t = t'$ and $t = t''$ in $\mathcal{L}_\omega$, then $t' = t''$ in $\mathcal{L}_\omega$, and hence, by the Soundness Lemma, $\delta(t') \cong_{\mathcal{L}} \delta(t'')$. If t' and t'' are in normal form, by the Auxiliary Lemma we obtain that t' and t'' are the same term. We conclude analogously that the $\mathcal{K}$-normal form is unique in the same sense with respect to $\mathcal{K}_\omega$.

8. The monoids $\mathcal{L}_n$ and $\mathcal{K}_n$

The monoid $\mathcal{L}_n$ has for every $i \in \{1, \ldots, n-1\}$ a generator h_i, called a *diapsis* (plural *diapsides*), and also for every ordinal $\alpha \in \varepsilon_0$ and every $k \in \{1, \ldots, n+1\}$ a generator c_k^α, called a *c-term*. The number n here could in principle be any natural number, but the interesting monoids $\mathcal{L}_n$ have $n \geqslant 2$. When n is 0 or 1, we have no diapsides. The diapsis h_i corresponds to the term $\lceil i \rceil \lfloor i \rfloor$ of $\mathcal{L}_\omega$. The terms of $\mathcal{L}_n$ are obtained from these generators and $\mathbf{1}$ by closing under multiplication.

We assume the following equations for $\mathcal{L}_n$:

$$
\begin{array}{ll}
(1) & \mathbf{1}t = t, \quad t\mathbf{1} = t,\\
(2) & t(uv) = (tu)v,\\[4pt]
(c1) & \mathbf{1} = c_k^0,\\
(c2) & c_k^\alpha c_k^\beta = c_k^{\alpha\sharp\beta},\\
(cc) & c_k^\alpha c_l^\beta = c_l^\beta c_k^\alpha, \quad \text{for } k \neq l,\\[4pt]
(h1) & h_i h_{j+2} = h_{j+2} h_i, \quad \text{for } i \leqslant j,\\
(h2) & h_i h_{i\pm 1} h_i = h_i,\\[4pt]
(hc1') & h_i c_k^\alpha = c_k^\alpha h_i, \quad \text{for } k \neq i+1,\\
(hc2') & h_i c_{i+1}^\alpha h_i = c_i^{\omega^\alpha} h_i,\\
(hc3) & c_i^\alpha h_i = c_{i+2}^\alpha h_i.
\end{array}
$$

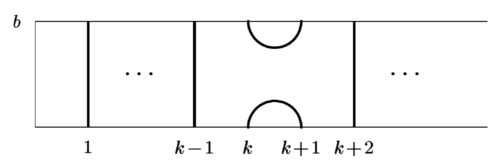
With the help of $(c2)$ we can derive (cc) for $k = l$ too.

With h_i defined as $\lceil i \rceil \lfloor i \rfloor$ and c_k^α defined as in Section 4, we can check easily that all the equations above hold in $\mathcal{L}_\omega$. We can make this checking also with friezes. So $\mathcal{L}_n$ is a submonoid of $\mathcal{L}_\omega$.

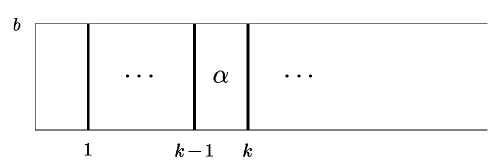
An *n-frieze* is a frieze such that for every $k \geqslant n+1$ we have a vertical thread identified with $[-k, k]$ and for every $k \geqslant n+2$ the ordinal of circular components assigned to the thread $[-k, k]$ is 0. Each n-frieze without circular components may be conceived up to $\mathcal{L}$-equivalence or $\mathcal{K}$-equivalence, which here coincide, as an element of the free (noncommutative) o-monoid generated by the empty set of generators (cf. Section 3). This is because the threads of each n-frieze without circular components are identified with a rooted subtree of S_Θ (see Section 5), whose root is $[-(n+1), n+1]$, and this rooted tree may be coded by a parenthetical word.

If $\mathcal{F}_n$ is the set of n-friezes, let $[\mathcal{F}_n]_\mathcal{L}$ be the set of $\mathcal{L}$-equivalence classes of these friezes, and analogously for $[\mathcal{F}_n]_\mathcal{K}$. The set $[\mathcal{F}_n]_\mathcal{L}$ has the structure of a monoid defined as for $[\mathcal{F}]_\mathcal{L}$.

Then it can be shown that the monoid $[\mathcal{F}_n]_\mathcal{L}$ is isomorphic to $\mathcal{L}_n$ with the help of a map $\delta : \mathcal{L}_n \to \mathcal{F}_n$ that maps a diapsis h_k into the *diapsidal n-frieze* H_k, which is the n-frieze in R_b, for some $b > 1$, without circular components, with a single semicircular cup with the end points (k, b) and $(k+1, b)$, and a single semicircular cap with the end points $(k, 0)$ and $(k+1, 0)$; all the other threads are vertical threads orthogonal to the x axis. A diapsidal n-frieze H_k looks as follows:



The c-term c_k^α is mapped by δ into the frieze



where α stands for an arbitrary circular form corresponding to α. We also have $\delta(\mathbf{1}) = I$ and $\delta(tu) = \delta(t) \circ \delta(u)$, as before. It is clear that the unit frieze I is an n-frieze for every $n \in \mathbf{N}$, and that the composition of two n-friezes is an n-frieze.

We will not go into details of the proof that we have an isomorphism here, because we don't have much use for $\mathcal{L}_n$ in this work. A great part of this proof is analogous to what we had for $\mathcal{L}_\omega$, or to what we have for $\mathcal{K}_n$ in [**2**] (see below). The essential part of the proof is the definition of unique normal form for elements of $\mathcal{L}_n$. Here is how such a normal form would look like.

For $1 \leqslant j \leqslant i \leqslant n-1$ and $\alpha, \beta \in \varepsilon_0$, let the *block* $h^{\alpha,\beta}_{[i,j]}$ be defined as

$$c^{\alpha}_{i+1} h_i h_{i-1} \dots h_{j+1} h_j c^{\beta}_{j+1}.$$

A term of $\mathcal{L}_n$ in *normal form* will be $\mathbf{1}$, or it looks as follows:

$$c^{\gamma_1}_{k_1} \dots c^{\gamma_l}_{k_l} h^{\alpha_1,\beta_1}_{[b_1,c_1]} \dots h^{\alpha_n,\beta_n}_{[b_n,c_n]},$$

where $n, l \geqslant 0$, $k_1 < \dots < k_l$, $b_1 < \dots < b_n$, and $c_1 < \dots < c_n$. All the c-terms on the left-hand side are such that they could be permuted with all the blocks, and pass to the right-hand side; i.e., they would not be "captured" by a block. We must also make a choice for the indices k_p of these c-terms to ensure uniqueness, and γ_p should not be 0.

One way to define the monoid $\mathcal{K}_n$ is to have the same generators as for $\mathcal{L}_n$, and the following equations, which we add to those of $\mathcal{L}_n$:

$$c^{\omega^\alpha}_k = c^{\alpha+1}_k,$$
$$c^{\alpha}_k = c^{\alpha}_{k+1}.$$

The first equation has the effect of collapsing the ordinals in ε_0 into natural numbers (as the equation (*solid*) of Section 3), while the second equation has the effect of making superfluous the lower index of c-terms.

An alternative, and simpler, axiomatization of $\mathcal{K}_n$ is obtained as follows. The monoid $\mathcal{K}_n$ has for every $i \in \{1, \dots, n-1\}$ a generator h_i, called again a *diapsis*, and also the generator c, called the *circle*. The terms of $\mathcal{K}_n$ are obtained from these generators and $\mathbf{1}$ by closing under multiplication. For $\mathcal{K}_n$ we assume the equations (1), (2), (h1), (h2) and the following two equations:

$$(hc1) \quad h_i c = c h_i,$$
$$(hc2) \quad h_i h_i = c h_i.$$

The equations (h1), (h2) and (hc2), which may be derived from Jones' paper [**9**, p. 13], and which appear in the form above in many works of Kauffman (see [**14**], [**13**, Section 6], and references therein), are usually tied to the presentation of Temperley-Lieb algebras. They may, however, be found in Brauer algebras too (see [**21**, pp. 180–181]).

With h_i defined as $\lceil i \rceil \lfloor i \rfloor$ and c defined as $\lfloor i \rfloor \lceil i \rceil$ we can check easily that $\mathcal{K}_n$ is a submonoid of $\mathcal{K}_\omega$.

For $1 \leqslant j \leqslant i \leqslant n-1$, let the *block* $h_{[i,j]}$ be defined as $h_i h_{i-1} \dots h_{j+1} h_j$. The block $h_{[i,i]}$, which is defined as h_i, will be called *singular*. (One could conceive $\lfloor i \rfloor$ as the infinite block $\dots h_{i+2} h_{i+1} h_i$, whereas $\lceil i \rceil$ would be $h_i h_{i+1} h_{i+2} \dots$) Let c^1 be c, and let c^{l+1} be $c^l c$.

A term is in *Jones normal form* iff it is either of the form $c^l h_{[b_1,a_1]} \dots h_{[b_k,a_k]}$ for $l, k \geqslant 0$, $l + k \geqslant 1$, $a_1 < \dots < a_k$ and $b_1 < \dots < b_k$, or it is the term $\mathbf{1}$ (see [**9**, §4.1.4, p. 14]). As before, if $l = 0$, then c^l is the empty sequence, and if $k = 0$, then $h_{[b_1,a_1]} \dots h_{[b_k,a_k]}$ is empty.

Then we can prove the following lemma as in [**2**]. (A lemma with the same content is established in a different manner in [**9**, pp. 13–14] and [**8**, pp. 87–89]).

NORMAL FORM LEMMA. *Every term of $\mathcal{K}_n$ is equal in $\mathcal{K}_n$ to a term in Jones normal form.*

We ascertained above that the unit frieze I of Section 5 is an n-frieze for every $n \in \mathbf{N}$. We have also defined there what is the diapsidal n-frieze H_i for $i \in \{1, \cdots, n-1\}$. The *circular* n-frieze C is the n-frieze that differs from the unit frieze I by having a single circular component, which, for the sake of definiteness, we choose to be a circle of radius $1/4$, with centre $(1/2, 1/2)$. We have also mentioned that the composition of two n-friezes is an n-frieze. Then we can prove the following lemma as in [**2**]. (Different, and more sketchy, proofs of this lemma may be found in [**20**, Chapter VIII, Section 26] and [**13**, Section 6]; in [**1**, Proposition 4.1.3] one may find a proof of something more general, and somewhat more complicated.)

GENERATING LEMMA. *Every n-frieze is $\mathcal{K}$-equivalent to an n-frieze generated from I, C and the diapsidal n-friezes H_i, for $i \in \{1, \cdots, n-1\}$, with the operation of composition $\circ$.*

Let $\mathcal{D}_n$ be the set of n-friezes. We define as follows a map δ from the terms of $\mathcal{K}_n$ into $\mathcal{D}_n$:

$$\begin{aligned} \delta(h_i) &= H_i, \\ \delta(c) &= C, \\ \delta(\mathbf{1}) &= I, \\ \delta(tu) &= \delta(t) \circ \delta(u). \end{aligned}$$

We can then prove easily the following.

SOUNDNESS LEMMA. *If $t = u$ in $\mathcal{K}_n$, then $\delta(t) \cong_{\mathcal{K}} \delta(u)$.*

We want to show that the homomorphism from $\mathcal{K}_n$ to $[\mathcal{F}_n]_{\mathcal{K}}$ defined via δ, whose existence is guaranteed by the Soundness Lemma, is an isomorphism. The Generating Lemma guarantees that this homomorphism is onto, and it remains to establish that it is one-one. The proof of that is based on the following lemmata, proved in [**2**].

KEY LEMMA. *If t is the term $h_{[b_1,a_1]} \ldots h_{[b_k,a_k]}$ with $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$, then $T_{\delta(t)}$ is $a_1, \ldots, a_k$ and $B_{\delta(t)}$ is $b_1, \ldots, b_k$.*

AUXILIARY LEMMA *If t and u are terms of $\mathcal{K}_n$ in Jones normal form and $\delta(t) \cong_{\mathcal{K}} \delta(u)$, then t and u are the same term.*

COMPLETENESS LEMMA. *If $\delta(t) \cong_{\mathcal{K}} \delta(u)$, then $t = u$ in $\mathcal{K}_n$.*

This last lemma is proved analogously to the Completeness Lemma of the preceding section by using the Normal Form Lemma, the Soundness Lemma and the Auxiliary Lemma of the present section. With this lemma we have established that $\mathcal{K}_n$ is isomorphic to $[\mathcal{F}_n]_{\mathcal{K}}$.

By reasoning as at the end of the preceding section, we can conclude that for every term t of $\mathcal{K}_n$ there is a unique term t' in Jones normal form such that $t = t'$ in $\mathcal{K}_n$.

9. The monoid $\mathcal{J}_\omega$

Let $\mathcal{J}_\omega$ be the monoid defined as $\mathcal{L}_\omega$ save that for every $k \in \mathbf{N}^+$ we require also

$$\lfloor k \rfloor \lceil k \rceil = \mathbf{1},$$

i.e., $[k] = \mathbf{1}$. It is clear that all the equations of $\mathcal{K}_\omega$ are satisfied in $\mathcal{J}_\omega$, but not conversely. In $\mathcal{J}_\omega$ circles are irrelevant.

The monoid $\mathcal{J}_n$ is obtained by extending $\mathcal{K}_n$ with $c_k^1 = \mathbf{1}$, or $c = \mathbf{1}$. Alternatively, we may omit c-terms, or the generator c, and assume only the equations (1), (2), ($h1$) and ($h2$) of the preceding section, together with the idempotency of h_i, namely, $h_i h_i = h_i$. (These axioms may be found in [**9**, p. 13]). The monoids $\mathcal{J}_n$ are submonoids of $\mathcal{J}_\omega$.

Let a *$\mathcal{J}$-frieze* be an ω-diagram with a finite number of cups and caps and denumerably many circular components. (Instead of "denumerably many circular components" we could put "κ circular components for a fixed infinite cardinal κ"; for the sake of definiteness, we chose κ to be the least infinite cardinal ω.) We define $\mathcal{K}$-equivalence of $\mathcal{J}$-friezes as for friezes, and we transpose other definitions of Section 5 to $\mathcal{J}$-friezes in the same manner. It is clear that the following holds, which means that circular components are irrelevant.

Remark 1$\mathcal{J}$. The $\mathcal{J}$-friezes D_1 and D_2 are $\mathcal{K}$-equivalent iff the end points of the threads in D_1 are identified with the same pairs of integers as the end points of the threads in D_2.

The unit $\mathcal{J}$-frieze is defined as the unit frieze I save that we assume that it has denumerably many circular components, which are located in some arbitrary regions. With composition of $\mathcal{J}$-friezes defined as before, the set of $\mathcal{K}$-equivalence classes of $\mathcal{J}$-friezes makes a monoid.

By adapting the argument in Sections 6 and 7, we can show that this monoid is isomorphic to $\mathcal{J}_\omega$. We don't need any more the Generating Circles Lemma, since circular $\mathcal{J}$-friezes are $\mathcal{K}$-equivalent to the unit $\mathcal{J}$-frieze. The cup and cap friezes V_k and Λ_k have now denumerably many circular components, which are located in some arbitrary regions.

A $\mathcal{J}$-n-frieze is defined as an n-frieze save that it has denumerably many circular components. Then we can show by adapting the argument in the preceding section that $\mathcal{J}_n$ is isomorphic to the monoid $[\mathcal{F}_{\mathcal{J}\text{-}n}]_\mathcal{K}$ of $\mathcal{K}$-equivalence classes of $\mathcal{J}$-n-friezes.

An alternative proof that the map from $\mathcal{J}_n$ to $[\mathcal{F}_{\mathcal{J}\text{-}n}]_\mathcal{K}$, defined analogously to what we had in the preceding section, is one-one may be obtained as follows. One can establish that the cardinality of $[\mathcal{F}_{\mathcal{J}\text{-}n}]_\mathcal{K}$ is the n-th Catalan number $(2n)!/(n!(n+1)!)$ (see the comment after the definition of n-frieze in the preceding section; see also [**13**, Section 6.1, and references therein]). Independently, one establishes as in [**9**, p. 14] that the number of terms of $\mathcal{J}_n$ in Jones normal form is also the n-th Catalan number. So, by the Normal Form Lemma of the preceding section, the cardinality of $\mathcal{J}_n$ is at most the n-th Catalan number. Since, by the Generating Lemma of that section, it is known that the map above is onto, it follows that it is one-one. This argument is on the lines of the argument in [**5**, Note C, pp. 464–465], which establishes that the standard presentation of symmetric groups is complete with respect to permutations. It can also be adapted to give an alternative proof of the Completeness Lemma of the preceding section, which is not based on the Key Lemma and the Auxiliary Lemma of that section.

10. The maximality of $\mathcal{J}_\omega$

We will now show that $\mathcal{J}_\omega$ is maximal in the following sense. Let t and u be terms of $\mathcal{L}_\omega$ such that $t = u$ does not hold in $\mathcal{J}_\omega$. If $\mathcal{X}$ is defined as $\mathcal{J}_\omega$ save that we require also $t = u$, then for every $k \in \mathbf{N}^+$ we have $\lceil k \rceil \lfloor k \rfloor = \mathbf{1}$ in $\mathcal{X}$. With the same assumptions, for some $n \in \mathbf{N}$ we have that $\mathcal{X}$ is isomorphic to the monoid $\mathbf{Z}/n$, i.e., the additive commutative monoid $\mathbf{Z}$ with equality modulo n.

For t a term of $\mathcal{L}_\omega$, and for $\delta(t)$ the corresponding $\mathcal{J}$-frieze, defined analogously to what we had in Section 7, let $cups(t) \in \mathbf{N}$ be the number of cups in $\delta(t)$, and $caps(t) \in \mathbf{N}$ the number of caps in $\delta(t)$. For t_1 and t_2 terms of $\mathcal{L}_\omega$, let the *balance* $\beta(t_1, t_2) \in \mathbf{N}$ of the pair (t_1, t_2) be defined by

$$\beta(t_1, t_2) = |cups(t_1) - cups(t_2) + caps(t_2) - caps(t_1)|.$$

Let $\mathcal{X}$ be defined as above. We will show that $\mathcal{X}$ is isomorphic to $\mathbf{Z}/\beta(t, u)$. In order to prove that we need first the following lemma.

BALANCE LEMMA. *If $t_1 = t_2$ holds in $\mathcal{X}$, then for some $n \in \mathbf{N}$ we have that $\beta(t_1, t_2) = n\beta(t, u)$.*

PROOF. We proceed by induction on the length of the derivation of $t_1 = t_2$ in $\mathcal{X}$. If $t_1 = t_2$ holds in $\mathcal{J}_\omega$, then $\beta(t_1, t_2) = 0 = 0 \cdot \beta(t, u)$, and if $t_1 = t_2$ is $t = u$, then $\beta(t_1, t_2) = \beta(t, u)$. It is easy to see that $\beta(t_1, t_2) = \beta(t_2, t_1)$. Next, if for some $n_1, n_2 \in \mathbf{N}$ we have $\beta(t_1, t_2) = n_1\beta(t, u)$ and $\beta(t_2, t_3) = n_2\beta(t, u)$, then for some $z_1, z_2 \in \mathbf{Z}$ such that $|z_1| = n_1$ and $|z_2| = n_2$

$$\begin{aligned} cups(t_1) - cups(t_2) + caps(t_2) - caps(t_1) &= z_1\beta(t, u), \\ cups(t_2) - cups(t_3) + caps(t_3) - caps(t_2) &= z_2\beta(t, u). \end{aligned}$$

Then

$$cups(t_1) - cups(t_3) + caps(t_3) - caps(t_1) = (z_1 + z_2)\beta(t, u),$$

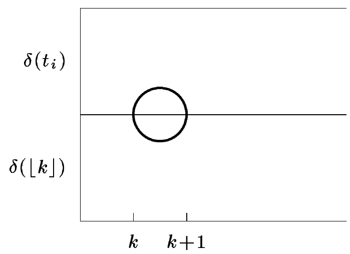
and hence $\beta(t_1, t_3) = |z_1 + z_2|\beta(t, u)$.

We also have that $\beta(t_1, t_2) = \beta(\lfloor k \rfloor t_1, \lfloor k \rfloor t_2)$. To show that, we have the following cases for $\delta(t_i)$, $i \in \{1, 2\}$:

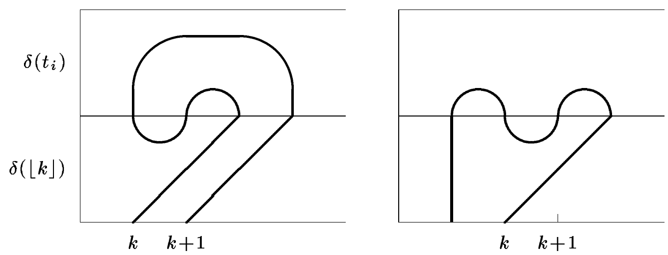
- **(1.1):** $(k, 0)$ and $(k+1, 0)$ are the end points of a single cap;
- **(1.2):** $(k, 0)$ and $(k+1, 0)$ are the end points of two caps;
- **(1.3):** one of $(k, 0)$ and $(k+1, 0)$ is the end point of a cap, and the other is the end point of a transversal thread;
- **(2):** $(k, 0)$ and $(k+1, 0)$ are the end points of two transversal threads.

Here we illustrate $\delta(\lfloor k \rfloor t_i)$ in these various cases:

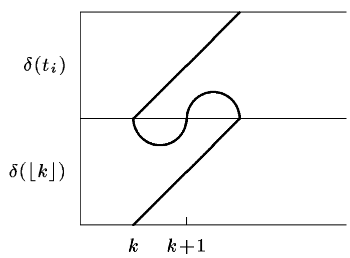
(1.1)



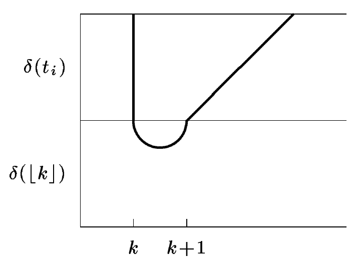
(1.2)



(1.3)



(2)



In cases (1.1), (1.2) and (1.3) we have

$$(1)\quad \begin{aligned} cups(t_i) &= cups(\lfloor k \rfloor t_i), \\ caps(t_i) &= caps(\lfloor k \rfloor t_i) + 1; \end{aligned}$$

and in case (2) we have

$$(2)\quad \begin{aligned} cups(t_i) &= cups(\lfloor k \rfloor t_i) - 1, \\ caps(t_i) &= caps(\lfloor k \rfloor t_i). \end{aligned}$$

When for both $\delta(t_1)$ and $\delta(t_2)$ we have (1), it is clear that $\beta(t_1,t_2)=\beta(\lfloor k\rfloor t_1,\lfloor k\rfloor t_2)$, and the same if for both we have (2). If for one of $\delta(t_i)$ we have (1) and for the other (2), we have again this equality of balances.

We show analogously that $\beta(t_1,t_2)=\beta(t_1\lceil k\rceil,t_2\lceil k\rceil)$, and since we have trivially that

$$\begin{aligned}\beta(t_1,t_2)&=\beta(\lceil k\rceil t_1,\lceil k\rceil t_2)\\&=\beta(t_1\lfloor k\rfloor,t_2\lfloor k\rfloor)\\&=\beta(\mathbf{1}t_1,\mathbf{1}t_2)\\&=\beta(t_1\mathbf{1},t_2\mathbf{1}),\end{aligned}$$

we can conclude that for every term s of $\mathcal{L}_\omega$

$$\begin{aligned}\beta(t_1,t_2)&=\beta(st_1,st_2)\\&=\beta(t_1s,t_2s).\end{aligned}$$

From that the lemma follows. □

We have seen in Section 5 that every $\mathcal{K}$-equivalence class of a frieze may be identified with a pair (S_Θ,l) where S_Θ is a set of nonoverlapping segments in $\mathbf{Z}$–$\{0\}$, and l is the number of circular components. For $\mathcal{J}$-friezes l is always ω, and hence every $\mathcal{J}$-frieze is identified up to $\mathcal{K}$-equivalence with S_Θ. To identify S_Θ of a $\mathcal{J}$-frieze it is enough to identify the rooted tree that makes the branching part of the tree of S_Θ, which is the part of S_Θ from the leaves down to the lowest node after which no node is branching. (This tree may consist of a single node.) We call this rooted subtree of S_Θ the *crown* of S_Θ. In the example in Section 5, the lowest node after which no node is branching is $[-11,7]$ and the crown is the tree above $[-11,7]$, whose root is this node.

The transversal thread in the $\mathcal{J}$-frieze that corresponds to the root of the crown will be called the *crown thread*. Every $\mathcal{J}$-frieze has a crown thread. If the end points of a crown thread are $(k,0)$ and (l,a), we call (k,l) the *crown pair*. So in our example the crown pair is $(11,7)$. All threads on the right-hand side of the crown thread are transversal threads identified with $[-(k+n),l+n]$, for $n\geqslant 1$ and (k,l) the crown pair.

Suppose $t=u$ does not hold in $\mathcal{J}_\omega$, and let $\mathcal{X}$ be as before $\mathcal{J}_\omega$ plus $t=u$. Let $\lceil j_1\rceil\dots\lceil j_m\rceil\lfloor i_1\rfloor\dots\lfloor i_n\rfloor$ be the normal form of t, and $\lceil k_1\rceil\dots\lceil k_p\rceil\lfloor l_1\rfloor\dots\lfloor l_q\rfloor$ the normal form of u. We show that there is a term v (built out of t and u) such that $v=\mathbf{1}$ in $\mathcal{X}$, but not in $\mathcal{J}_\omega$.

Let s_1 be $\lfloor k_p+1\rfloor\dots\lfloor k_1+1\rfloor$, let s_2 be $\lceil l_q+1\rceil\dots\lceil l_1+1\rceil$, let s_1' be $\lfloor j_m+1\rfloor\dots\lfloor j_1+1\rfloor$, and, finally, let s_2' be $\lceil i_n+1\rceil\dots\lceil i_1+1\rceil$. It is clear that by (*cup-cap* 3) we have $s_1us_2=\mathbf{1}$ and $s_1'ts_2'=\mathbf{1}$ in $\mathcal{J}_\omega$, while $s_1ts_2=\mathbf{1}$ and $s_1'us_2'=\mathbf{1}$ hold in $\mathcal{X}$.

(*i*) If $m>p$, then $s_1ts_2=\mathbf{1}$ cannot hold in $\mathcal{J}_\omega$, because the normal form of s_1ts_2 has at least one cap, and analogously if $n>q$, because then the normal form of s_1ts_2 has at least one cup. If $m<p$ or $n<q$, then $s_1'us_2'=\mathbf{1}$ cannot hold in $\mathcal{J}_\omega$.

(*ii*) If $m = p$ and $n = q$, then we proceed by induction on $m+n$. If $m+n = 1$, then we have in $\mathcal{X}$ either $\lceil j_1 \rceil = \lceil k_1 \rceil$ for $j_1 \neq k_1$, or $\lfloor i_1 \rfloor = \lfloor l_1 \rfloor$ for $i_1 \neq l_1$. If $j_1 < k_1$, then we have $\lceil j_1 \rceil \lfloor k_1 - 1 \rfloor = \mathbf{1}$ in $\mathcal{X}$, but not in $\mathcal{J}_\omega$. We proceed analogously in the other cases of $j_1 \neq k_1$ and $i_1 \neq l_1$.

Suppose now $m+n > 1$ and $m \geqslant 1$. Let a *cap-block* $\lceil r, \ldots, r-k \rceil$ be $\lceil r \rceil \lceil r-1 \rceil \ldots \lceil r-k \rceil$ for $r \in \mathbf{N}^+$ and $k \in \mathbf{N}$. Then the sequence of caps $\lceil j_1 \rceil \ldots \lceil j_m \rceil$ can be written in terms of cap-blocks as

$$\lceil r_1, \ldots, r_1 - k_1 \rceil \ldots \lceil r_h, \ldots, r_h - k_h \rceil$$

such that $1 \leqslant h \leqslant m$, $r_1 = j_1$, $r_h - k_h = j_m$ and $r_i - k_i - r_{i+1} \geqslant 2$. Let $\lceil r'_1, \ldots, r'_1 - k'_1 \rceil$ be the leftmost cap-block of $\lceil k_1 \rceil \ldots \lceil k_m \rceil$, as $\lceil r_1, \ldots, r_1 - k_1 \rceil$ is the leftmost cap-block of $\lceil j_1 \rceil \ldots \lceil j_m \rceil$. We have

$$\lfloor r_1 + k_1 + 1 \rfloor \lceil r_1, \ldots, r_1 - k_1 \rceil = \lceil r_1, \ldots, r_1 - k_1 + 1 \rceil.$$

If $r_1 + k_1 = r'_1 + k'_1$, then

$$\lfloor r_1 + k_1 + 1 \rfloor \lceil r'_1, \ldots, r'_1 - k'_1 \rceil = \lceil r'_1, \ldots, r'_1 - k'_1 + 1 \rceil,$$

and from $t = u$ in $\mathcal{X}$, we obtain $\lfloor r_1 + k_1 + 1 \rfloor t = \lfloor r_1 + k_1 + 1 \rfloor u$ in $\mathcal{X}$, but not in $\mathcal{J}_\omega$, since the difference in the normal forms of t and u persists. We can then apply the induction hypothesis, since the new m has decreased. If $r_1 + k_1 > r'_1 + k'_1$, then for some r'

$$\lfloor r_1 + k_1 + 1 \rfloor \lceil k_1 \rceil \ldots \lceil k_m \rceil = \lceil k_1 \rceil \ldots \lceil k_m \rceil \lfloor r' \rfloor$$

(as can be ascertained from the corresponding $\mathcal{J}$-friezes), and with $\lfloor r_1 + k_1 + 1 \rfloor t = \lfloor r_1 + k_1 + 1 \rfloor u$ we are in case (*i*). All the other cases, where $r_1 + k_1 < r'_1 + k'_1$, and where $n \geqslant 1$, are dealt with analogously.

We can verify that $\beta(t, u) = \beta(v, \mathbf{1})$. If v is $s_1 t s_2$, then

$$\beta(t, u) = \beta(s_1 t s_2, s_1 u s_2),$$

as we have seen in the proof of the Balance Lemma, and the right-hand side is equal to $\beta(v, \mathbf{1})$. We reason analogously for the other possible forms of v.

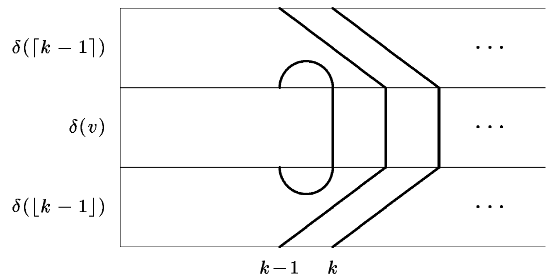
If (k, l) is the crown pair of $\delta(v)$, then $|k - l| = 2\beta(v, \mathbf{1})$. The numbers k and l cannot both be 1; otherwise, $v = \mathbf{1}$ would hold in $\mathcal{J}_\omega$. We have the following cases.

(1) Suppose $k = l > 1$. If for some i we have $v = h_i$ in $\mathcal{J}_\omega$, where h_i abbreviates $\lceil i \rceil \lfloor i \rfloor$, then in $\mathcal{X}$ we have $h_i = \mathbf{1}$, and hence

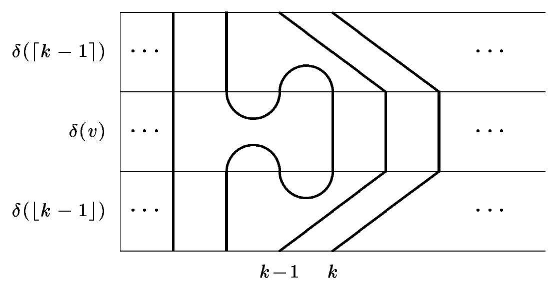
$$\begin{aligned} h_i &= h_i h_{i\pm 1} h_i, \quad \text{by } (h2) \\ &= h_{i\pm 1}, \quad \text{by } (1). \end{aligned}$$

So for every $k \in \mathbf{N}^+$ we have $h_k = \mathbf{1}$ in $\mathcal{X}$.

If for every i we don't have $v = h_i$ in $\mathcal{J}_\omega$, then $\lfloor k-1 \rfloor v \lceil k-1 \rceil = \mathbf{1}$ in $\mathcal{X}$, and in the crown pair (k', l') of $\lfloor k-1 \rfloor v \lceil k-1 \rceil$ we have $k' < k$ and $l' < l$. This is clear from the following picture:

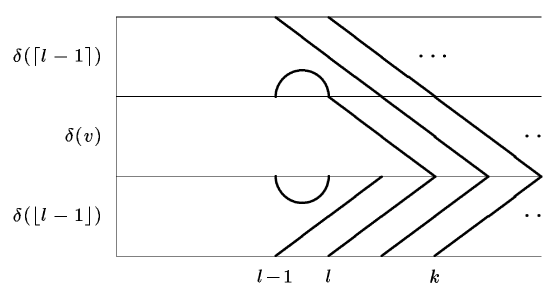


We would have $\lfloor k-1\rfloor v\lceil k-1\rceil = \mathbf{1}$ in $\mathcal{J}_\omega$ only if we had $v = h_{k-2}$ in $\mathcal{J}_\omega$, as can be seen from the following picture:



There are several straightforward cases to consider in order to prove this assertion. So $\lfloor k-1\rfloor v\lceil k-1\rceil = \mathbf{1}$ does not hold in $\mathcal{J}_\omega$, and by induction we obtain $h_i = \mathbf{1}$ in $\mathcal{X}$ for some i. So, as above, for every $k \in \mathbf{N}^+$ we have $h_k = \mathbf{1}$ in $\mathcal{X}$.

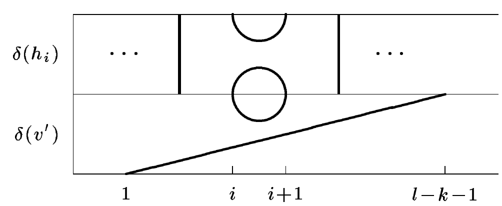
(2) If $k \neq l$, and $\min(k,l) = m_1 > 1$, then $\lfloor m_1-1\rfloor v\lceil m_1-1\rceil = \mathbf{1}$ in $\mathcal{X}$, and in the crown pair (k',l') of $\lfloor m_1-1\rfloor v\lceil m_1-1\rceil$ we have $k' < k$ and $l' < l$. This is clear from the following picture:



We cannot have $\lfloor m_1-1\rfloor v\lceil m_1-1\rceil = \mathbf{1}$ in $\mathcal{J}_\omega$ because we must have $|k'-l'| = |k-l|$, as it is clear from the picture. We continue in the same manner until we reach a term $\lfloor m_p-1\rfloor \dots \lfloor m_1-1\rfloor v\lceil m_1-1\rceil \dots \lceil m_p-1\rceil$, which we abbreviate by v', such that $v' = \mathbf{1}$ in $\mathcal{X}$, and the crown pair of $\delta(v')$ is either $(1, l-k+1)$ or $(k-l+1, 1)$. Suppose $l > k$. Then in $\delta(v')$ there must be a cup $[i, i+1]$, and we have

$$h_i = v'h_i = v' = \mathbf{1}.$$

That $v'h_i = v'$ is clear from the picture



We proceed analogously when $k > l$. As before, from $h_i = \mathbf{1}$ we derive that for every $k \in \mathbf{N}^+$ we have $h_k = \mathbf{1}$ in $\mathcal{X}$.

So, both in case (1) and in case (2), for every $k \in \mathbf{N}^+$ we have $h_k = \mathbf{1}$ in $\mathcal{X}$. Since

$$\lfloor i \rfloor \lceil i+1 \rceil \lfloor i+1 \rfloor = \lfloor i+1 \rfloor, \quad \text{by } (\textit{cup-cap } 3),$$

with $h_{i+1} = \mathbf{1}$, we obtain $\lfloor i \rfloor = \lfloor i+1 \rfloor$ in $\mathcal{X}$ for every $i \in \mathbf{N}^+$. Analogously, we obtain $\lceil i \rceil = \lceil i+1 \rceil$ in $\mathcal{X}$ for every $i \in \mathbf{N}^+$.

If $\lfloor 1 \rfloor^0$ and $\lceil 1 \rceil^0$ are $\mathbf{1}$, while $\lfloor 1 \rfloor^{k+1}$ is $\lfloor 1 \rfloor^k \lfloor 1 \rfloor$ and $\lceil 1 \rceil^{k+1}$ is $\lceil 1 \rceil^k \lceil 1 \rceil$, then every element of $\mathcal{X}$ is either of the form $\lfloor 1 \rfloor^k$ or of the form $\lceil 1 \rceil^k$ for some $k \in \mathbf{N}$. To see that, start from the normal form of an element, identify all cups with $\lfloor 1 \rfloor$, all caps with $\lceil 1 \rceil$, and then use $\lceil 1 \rceil \lfloor 1 \rfloor = \mathbf{1}$.

So all the elements of $\mathcal{X}$ are the following

$$\ldots, \lceil 1 \rceil^3, \lceil 1 \rceil^2, \lceil 1 \rceil^1, \mathbf{1}, \lfloor 1 \rfloor^1, \lfloor 1 \rfloor^2, \lfloor 1 \rfloor^3, \ldots$$

If $\beta(t,u) = 0$, then the Balance Lemma guarantees that all the elements of $\mathcal{X}$ above are mutually distinct. Composition in $\mathcal{X}$ then behaves as addition of integers, where $\mathbf{1}$ is zero, and so $\mathcal{X}$ is isomorphic to $\mathbf{Z} = \mathbf{Z}/0$.

In case $\beta(t,u) = n > 0$, the elements of $\mathcal{X}$ above are not all mutually distinct. We are then in case (2), and from $v' = \mathbf{1}$ we can infer $\lfloor 1 \rfloor^n = \mathbf{1}$, and also $\lceil 1 \rceil^n = \mathbf{1}$, in $\mathcal{X}$. Hence every element of $\mathcal{X}$ is equal to one of the following

$$\mathbf{1}, \lfloor 1 \rfloor^1, \ldots \lfloor 1 \rfloor^{n-1},$$

and by the Balance Lemma these are all mutually distinct. Composition in $\mathcal{X}$ then behaves as addition of integers modulo n, where $\mathbf{1}$ is zero, and so $\mathcal{X}$ is isomorphic to $\mathbf{Z}/n$. So, in any case, we have that $\mathcal{X}$ is isomorphic to $\mathbf{Z}/\beta(t,u)$.

11. The monoids $\mathcal{L}_{\pm\omega}$ and $\mathcal{K}_{\pm\omega}$

The monoid $\mathcal{L}_{\pm\omega}$ is defined as $\mathcal{L}_\omega$ save that for every $k \in \mathbf{Z}$ there is a generator $\lfloor k \rfloor$ and a generator $\lceil k \rceil$, and there are two additional generators // and \\. The equations of $\mathcal{L}_{\pm\omega}$ are those of $\mathcal{L}_\omega$ plus

$$\lfloor k \rfloor // = // \lfloor k+1 \rfloor,$$
$$\lceil k \rceil // = // \lceil k+1 \rceil,$$
$$\backslash\backslash // = // \backslash\backslash = \mathbf{1}.$$

We derive easily

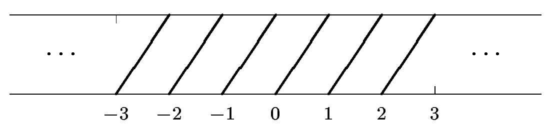
$$\lfloor k+1 \rfloor \backslash\backslash = \backslash\backslash \lfloor k \rfloor,$$
$$\lceil k+1 \rceil \backslash\backslash = \backslash\backslash \lceil k \rceil.$$

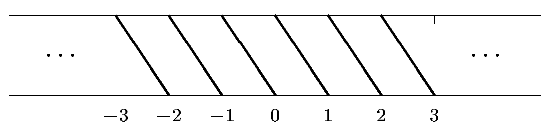
The monoid $\mathcal{K}_{\pm\omega}$ has in addition the equation (*cup-cap* 4).

Let $//^0$ be the empty sequence, while $//^{k+1}$ is $//^k//$, and analogously for $\backslash\backslash^k$. We can define the normal form for terms of $\mathcal{L}_{\pm\omega}$ as the normal form for $\mathcal{L}_\omega$ of Section 4 prefixed with either $//^k$ or $\backslash\backslash^k$ for $k \in \mathbf{N}$. It is clear that combined with the Normal Form Lemma of Section 4 the equations above enable us to reduce every term to a term in normal form equal to the original term.

Let now R_a be $(-\infty, \infty) \times [0, a]$. An $\pm\omega$-*diagram* in R_a is defined as an ω-diagram save that $\mathbf{N}^+$ is replaced by $\mathbf{Z}$. A $\pm$ *frieze* is a $\pm\omega$-diagram with a finite number of cups, caps and circular components.

The cups and caps $\lfloor k \rfloor$ and $\lceil k \rceil$ are mapped to $\pm$ friezes analogously to what we had before, while $//$ and $\backslash\backslash$ are mapped into the $\mathcal{L}$-equivalence classes of the following $\pm$ friezes:





One could conceive $////$, i.e., $//$ multiplied with itself, as the infinite block $\ldots h_2 h_1 h_0 h_{-1} h_{-2} \ldots$, and $\backslash\backslash\backslash\backslash$ as the infinite block $\ldots h_{-2} h_{-1} h_0 h_1 h_2 \ldots$

That the monoid $\mathcal{L}_{\pm\omega}$ is isomorphic to a monoid made of $\mathcal{L}$-equivalence classes of $\pm$ friezes is shown analogously to what we had for $\mathcal{L}_\omega$, and the same can be shown with $\mathcal{L}$ replaced by $\mathcal{K}$. From these isomorphisms it follows that $\mathcal{L}_\omega$ can be embedded in $\mathcal{L}_{\pm\omega}$, and analogously with $\mathcal{K}$. This is because we can identify every frieze with a $\pm$ frieze such that for every $z \leqslant 0$ we have a vertical thread with the end points $(z, 0)$ and (z, a).

The monoid $\mathcal{L}_n^{\mathrm{cyl}}$ is defined as the monoid $\mathcal{L}_n$ save that we have also the diapsides h_0 and h_n. To the equations of $\mathcal{L}_n$ we add the equations

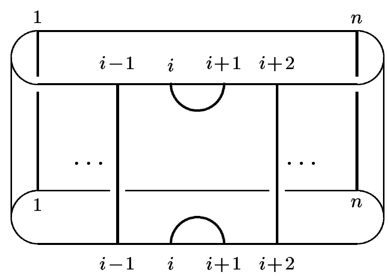
$$\begin{aligned} & h_0 = h_n, \\ & c_1^\alpha = c_{n+1}^\alpha, \\ & h_k// = //h_{k+1}, \quad \text{for } k \in \{0, \ldots, n-1\} \\ & \backslash\backslash// = //\backslash\backslash = \mathbf{1}. \end{aligned}$$

The monoid $\mathcal{K}_n^{\mathrm{cyl}}$ is obtained from $\mathcal{L}_n^{\mathrm{cyl}}$ as $\mathcal{K}_n$ is obtained from $\mathcal{L}_n$.

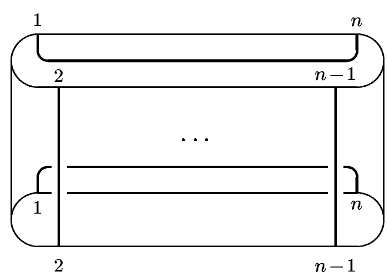
Note that while $\mathcal{L}_n$ was a submonoid of $\mathcal{L}_\omega$, because of $h_0 = h_n$ we don't have that $\mathcal{L}_n^{\mathrm{cyl}}$ is a submonoid of $\mathcal{L}_{\pm\omega}$. For the same reason $\mathcal{L}_n^{\mathrm{cyl}}$ is not a submonoid of

$\mathcal{L}^{\mathrm{cyl}}_{n+1}$, while $\mathcal{L}_n$ was isomorphic to two submonoids of $\mathcal{L}_{n+1}$ (we map h_i either to h_i or to h_{i+1}). The same holds when $\mathcal{L}$ is replaced by $\mathcal{K}$.

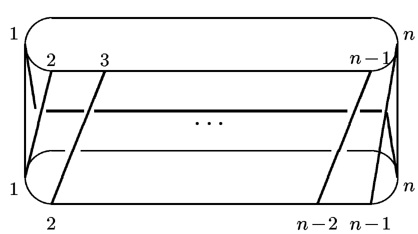
The monoid $\mathcal{L}^{\mathrm{cyl}}_n$ may be shown isomorphic to a monoid made of equivalence classes of cylindric friezes, which are roughly defined as follows. Instead of diagrams in R_a we now have diagrams in cylinders where the top and bottom are copies of a circle with n points labelled counterclockwise with the numbers from 1 to n. We interpret h_i for $i \in \{1, \dots, n-1\}$ by

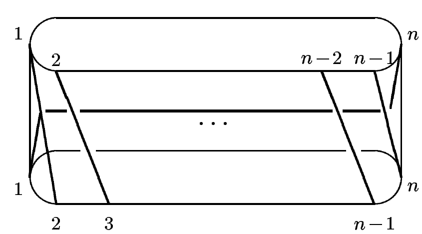


while h_0 and h_n are interpreted by



We interpret // and \\ by





Cylindric friezes are special three-dimensional tangles, whereas with friezes and ± friezes we had only two-dimensional tangles. In these special three-dimensional tangles we have only "cyclic braidings" or "torsions" like those obtained from the last two pictures, and further "cyclic braidings" obtained by composing these. Diagrams like our cylindric friezes were considered in [**10**].

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