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GRAPHS WITH LEAST EIGENVALUE AT LEAST $-\sqrt{3}$

Dragoš Cvetković and Dragan Stevanović

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ABSTRACT. We determine the graphs whose least eigenvalue is at least $-\sqrt{3}$.

1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices. We write $V(G)$ for the vertex set of G, and $E(G)$ for the edge set of G.

The *complement* of a graph G is denoted by $\overline{G}$. For $v \in V(G)$, $G - v$ denotes the graph obtained from G by deleting the vertex v and all edges incident with v. More generally, for $U \subseteq V(G)$, $G - U$ is the subgraph of G induced by $V(G) \setminus U$.

The characteristic polynomial $\det(xI - A)$ of the adjacency matrix A of G is called the *characteristic polynomial of* G and denoted by $P_G(x)$. The eigenvalues of A (i.e.,the zeros of $\det(xI - A)$) and the spectrum of A (which consists of the n eigenvalues) are also called the *eigenvalues* and the *spectrum* of G, respectively. The eigenvalues of G are usually denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$; they are real because A is symmetric. Unless we indicate otherwise, we shall assume that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and use the notation $\lambda_i = \lambda_i(G)$ for $i = 1, 2, \ldots, n$. The least eigenvalue $\lambda_n(G)$ of a graph G will also be denoted by $\lambda(G)$.

As usual, K_n, C_n and P_n denote respectively the *complete graph*, the *cycle* and the *path* on n vertices. Further, $K_{m,n}$ denotes the *complete bipartite* graph on $m+n$ vertices. The graph $K_{1,n}$ is called a *star* and its vertex of maximal degree is denoted as *central.* A *double star* $D_{m,n}$ is the graph formed by adding an edge between the central vertices of stars $K_{1,m}$ and $K_{1,n}$.

The *cocktail-party graph* $CP(n)$ is the unique regular graph with $2n$ vertices of degree $2n - 2$; it is obtained from K_{2n} by deleting n mutually non-adjacent edges.

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A connected graph with n vertices is said to be a *tree* if it has $n-1$ edges. If T is a tree, a vertex of T of degree one is called a *leaf*. An *end-edge* of T is an edge one of whose endvertices is a leaf.

A connected graph with n vertices is said to be *unicyclic* if it has n edges. It is called *even* (*odd*) if its unique cycle is even (odd).

The *join* $G \bigtriangledown H$ of graphs G and H is obtained from G and H by joining with an edge each vertex of G to each vertex of H.

If G is a graph of order n, the *corona* $G \otimes H$ of graphs G and H is obtained from G and n copies of the graph H by adding edges between the i-th vertex of G and each vertex in the i-th copy of H $(i = 1, 2, \ldots, n)$.

The *line graph* $L(H)$ of any graph H is defined as follows. The vertices of $L(H)$ are the edges of H and two vertices of $L(H)$ are adjacent whenever the corresponding edges of H have a vertex of H in common.

A *generalized line graph* $L(H; a_1, \ldots, a_n)$ is defined for graphs H with vertex set $\{1, \ldots, n\}$ and non-negative integers $a_1, \ldots, a_n$ by taking the graphs $L(H)$ and $CP(a_i)$ $(i = 1, \ldots, n)$ and adding extra edges: a vertex e in $L(H)$ is joined to all vertices in $CP(a_i)$ if i is an end-vertex of e as an edge of H. We include as special cases an ordinary line graph $(a_1 = a_2 = \cdots = a_n = 0)$ and the cocktail-party graph $CP(n)$ $(n = 1$ and $a_1 = n)$.

An *exceptional* graph is a connected graph with least eigenvalue greater than or equal to -2 which is not a generalized line graph.

The following result of M. Doob and D. Cvetković [**11**] is our starting point. (It appears as Theorem 1.3 of [**3**] with a misprint in part (v).)

THEOREM 1. *If G is a connected graph with least eigenvalue greater than -2 then one of the following holds:*
(i) $G = L(T; 1, 0, \ldots, 0)$ *where T is a tree;*
(ii) $G = L(H)$ *where H is a tree or an odd unicyclic graph;*
(iii) *G is one of* 20 *graphs on* 6 *vertices represented in the root system E_6;*
(iv) *G is one of* 110 *graphs on* 7 *vertices represented in the root system E_7;*
(v) *G is one of* 443 *graphs on* 8 *vertices represented in the root system E_8.*

The exceptional graphs with least eigenvalue greater than -2 are those appearing in parts (iii)–(v) of Theorem 1 (573 in total). Those of type (v) are one-vertex extensions of graphs of type (iv), which are in turn one-vertex extensions of graphs of type (iii). The 443 graphs of type (v) are tabulated in [**1**]. The 110 graphs of type (iv) are identified in [**5**] by means of the list of 7-vertex graphs in [**3**]. The twenty 6-vertex graphs of type (iii) are identified in [**7**]. All 573 exceptional graphs with least eigenvalue greater than -2 are also given in the technical report [**6**] together with related data.

By the well-known interlacing theorem for graph eigenvalues (cf., e.g., [**4**, p. 19]), the property $\lambda(G) \geqslant a$ for a fixed real a, is a hereditary property.

It is shown in [**15**] that, for $n \geqslant 4$, if G is not a complete graph on n vertices, then

$$\lambda(G) < -\frac{1}{2}\left(1 + \sqrt{1 + 4\frac{n-3}{n-1}}\right)$$

When n tends to infinity, this upper bound tends to $\tau = -(1+\sqrt{5})/2 \approx -1.61803$. We are interested to find such graphs G whose smallest eigenvalue $\lambda(G)$ falls in the gap between $\tau = -(1+\sqrt{5})/2$ and this upper bound, i.e., that satisfy $\lambda_n \geqslant -(1+\sqrt{5})/2$. Such graphs will be called *τ-graphs.*

Recall that x is a *limit point* of a set S of reals if any open interval containing x contains an element of S different from x.

The value τ is the largest limit point of the least eigenvalue of graphs. The second largest limit point is $\omega = -\sqrt{3}$. This follows from some results of A. J. Hoffman who determined in [**12**] all reals exceeding -2 which are limit points of the set Λ of least eigenvalues of graphs. Let T be a tree with at least two edges, e an end-edge of T. Let $\hat{A}(T,e)$ be the adjacency matrix of $L(T)$, modified by putting -1 in the diagonal position corresponding to e. We will say that the pair (T,e) is *proper* provided $\lambda(\hat{A}(T,e)) < \lambda(L(T))$. (It was conjectured in [**12**] that every (T,e) is proper, but so far there is no proof.) The main result of [**12**] is given in the following theorem.

THEOREM 2. *If (T,e) is proper, $\lambda(\hat{A}(T,e))$ is a limit point of Λ. Conversely, if $\lambda > -2$ is a limit point of Λ, then $\lambda = \lambda(\hat{A}(T,e))$ for some proper (T,e).*

The limit point τ is obtained if $T = K_{1,2}$ while the next limit point ω is obtained for $T = K_{1,3}$. We will also determine all ω-graphs.

Before [**15**] it was established in [**13**] that if we order connected graphs on n $(n > 2)$ vertices by decreasing least eigenvalues the first graph is K_n and the second one is K_{n-1} with a pendant edge attached, which is here denoted by L_n. The sequence $\lambda(L_n)$ can be easily calculated and it is decreasing and tends to τ.

τ-graphs are related to the problem of characterizing graphs with $\lambda_2 \leqslant (\sqrt{5}-1)/2 = \sigma \approx 0.61803$ [**9**], [**10**]. For let $\overline{\lambda}_1 \geqslant \overline{\lambda}_2 \geqslant \cdots \geqslant \overline{\lambda}_n$ be the eigenvalues of $\overline{G}$. The Courant-Weyl inequalities [**14**, Theorem 34.2.1], imply that $\lambda_2 + \overline{\lambda}_{n-1} \geqslant -1$, while $\lambda_2 + \overline{\lambda}_n \leqslant -1$ (cf. [**2**]). This shows that if $\lambda_n \geqslant -(1+\sqrt{5})/2$, then $\overline{\lambda}_2 \leqslant (\sqrt{5}-1)/2$. Hence, the complements of τ-graphs have $\lambda_2 \leqslant \sigma$.

Let $\alpha > -2$ and let S_α denote the set of all graphs G satisfying $\lambda(G) \geqslant \alpha$. From Theorem 1 it follows that a graph G from S_α is:

- one of 573 exceptional graphs, or
- the generalized line graph $L(T;1,0,\dots,0)$ for some tree T, or
- the line graph $L(T)$ for some tree T, or
- the line graph $L(H)$ for some odd unicyclic graph H.

Suppose that G is isomorphic to either $L(T;1,0,\dots,0)$ or $L(T)$ for some tree T. Note that if T' is an induced subgraph of T, then $L(T')$ is an induced subgraph of $L(T)$. Since the sequence $\lambda_d(P_d)$ is monotonic decreasing and $\lim_{d\mapsto\infty} \lambda_d(P_d) = -2$, it follows that there is $d_\alpha \in \mathbb{N}$ such that $\alpha > \lambda_{d_\alpha}(P_{d_\alpha})$. If T has a diameter at least d_α then it contains $P_{d_\alpha+1}$ as an induced subgraph, and $G = L(T)$ contains $P_{d_\alpha} = L(P_{d_\alpha+1})$ as an induced subgraph, which is contradiction, since from the interlacing theorem it follows that $\alpha > \lambda_{d_\alpha}(P_{d_\alpha}) \geqslant \lambda_n(G)$. Therefore, we conclude that a tree T has diameter at most $d_\alpha - 1$.

Now, suppose that G is isomorphic to $L(H)$ for some odd unicyclic graph H. Since the sequence $\lambda_{2l+1}(C_{2l+1})$ is monotonic decreasing and $\lim_{l\mapsto\infty}\lambda_{2l+1}(C_{2l+1}) = -2$, it follows that there is $l_\alpha \in \mathbb{N}$ such that $\alpha > \lambda_{2l_\alpha+1}(C_{2l_\alpha+1})$. If H contains (as an induced subgraph) an odd cycle of length at least $2l_\alpha+1$, then $G = L(H)$ contains $C_{2l_\alpha+1} = L(C_{2l_\alpha+1})$ as an induced subgraph too, which is contradiction, since from the interlacing theorem it follows that $\alpha > \lambda_{2l_\alpha+1}(C_{2l_\alpha+1}) \geqslant \lambda_n(G)$. Therefore, we conclude that H has a cycle of length at most $2l_\alpha - 1$.

In the following two sections, we apply previous considerations to determine the sets S_τ and S_ω.

2. The set S_τ

LEMMA 1. *The wheel W_5, shown in Fig. 1a, is the only exceptional graph which belongs to S_τ.*

PROOF. Looking at the tables of [**6**] we see that out of 573 exceptional graphs with least eigenvalue greater than -2 only the wheel W_5, shown in Fig. 1a, belongs to S_τ. In fact, W_5 has least eigenvalue equal to τ. □

LEMMA 2. *An odd unicyclic graph H such that $L(H) \in S_\tau$ contains an odd cycle of length either 3 or 5.*

PROOF. Since $\lambda_7(C_7) \approx -1.8019 < \tau$, we conclude that an odd unicyclic graph H such that $L(H) \in S_\tau$ contains an odd cycle of length either 3 or 5. □

LEMMA 3. *The only unicyclic graph H with a cycle C of length 5 for which $L(H)$ belongs to S_τ is the cycle C_5 itself, shown in Fig. 1b.*

PROOF. Suppose that $G = L(H) \in S_\tau$ where H is a unicyclic graph with a cycle C of length 5. If there exists a vertex v of H adjacent to a vertex of C, then H contains as an induced subgraph the graph B_1 from Fig. 2, which is a contradiction, since then $\lambda_n(G) \leqslant \lambda_6(L(B_1)) \approx -1.7566$. Therefore, H does not have any vertex adjacent to a vertex from C. The cycle C_5, shown in Fig. 1b, has the smallest eigenvalue equal to τ and it belongs to S_τ. □

LEMMA 4. *The line graphs of a unicyclic graph with a cycle C of length 3 which belong to S_τ, are shown in Fig. 1c–f.*

PROOF. Let $G = L(H)$ and suppose that H is a unicyclic graph with a cycle C of length 3 consisting of vertices c_1, c_2, c_3. If any of these vertices has degree at least 4, then H contains as an induced subgraph the graph B_2 from Fig. 2, which is a contradiction, since then $\lambda_n(G) \leqslant \lambda_5(L(B_2)) \approx -1.6813$. Therefore, each of vertices c_1, c_2, c_3 has degree either 2 or 3. If there is a vertex v of H adjacent to vertex c_i for some $i \in \{1,2,3\}$, and the degree of v is at least 2, then H contains as an induced subgraph the graph B_3 from Fig. 2, which is also a contradiction, since then $\lambda_n(G) \leqslant \lambda_5(L(B_3)) \approx -1.7757$. Therefore, possible neighbors of vertices c_1, c_2, c_3 may be only pendant vertices and we conclude that there are four possibilities, the line graphs of which all belong to S_τ, and which are shown in Fig. 1c–f. □

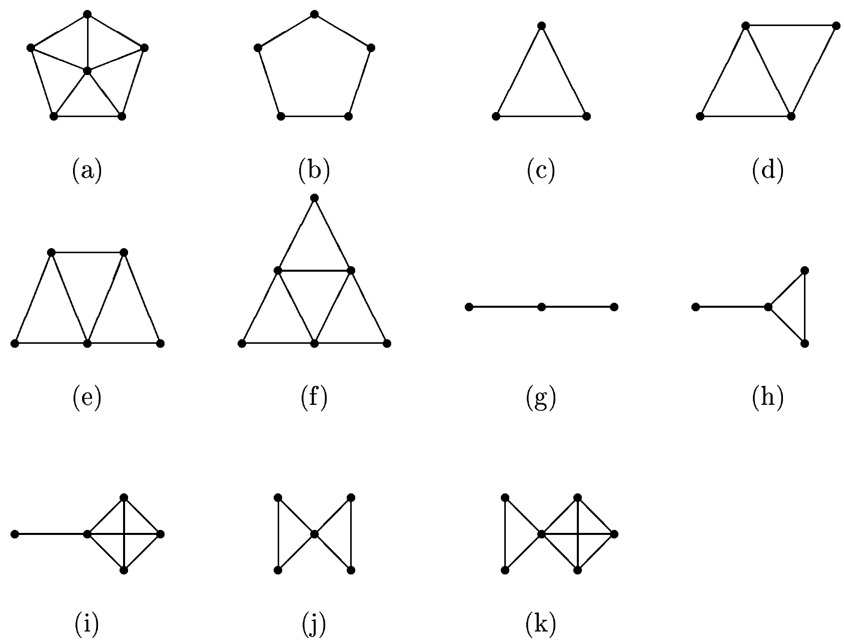

FIGURE 1. Some τ-graphs.

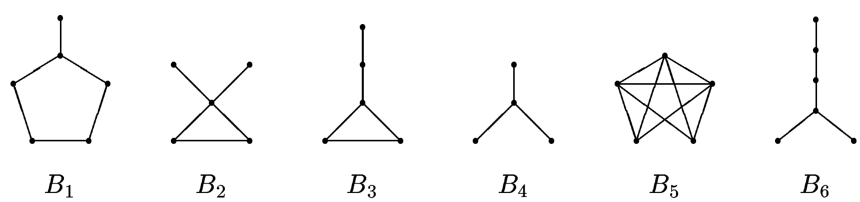

FIGURE 2. Forbidden subgraphs.

LEMMA 5. *Suppose that $G = L(T; 1, 0, \ldots, 0) \in S_\tau$, where T is a tree with at least one edge. Then G is either graph (g) or graph (d) as shown in Fig. 1.*

PROOF. Suppose that $L(T; 1, 0, \ldots, 0) \in S_\tau$, where T is a tree with at least one edge. Consider the vertex v_1 of T. The vertices of $L(T; 1, 0, \ldots, 0)$, corresponding to the edges e of T having v_1 as an endvertex, are adjacent to both vertices of $CP(1) \cong K_2$ in $L(T; 1, 0, \ldots, 0)$. If there is a vertex w at distance 2 from v_1 in T, then $L(T; 1, 0, \ldots, 0)$ contains as an induced subgraph the graph $B_4 \cong K_{1,3}$ from Fig. 2, which is a contradiction, since then $\lambda_n(L(T; 1, 0, \ldots, 0)) \leqslant \lambda_3(B_4) \approx -1.7321$. Therefore, all other vertices of T are neighbors of v_1. If v_1 has one neighbor, i.e., if $T \cong K_2$, then $L(T; 1, 0) \cong P_3 \in S_\tau$ and it is shown in Fig. 1g. If v_1

has two neighbors, i.e., if $T \cong P_3$, then $L(T;1,0,0)$ is isomorphic to the graph in Fig. 1d. However, if v_1 has at least three neighbors, then $L(T;1,0,\ldots,0)$ contains as an induced subgraph the graph B_5 from Fig. 2, which is a contradiction, since then $\lambda_n(L(T;1,0,\ldots,0)) \leqslant \lambda_5(B_5) \approx -1.6458$. □

LEMMA 6. *Suppose that $G = L(T) \in S_\tau$, where T is a tree with at least one edge. Then G is one of the graphs (g)–(k) shown in Fig. 1 or belongs to the family $Y_{k,l}$, shown in Fig. 3b.*

PROOF. Suppose that $G = L(T) \in S_\tau$, where T is a tree with at least one edge. Since $\lambda_5(P_5) = -\sqrt{3} < \tau$, from observations in previous section, we conclude that T has diameter at most 4.

If T has diameter 1, then $T \cong K_2$ and $L(T) \cong K_1$, which has no edges. If T has diameter 2, then for some $n \in \mathbb{N}$ we have $T \cong K_{1,n}$ and $L(T) \cong K_n$. The complete graphs have smallest eigenvalue equal to -1 and they belong to S_τ. However, they form a subfamily of a larger family which we later find is contained in S_τ.

If T has diameter 3, then T is isomorphic to a *double star* $D_{m,n}$ for some $m \geqslant n \geqslant 1$. In that case, $L(T)$ is isomorphic to a graph formed by identifying a pair of vertices of complete graphs K_{m+1} and K_{n+1}. Since $\lambda_7(L(D_{4,2})) \approx -1.6262$, in order that $L(D_{m,n}) \in S_\tau$ we must have that either $n = 1$ or $m \leqslant 3$. For $n = 1$ the graphs $L(D_{m,1})$ form a subfamily of a larger family which we later find is contained in S_τ. If $m \leqslant 3$, then we have in all six possibilities for the pairs (m,n) and double stars $D_{m,n}$. Their line graphs all belong to S_τ, except for the case $m = n = 3$, and they are shown in Fig. 1g–k (since $L(D_{1,1}) \cong P_3$, which is already shown in Fig. 1g).

Finally, suppose that T has diameter 4, let u and v be two vertices of T with $d(u,v) = 4$ and let c be the unique vertex of T such that $d(c,u) = d(c,v) = 2$. If there is a vertex w of T such that $d(c,w) = 2$ and either $d(u,w) \leqslant 2$ or $d(v,w) \leqslant 2$, then T contains as an induced subgraph the graph B_6 from Fig. 2, which is a contradiction, since then $\lambda_n(L(T)) \leqslant \lambda_5(B_6) \approx -1.6751$. Therefore, for each vertex w of T such that $d(c,w) = 2$ we conclude that $d(u,w) = d(v,w) = 4$. Thus, there exist non-negative integers k, l $(k \geqslant 2)$, such that T is isomorphic to the tree $X_{k,l}$, shown in Fig. 3a, while $L(T)$ is isomorphic to the graph $Y_{k,l}$, shown in Fig. 3b. □

Note that the complete graph K_n is just $Y_{0,n}$, while the graph $L(D_{m,1})$ is just $Y_{1,m}$.

All graphs $Y_{k,l}$ belong to S_τ. To see this, it is enough to show that $Y_{k,0} \in S_\tau$, since $Y_{k,l}$ is an induced subgraph of $Y_{k+l,0}$. Actually, the graphs $Y_{k,0}$ may be obtained by adding a pendant vertex to each vertex of K_k. On page 60 of [**4**] one can find a formula for the characteristic polynomial of a graph obtained in this way (alternatively, one can use a more general formula for the corona of two graphs in the next section):

$$P(Y_{k,0};\lambda) = \lambda^k P(K_k;\lambda - 1/\lambda) = (\lambda^2 - (k-1)\lambda - 1)(\lambda^2 + \lambda - 1)^{k-1}.$$

Thus, the eigenvalues of $Y_{k,0}$ are simple eigenvalues $(k - 1 \pm \sqrt{(k-1)^2 + 4})/2$, and eigenvalues $(\sqrt{5} - 1)/2$ and $-(1 + \sqrt{5})/2$, each with multiplicity $k - 1$.

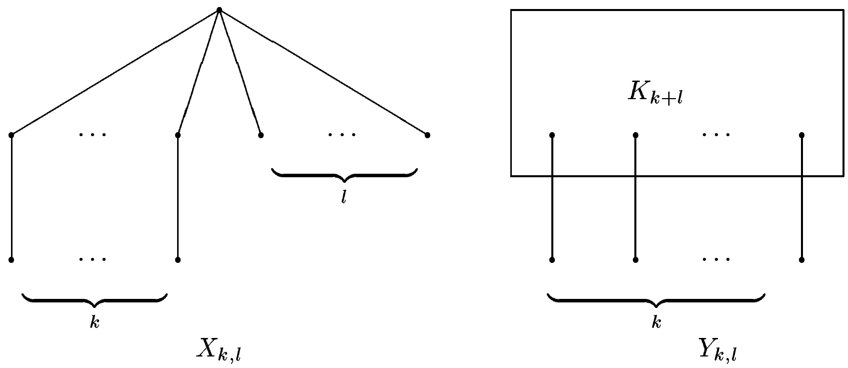

FIGURE 3. A family of graphs.

This ends our search and establishes the following theorem.

THEOREM 3. *The set S_τ consists of connected induced subgraphs of the following graphs:*

(1) *graph (a) of Fig. 1 (i.e., the wheel W_5),*
(2) *graph (f) of Fig. 1,*
(3) *graph (k) of Fig. 1,*
(4) *the graph $Y_{n,0}$ for some $n = 1, 2, \dots$.*

3. The set S_ω

We obviously have $S_\tau \subseteq S_\omega$, since $\tau = -(1+\sqrt{5})/2 > -\sqrt{3} = \omega$. Thus, in order to save space, in Lemmas 7–12 below we will specify the graphs belonging to $S_\omega \setminus S_\tau$ only.

LEMMA 7. *The exceptional graphs belonging to $S_\omega \setminus S_\tau$ are $P_4 \bigtriangledown K_2$ and $C_5 \bigtriangledown 2K_1$.*

PROOF. Looking at the tables of [**6**] we see that out of 573 exceptional graphs with least eigenvalue greater than -2, only the graphs $P_4 \bigtriangledown K_2$ and $C_5 \bigtriangledown 2K_1$ belong to $S_\omega \setminus S_\tau$. □

LEMMA 8. *An odd unicyclic graph H such that $L(H) \in S_\omega$ contains an odd cycle of length either 3 or 5.*

PROOF. As in Lemma 2, it is impossible that H contains a cycle of length at least 7, since for $k \geqslant 3$ we have $\lambda_{2k+1}(C_{2k+1}) \leqslant \lambda_7(C_7) \approx -1.8019 < \omega$. □

LEMMA 9. *There exists no unicyclic graph H with a cycle C of length 5 for which $L(H)$ belongs to $S_\omega \setminus S_\tau$.*

PROOF. This follows from the proof of Lemma 3, where it is shown that the line graph of such H has smallest eigenvalue at most $\lambda_6(L(B_1)) \approx -1.7566 < \omega$. □

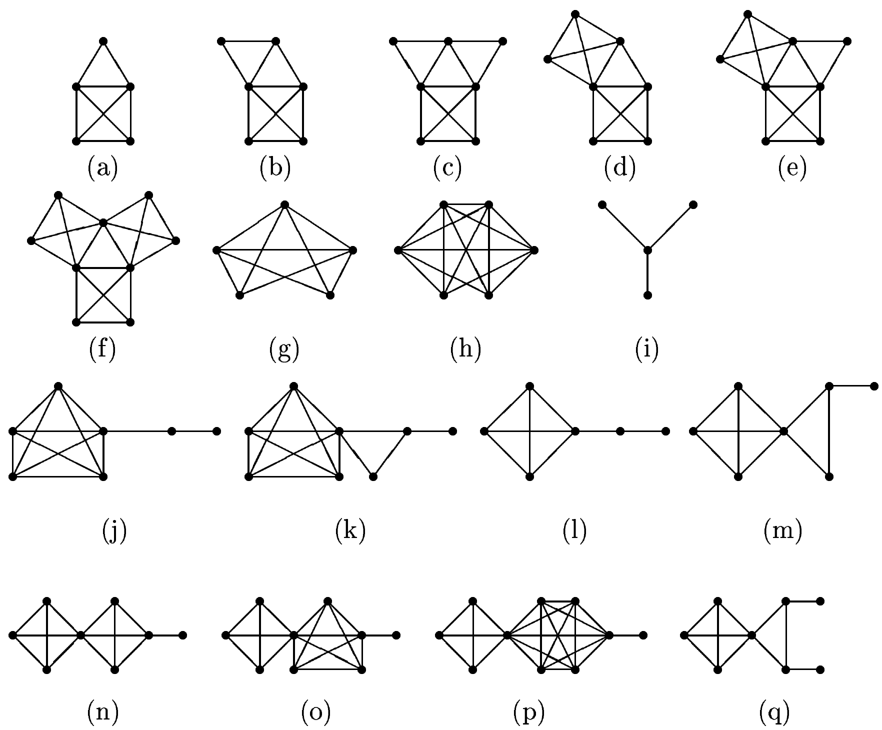

FIGURE 4. Some ω-graphs.

LEMMA 10. *The line graphs of a unicyclic graph with a cycle C of length 3 which belong to $S_\omega \setminus S_\tau$ are those shown in Fig. 4a–f.*

PROOF. Let $G = L(H)$ and suppose that H is a unicyclic graph with a cycle C of length 3 consisting of vertices c_1, c_2, c_3. If any of these vertices has degree at least 5, then H contains as an induced subgraph the graph B_7 from Fig. 5, which is a contradiction, since then $\lambda_n(G) \leqslant \lambda_6(L(B_7)) \approx -1.7466$. Therefore, each of vertices c_1, c_2, c_3 has degree at most 4.

As in the proof of Lemma 4, we may conclude that the possible neighbors of vertices c_1, c_2, c_3 must be pendant vertices. We have already seen in the proof of Lemma 4 that if the degrees of c_1, c_2 and c_3 are at most 3, then all the corresponding line graphs belong to S_τ. Therefore, we only need to consider those graphs where one of these vertices has degree 4. There are six such nonisomorphic graphs, coded by the nonincreasing degrees of vertices c_1, c_2 and c_3:

$$\{(4,2,2),(4,3,2),(4,3,3),(4,4,2),(4,4,3),(4,4,4)\}.$$

The line graphs of all these graphs have least eigenvalue at least $-\sqrt{3}$ (three of them have least eigenvalue strictly greater than $-\sqrt{3}$ while the other three have least eigenvalue equal to $-\sqrt{3}$), and they are shown in Fig. 4a–f. □

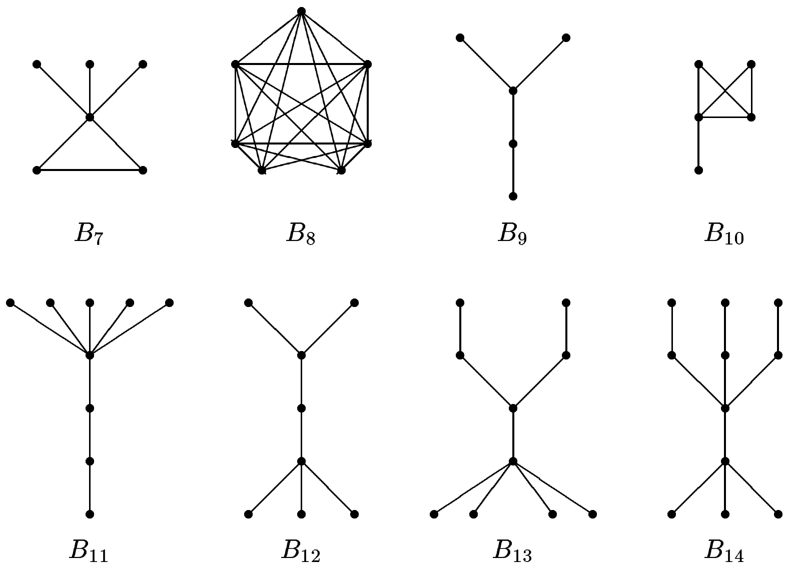

FIGURE 5. Additional forbidden subgraphs.

LEMMA 11. *Suppose that $G = L(T; 1, 0, \dots, 0) \in S_\omega \setminus S_\tau$, where T is a tree with at least one edge. Then G is either one of the graphs (g)–(i) or the graph (d) shown in Fig.* 4.

PROOF. Suppose that $L(T; 1, 0, \dots, 0) \in S_\omega \setminus S_\tau$, where T is a tree with at least one edge. Consider the vertex v_1 of T. The vertices of $L(T; 1, 0, \dots, 0)$, corresponding to the edges e of T having v_1 as its endvertex, are adjacent to both vertices of $CP(1) \cong K_2$ in $L(T; 1, 0, \dots, 0)$.

The vertex v_1 has at most four neighbors in T, since otherwise $L(T; 1, 0, \dots, 0)$ contains as an induced subgraph the graph B_8 from Fig. 5, which is a contradiction, since then $\lambda_n(L(T; 1, 0, \dots, 0)) \leqslant \lambda_7(B_8) \approx -1.7417$.

If there is a vertex w at distance 3 from v_1 in T, then $L(T; 1, 0, \dots, 0)$ contains as an induced subgraph the graph B_9 from Fig. 5, which is a contradiction, since then $\lambda_n(L(T; 1, 0, \dots, 0)) \leqslant \lambda_5(B_9) \approx -1.8478$. Therefore, all other vertices of T are at distance at most 2 from v_1.

Suppose that w is a vertex at distance 2 from v_1 in T, and let v be a common neighbor of w and v_1. If v_1 has another neighbor, say u, then $L(T; 1, 0, \dots, 0)$ contains as an induced subgraph the graph B_{10} from Fig. 5, which is a contradiction, since then $\lambda_n(L(T; 1, 0, \dots, 0)) \leqslant \lambda_5(B_{10}) \approx -1.7491$.

Therefore, either all neighbors of v_1 are pendant vertices, or v_1 is a pendant vertex of a nontrivial star. In the first case, the generalized line graph is isomorphic to a complete graph with one edge deleted. If v_1 has degree 1 or 2, then the two corresponding generalized line graphs belong to S_τ (shown in Fig. 1g and Fig. 1d, respectively). If v_1 has degree 3 or 4, then the two corresponding generalized line graphs belong to $S_\omega \setminus S_\tau$ and they are shown in Fig. 4g and Fig. 4h, respectively.

In the second case, the generalized line graph is isomorphic to a complete graph K_n with two pendant vertices attached to one vertex of K_n. For these graphs, the least eigenvalue is monotonic non-increasing as n increases. For $n = 2$ the corresponding graph has the least eigenvalue $-\sqrt{3}$ and it belongs $S_\omega \setminus S_\tau$ (shown in Fig. 4i). For $n = 3$ the corresponding graph has least eigenvalue -1.8136 and thus none of these graphs belong to $S_\omega \setminus S_\tau$ for $n \geqslant 3$. □

LEMMA 12. *Suppose that $G = L(T) \in S_\omega$, where T is a tree with at least one edge. Then G is one of the following a) a complete graph, b) a graph formed by identifying a pair of vertices of complete graphs K_{m+1} and K_{n+1} where either $n = 2$ and m is arbitrary, or $n = 3$ and $3 \leqslant m \leqslant 9$, or $n = 4$ and $4 \leqslant m \leqslant 5$, c) G is one of the graphs shown in Fig. 4j–q, d) G belongs to the family Y_{k,l_2,l_1}, shown in Fig. 6.*

PROOF. Suppose that $G = L(T) \in S_\omega \setminus S_\tau$, where T is a tree with at least one edge. If T has diameter at least 5, then it contains P_6 as an induced subgraph, which is impossible, since $\lambda_6(P_6) \approx -1.8019 < \omega$. Therefore, T has diameter at most 4.

If T has diameter 1, then $T \cong K_2$ and $L(T) \cong K_1$, which has no edges. If T has diameter 2, then for some $n \in \mathbb{N}$ we have $T \cong K_{1,n}$ and $L(T) \cong K_n$. The complete graphs have smallest eigenvalue equal to -1 and they already belong to S_τ.

If T has diameter 3, then T is isomorphic to a *double star* $D_{m,n}$ for some $m \geqslant n \geqslant 1$. In that case, $L(T)$ is isomorphic to a graph formed by identifying a pair of vertices of complete graphs K_{m+1} and K_{n+1}. We have already seen that the graphs $L(D_{m,1})$ belong to S_τ. Later we will show that the graphs $L(D_{m,2})$ belong to S_ω. If $n \geqslant 3$, then we have that $D_{10,3}$, $D_{6,4}$ and $D_{5,5}$ are the minimal double stars whose least eigenvalue is less than $-\sqrt{3}$. Thus, as new graphs in S_ω we will have the graphs of the form $L(D_{m,2})$ for all $m \in \mathbb{N}$, as well as $L(D_{m,3})$ for $3 \leqslant m \leqslant 9$ and $L(D_{m,4})$ for $4 \leqslant m \leqslant 5$.

Finally, suppose that T has diameter 4, let u and v be two vertices of T with $d(u, v) = 4$ and let c be the unique vertex of T such that $d(c, u) = d(c, v) = 2$. Denote the neighbors of c by $w_1, w_2, \ldots, w_k$. Each of the vertices w_i may be adjacent to at most 4 leaves; otherwise, T contains as an induced subgraph the graph B_{11} from Fig. 5, which has $\lambda_8(L(B_{11})) \approx -1.7350$. Further, if w_i is adjacent to at least 3 leaves for some i, then w_j, for $j \neq i$, may be adjacent to at most one leaf; otherwise, T contains as an induced subgraph the graph B_{12} from Fig. 5, which has $\lambda_7(L(B_{12})) \approx -1.7616$.

Without loss of generality, suppose that the vertices $w_1, w_2, \ldots, w_k$ are ordered by nonincreasing degrees. Then the following cases are possible:

a) w_1 is adjacent to 4 leaves. Then we may suppose that $w_2, \ldots, w_l$ $(2 \leqslant l \leqslant k)$ are each adjacent to one leaf, while $w_{l+1}, \ldots, w_k$ are themselves leaves.

It must be that $l = 2$; otherwise, if $l \geqslant 3$ then T contains as an induced subgraph the graph B_{13} from Fig. 5, which has $\lambda_9(L(B_{13})) \approx -1.7558$.

With $l = 2$, the least eigenvalue of these graphs is monotonic as k increases. For $k = 2$ and $k = 3$ we obtain graphs in S_ω, shown in Fig. 4j and k. The graphs obtained for $k \geqslant 5$ have least eigenvalue less than $-\sqrt{3}$.

b) w_1 is adjacent to 3 leaves. Then we may suppose that $w_2, \dots, w_l$ $(2 \leqslant l \leqslant k)$ are each adjacent to one leaf, while $w_{l+1}, \dots, w_k$ are themselves leaves.

It must be that $l \leqslant 3$; otherwise, if $l \geqslant 4$ then T contains as an induced subgraph the graph B_{14} from Fig. 5, which has $\lambda_{10}(L(B_{14})) \approx -1.7462$.

With l fixed, the least eigenvalue of these graphs is monotonic as k increases. Thus, once the least eigenvalue of a graph from this sequence drops below $-\sqrt{3}$, then all the graphs following it also have the least eigenvalue less than $-\sqrt{3}$.

If $l = 2$, then the corresponding graph belongs to S_ω only for $k \leqslant 6$, giving 5 new graphs in S_ω; they are shown in Fig. 4l-p.

If $l = 3$, then the corresponding graph belongs to S_ω only for $k = 3$, giving one new graph in S_ω; it is shown in Fig. 4q. □

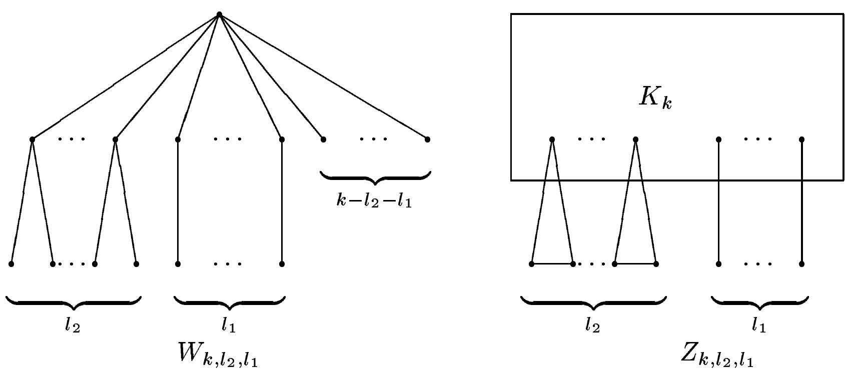

FIGURE 6. Another family of graphs.

Vertices $w_1, \dots, w_{l_2}$ are each adjacent to two leaves, vertices $w_{l_2+1}, \dots, w_{l_2+l_1}$ are each adjacent to one leaf, and vertices $w_{l_2+l_1+1}, \dots, w_k$ are themselves leaves.

We denote such a graph by W_{k,l_2,l_1} (see Fig. 6) and prove that its line graph Z_{k,l_2,l_1} (see Fig. 6) always belongs to S_ω. The easiest way to show this is first to notice that Z_{k,l_2,l_1} is always an induced subgraph of $Z_{k,k,0}$, and then to show that $Z_{k,k,0}$ belongs to S_ω. The graph $Z_{k,k,0}$ can be represented as $K_k \otimes K_2$, where $\otimes$ denotes the corona of graphs. A formula for the characteristic polynomial of the corona of two regular graphs is given in [**3**, p. 50], according to which we have

$$\begin{aligned} &P_{K_p \otimes K_q}(\lambda) \\ &= \Big(\lambda - \frac{q}{\lambda - q + 1} - p + 1\Big)\Big(\lambda - \frac{q}{\lambda - q + 1} + 1\Big)^{p-1} (\lambda - q + 1)^p (\lambda + 1)^{p(q-1)} \\ &= \left(\lambda^2 - (p + q - 2)\lambda + pq - p - 2q + 1\right)\left(\lambda^2 - (q-2)\lambda - 2q + 1\right)^{p-1} (\lambda + 1)^{p(q-1)}. \end{aligned}$$

The first factor above yields simple eigenvalues $\frac{1}{2}(p+q-2 \pm \sqrt{(p-q)^2+4q})$, and the second factor yields eigenvalues $\frac{1}{2}(q-2 \pm \sqrt{q^2+4q})$ of the multiplicity $p-1$. For fixed q, the function $\frac{1}{2}(p+q-2 \pm \sqrt{(p-q)^2+4q})$ is monotone increasing and thus, the least eigenvalue of $K_p \otimes K_q$ is $\frac{1}{2}(q-2-\sqrt{q^2+4q})$, equal to τ for $q=1$ and ω for $q=2$.

Notice also that $L(D_{m,2})$, mentioned above, is just $Z_{m+1,1,0}$, and thus it also belongs to S_ω for arbitrary m.

This ends our search and establishes the following theorem.

THEOREM 4. *The set S_ω consists of connected induced subgraphs of the following graphs:*

(1) *the exceptional graphs $P_4 \bigtriangledown K_2$ and $C_5 \bigtriangledown 2K_1$,*
(2) *one of the graphs (f), (h), (i), (k), (p) and (q) of Fig. 4,*
(3) *the graph formed by identifying a pair of vertices of the complete graphs K_{m+1} and K_{n+1} where $(m,n)=(9,3)$ or $(m,n)=(5,4)$,*
(4) *graph $Z_{k,k,0}$ of Fig. 6 for some $k=1,2,\dots$.*

Theorems 3 and 4 show the existence of some interesting points of a different type when compared to limit points considered by A.J. Hoffman. Namely, the sequence

$$\lambda_{n(q+1)}(K_n \otimes K_q) = \frac{1}{2}\left(q-2-\sqrt{q^2+4q}\right), \qquad n=1,2,\dots$$

is constant for fixed q. We do not know whether other non-trivial such graph sequences of line graphs of trees exist, or what is the relation between the points defined by constant sequences and the limit points considered by A.J. Hoffman.

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References

[1] F. C. Bussemaker, A. Neumaier, *Exceptional graphs with smallest eigenvalue −2 and related problems*, Math. Comput. 59 (1992), 583–608.
[2] D. Cvetković, *On graphs whose second largest eigenvalue does not exceed 1*, Publ. Inst. Math. (Beograd) 31(45) (1982), 15–20.
[3] D. Cvetković, M. Doob, I. Gutman, A. Torgašev, *Recent Results in the Theory of Graph Spectra*, North-Holland, Amsterdam, 1988.
[4] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, 3rd edition, Johann Ambrosius Barth Verlag, Heidelberg, 1995.
[5] D. Cvetković, M. Lepović, P. Rowlinson, S. Simić, *A database of star complements of graphs*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. 9 (1998), 103–112.
[6] D. Cvetković, M. Lepović, P. Rowlinson, S. Simić, *Computer investigations of the maximal exceptional graphs*, University of Stirling, Technical Report CSM-160, Stirling, 2001.
[7] D. Cvetković, M. Petrić, *A table of connected graphs on six vertices*, Discrete Math. 50 (1984), 37–49.
[8] D. Cvetković, P. Rowlinson, S. Simić, *Spectral Generalizations of Line Graphs; A Research Monograph on Graphs with Least Eigenvalue* −2, Cambridge University Press, Cambridge, to appear.
[9] D. Cvetković, S. Simić, *On the graphs whose second largest eigenvalue does not exceed* $(\sqrt{5}-1)/2$, Discrete Math. 138 (1995), 213–227.

[10] D. Cvetković, S. Simić, *The second largest eigenvalue of a graph – a survey*, Filomat 9(3) (1995), Int. Conf. on Algebra, Logic and Discrete Math., Niš, April 14–16, 1995, ed. S. Bogdanović, M. Ćirić, Ž. Perović, 449–472.

[11] M. Doob, D. Cvetković, *On Spectral Characterizations and Embedding of Graphs*, Linear Algebra Appl. 27 (1979), 17–26.

[12] A. J. Hoffman, *On limit points of the least eigenvalue of a graph*, Ars Combinatoria 3 (1977), 3–14.

[13] Y. Hong, *On the least eigenvalue of a graph*, System Sci. Math. Sci. 6 (1993), 269–275.

[14] V. V. Prasolov, *Problems and Theorems in Linear Algebra*, Amer. Math. Soc., Providence, RI, 1994.

[15] X. Yong, *On the distribution of eigenvalues of a simple undirected graph*, Linear Algebra Appl. 295 (1999), 73–80.

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Elektrotehnički fakultet
11000 Beograd
Serbia

ecvetkod@etf.bg.ac.yu

Service de Mathématiques de la Gestion
Université Libre de Bruxelles
Brussels
Belgium

dragance106@yahoo.com