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SOME CLASSES OF INTEGRAL GRAPHS WHICH BELONG TO THE CLASS $\overline{\alpha K_a \cup \beta K_{b,b}}$

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Abstract. Let G be a simple graph and let $\overline{G}$ denote its complement. We say that G is integral if its spectrum consists of integral values. We have recently established a characterization of integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_{b,b}}$, where mG denotes the m-fold union of the graph G. In this work we investigate integral graphs from the class $\overline{\alpha K_a \cup \beta K_{b,b}}$ with $\overline{\lambda}_1 = a+b$, where $\overline{\lambda}_1$ is the largest eigenvalue of $\overline{\alpha K_a \cup \beta K_{b,b}}$.

In this work we consider only simple graphs. The spectrum of a simple graph G of order n contains the eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ of the ordinary adjacency matrix of G, and is denoted by $\sigma(G)$. A graph G is called integral if its spectrum $\sigma(G)$ consists only of integers [**1**].

An eigenvalue μ of G is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j} \rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos\alpha|$ is called the main angle of μ.

Let K_n and $K_{m,n}$ denote the complete graph and the complete bipartite graph, respectively. We have recently described all integral graphs which belong to the classes $\overline{\alpha K_a \cup \beta K_b}$, $\overline{\alpha K_{a,b}}$, $\overline{K_{a,a} \cup K_{b,b}}$ and $\overline{\alpha K_a \cup \beta K_{b,b}}$ (see [**2**], [**3**], [**4**] and [**5**], respectively), where $\overline{G}$ and mG denote the complementary graph of G and the m-fold union of the graph G, respectively.

The characterization of integral graphs which is related to the class $\overline{\alpha K_a \cup \beta K_{b,b}}$ is reduced to the problem of finding the most general integral solution of the following Diophantine equation [**5**]

$$\big[(\alpha+1)a + (2\beta-1)b - 1\big]^2 - 4\alpha a(a-b-1) = \delta^2\,. \tag{1}$$

In other words, $\overline{\alpha K_a \cup \beta K_{b,b}}$ is integral if and only if $(\alpha, \beta, a, b, \delta)$ represents a positive integral solution of the equation (1). We note that $\alpha K_a \cup \beta K_{b,b}$ is an

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integral graph with two main eigenvalues $\mu_a = a-1$ and $\mu_b = b$ for any $\alpha, \beta, a, b \in \mathbb{N}$ with $a \neq (b+1)$.

REMARK 1. We know that $\lambda_1 + \overline{\lambda}_1 \geqslant n-1$ for any graph G of order n with equality if and only if G is regular [**1**], where $\overline{\lambda}_1$ is the largest eigenvalue of $\overline{G}$. If $G = \alpha K_a \cup \beta K_{b,b}$ we obtain (i) $\overline{\lambda}_1 \geqslant 2b+1$ if $a > (b+1)$ and (ii) $\overline{\lambda}_1 \geqslant a+b$ if $a \leqslant b$.

In the sequel the symbol (m,n) denotes the greatest common divisor of integers m, n while $m \mid n$ means that m divides n.

THEOREM 1 (Lepović [**5**]). *If* $\overline{\alpha K_a \cup \beta K_{b,b}}$ *is integral then it belongs to one of the following classes of integral graphs:*

$$\overline{\Big[\pm \frac{2kt}{\tau}\, x_0 + \frac{4mt}{\tau}\, z\Big] K_a \cup \Big[\pm \frac{2kt}{\tau}\, y_0 + \frac{a}{\tau}\, z\Big](2n-1) K_{b,b}}\,, \tag{2}$$

where (i) $a = \pm\big[2t + (2\ell-1)(2n-1)\big]k + (2\ell-1)m + 1$ *and* $b = (2\ell-1)m$*; (ii)* $t, k, \ell, m, n \in \mathbb{N}$ *such that* $(m, 2n-1) = 1$, $(2n-1, 2t) = 1$ *and* $(2\ell-1, 2t) = 1$*; (iii)* $\tau = (a, 4mt)$ *such that* $\tau \mid 2kt$*; (iv)* x_0 *and* y_0 *is a particular solution of the linear Diophantine equation* $ax - (4mt)y = \tau$ *and (v)* $z \geqslant z_0$, *where* $z_0 = \min \mathbb{Z}$ *such that* $(\pm\frac{2kt}{\tau}\, x_0 + \frac{4mt}{\tau}\, z_0) \geqslant 1$ *and* $\big(\pm \frac{2kt}{\tau}\, y_0 + \frac{a}{\tau}\, z_0\big) \geqslant 1$*;*

$$\overline{\Big[\pm \frac{(2t-1)k}{\tau}\, x_0 + \frac{2m(2t-1)}{\tau}\, z\Big] K_a \cup \Big[\pm \frac{(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z\Big](2n-1) K_{b,b}}\,, \tag{3}$$

where (i) $a = \pm\big[(2t-1) + (2\ell-1)(2n-1)\big]k + (2\ell-1)m + 1$ *and* $b = (2\ell-1)m$*; (ii)* $t, k, \ell, m, n \in \mathbb{N}$ *such that* $(m, 2n-1) = 1$, $(2n-1, 2t-1) = 1$ *and* $(2\ell-1, 2t-1) = 1$*; (iii)* $\tau = (a, 2m(2t-1))$ *such that* $\tau \mid (2t-1)k$*; (iv)* x_0 *and* y_0 *is a particular solution of the linear Diophantine equation* $ax - 2m(2t-1)y = \tau$ *and (v)* $z \geqslant z_0$ *where* $z_0 = \min \mathbb{Z}$ *such that* $(\pm\frac{(2t-1)k}{\tau}\, x_0 + \frac{2m(2t-1)}{\tau}\, z_0) \geqslant 1$ *and* $\big(\pm \frac{(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z_0\big) \geqslant 1$*;*

$$\overline{\Big[\pm \frac{(2t-1)k}{\tau}\, x_0 + \frac{(2t-1)m}{\tau}\, z\Big] K_a \cup \Big[\pm \frac{(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z\Big] n K_{b,b}}\,, \tag{4}$$

where (i) $a = \pm\big[(2t-1) + 2\ell n\big]k + \ell m + 1$ *and* $b = \ell m$*; (ii)* $t, k, \ell, m, n \in \mathbb{N}$ *such that* $(m,n) = 1$, $(n, 2t-1) = 1$ *and* $(\ell, 2t-1) = 1$*; (iii)* $\tau = (a, (2t-1)m)$ *such that* $\tau \mid (2t-1)k$*; (iv)* x_0 *and* y_0 *is a particular solution of the linear Diophantine equation* $ax - (2t-1)my = \tau$ *and (v)* $z \geqslant z_0$ *where* $z_0 = \min \mathbb{Z}$ *with* $(\pm\frac{(2t-1)k}{\tau}\, x_0 + \frac{(2t-1)m}{\tau}\, z_0) \geqslant 1$ *and* $\big(\pm \frac{(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z_0\big) \geqslant 1$. *In these classes the symbol '$\pm$' is related to '$+$' if* $a > (b+1)$*; and '$\pm$' is related to '$-$' if* $a \leqslant b$.

If $\overline{\alpha K_a \cup \beta K_{b,b}}$ is an integral graph then it uniquely determines the parameters t, τ, k, ℓ, m, n. However, if x_0 and y_0 is obtained by using the EUCLID algorithm then a fixed integral graph $\overline{\alpha K_a \cup \beta K_{b,b}}$ also uniquely determines the parameters x_0, y_0, z_0, z (see [**5**]).

Using Theorem1 we proved in [**5**] the following results: (i) if $\overline{\alpha K_a \cup \beta K_{b,b}}$ is integral with $\overline{\lambda}_1 = 2b+1$ and $a > (b+1)$ then it is $\overline{K_5 \cup K_{2,2}}$; (ii) if $\overline{\alpha K_a \cup \beta K_{b,b}}$

is integral with $\overline{\lambda}_1 = 2b+1$ and $a \leqslant b$ then it belongs to the class of integral graphs $\overline{3K_t \cup K_{4t-2,4t-2}}$, where $t \in \mathbb{N}$ and (iii) if $\overline{\alpha K_a \cup \beta K_{b,b}}$ is integral with $\overline{\lambda}_1 = a+b$ and $a \leqslant b$ then it is one of the following two integral graphs $\overline{K_2 \cup K_{6,6}}$ or $\overline{K_3 \cup K_{6,6}}$.

The characterization of integral graphs with $\overline{\lambda}_1 = a+b$ and $\lambda_1 = a-1$ is reduced to the problem of finding the most general positive solution of the equation $x^2 - dy^2 = c$, where d is not a perfect square. It is based on the concept of continued fractions and some basic results which are related to $x^2 - dy^2 = c$ (see [**6**]).

Let $a_0, a_1, \ldots, a_n$ be a sequence of integers with $a_i > 0$ for $i \geqslant 1$. Then the term $[a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots, a_{n-1} + \frac{1}{a_n}]$ is called the simple continued fraction, where $[a_0; a_1] = a_0 + \frac{1}{a_1}$. If $a_0, a_1, a_2, \ldots$ is an infinite sequence of integers with $a_i > 0$ for $i \geqslant 1$, the expression $[a_0; a_1, a_2, \ldots] = \lim_{n \to +\infty} [a_0; a_1, \ldots, a_n]$ is called the infinite simple continued fraction. We say that $[a_0; a_1, \ldots, a_{m-1}, \overline{a_m, \ldots, a_{m+r-1}}]$ is an infinite simple continued fraction of periodic r if r is the least positive integer such that $a_{r+n} = a_n$ for any $n \geqslant m$.

Let $a_0, a_1, \ldots$ be a sequence of integers with $a_i > 0$ for $i \geqslant 1$. We then define two associated sequences $\{p_n\}$ and $\{q_n\}$ by $p_i = a_i p_{i-1} + p_{i-2}$ and $q_i = a_i q_{i-1} + q_{i-2}$ for $i \geqslant 0$, where $p_{-2} = 0$, $p_{-1} = 1$ and $q_{-2} = 1$, $q_{-1} = 0$. The rational number $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ is called the n-th convergent to the infinite simple continued fraction.

Next, the general solution of the Pell equation $x^2 - dy^2 = 1$ is given in the form $x_i + y_i\sqrt{d} = (x_1 + y_1\sqrt{d})^i$, where $x_1 + y_1\sqrt{d}$ is its fundamental solution. We know that $x_1 + y_1\sqrt{d} = p_{r-1} + q_{r-1}\sqrt{d}$ if r is even, and $x_1 + y_1\sqrt{d} = p_{2r-1} + q_{2r-1}\sqrt{d}$ if r is odd, where r is the period length of $\sqrt{d}$. If $\rho_0 + \varphi_0\sqrt{d}$ is a fundamental solution of the equation $x^2 - dy^2 = c$, then

$$\rho_i + \varphi_i\sqrt{d} = (\rho_0 + \varphi_0\sqrt{d})(x_1 + y_1\sqrt{d})^i$$

represents a class of solutions of $x^2 - dy^2 = c$. Using the last relation we easily find that $\rho_i = \rho_0 x_i + d\varphi_0 y_i$ and $\varphi_i = \varphi_0 x_i + \rho_0 y_i$ for any $i \geqslant 0$, understanding that $x_0 = 1$ and $y_0 = 0$. Besides, we have

$$\rho_i = \frac{\rho_0 + \varphi_0\sqrt{d}}{2}\left(x_1 + y_1\sqrt{d}\right)^i + \frac{\rho_0 - \varphi_0\sqrt{d}}{2}\left(x_1 - y_1\sqrt{d}\right)^i; \tag{5}$$

$$\varphi_i = \frac{\rho_0 + \varphi_0\sqrt{d}}{2\sqrt{d}}\left(x_1 + y_1\sqrt{d}\right)^i - \frac{\rho_0 - \varphi_0\sqrt{d}}{2\sqrt{d}}\left(x_1 - y_1\sqrt{d}\right)^i. \tag{6}$$

Finally, for any fundamental solution $\rho_0 + \varphi_0\sqrt{d}$ of the equation $x^2 - dy^2 = c$, the following two relations are satisfied [**6**]

$$0 \leqslant |\rho_0| \leqslant \sqrt{\frac{c(x_1+1)}{2}} \quad \text{and} \quad 0 \leqslant \varphi_0 \leqslant y_1\sqrt{\frac{c}{2(x_1+1)}}. \tag{7}$$

Using the concept of continued fractions we proved in [**5**] that there is no integral graph from the class $\overline{\alpha K_a \cup (3\beta+2)K_{b,b}}$ with $\overline{\lambda}_1 = a+b$ and $a > (b+1)$ for any $\beta \in \mathbb{N}$. It is also observed that there is no integral graph from the class $\overline{\alpha K_a \cup \beta K_{b,b}}$ with $\overline{\lambda}_1 = a+b$ and $a > (b+1)$ for $\beta = 1$.

The characterization of integral graphs with $\overline{\lambda}_1 = a + b$ and $\lambda_1 = a - 1$ is reduced to the problem of finding the most general positive integral solution of the following two Diophantine equations:

$$[8\eta(\eta\dot{n} - 1)m - k]^2 - [16\eta\dot{n}(\eta\dot{n} - 1) + 1]k^2 = 16\eta(\eta\dot{n} - 1)\,, \tag{8}$$

where $\dot{n} = 2n - 1$ and $\beta = \eta\dot{n}$; (1.1) $a = (2\ell - 1)(2t - 1)$; (1.2) $b = (2\ell - 1)m$; (1.3) $(2\ell - 1) = 2\eta m + k$ and (1.4) $(2t - 1) = (2\ell - 1)\dot{n} - m$; and

$$[4\eta(\eta n - 1)m - k]^2 - [16\eta n(\eta n - 1) + 1]k^2 = 8\eta(\eta n - 1)\,, \tag{9}$$

where $\beta = \eta n$ and (2.1) $a = (2t - 1)\ell$; (2.2) $b = \ell m$; (2.3) $\ell = \eta m + k$ and (2.4) $(2t - 1) = 2\ell n - m$ (see [**5**]).

Further, let $x = 8\eta(\eta\dot{n} - 1)m - k$ and let $y = k$. Let $d = 16\eta\dot{n}(\eta\dot{n} - 1) + 1$ and let $\rho_0 + \varphi_0\sqrt{d}$ be a fundamental solution of $x^2 - dy^2 = 16\eta(\eta\dot{n} - 1)$. Then $k = \varphi_i$ and $m = \frac{\rho_i + \varphi_i}{8\eta(\eta\dot{n} - 1)}$, understanding that $\rho_i + \varphi_i\sqrt{d}$ is the i-th solution which belongs to the class with respect to $\rho_0 + \varphi_0\sqrt{d}$. It was proved in [**5**] that $8\eta(\eta\dot{n} - 1) \mid (\rho_i + \varphi_i)$ if and only if $8\eta(\eta\dot{n} - 1) \mid (\rho_0 + \varphi_0)$. Consequently, the most general integral solution of (8) is reduced to the positive fundamental solutions $\rho_0 + \varphi_0\sqrt{d}$ for which $8\eta(\eta\dot{n} - 1) \mid (\rho_0 + \varphi_0)$. Similarly, the most general integral solution of (9) is reduced to the positive fundamental solutions $\rho_0 + \varphi_0\sqrt{d}$ for which $4\eta(\eta n - 1) \mid (\rho_0 + \varphi_0)$.

We now proceed to establish a characterization of integral graphs $\overline{\alpha K_a \cup \beta K_{b,b}}$ with $\overline{\lambda}_1 = a + b$ and $a > (b + 1)$ for $\beta = 2, 3, 4$. We note first if $\overline{\alpha K_a \cup \beta K_{b,b}}$ is an integral graph with $\overline{\lambda}_1 = a + b$ and $\lambda_1 = a - 1$ then $(a + b) + (a - 1) \geqslant \alpha a + 2\beta b$ (see Remark1), which implies that $\alpha = 1$.

Proposition 1. *If $\overline{\alpha K_a \cup 2K_{b,b}}$ is integral with $\overline{\lambda}_1 = a + b$ and $a > (b + 1)$ then it belongs to the following class of integral graphs*

$$\overline{K_{a_+ z_+^{2i} + a_- z_-^{2i} + \frac{1}{33}} \cup 2K_{b_+ z_+^{2i} + b_- z_-^{2i} + \frac{7}{33},\, b_+ z_+^{2i} + b_- z_-^{2i} + \frac{7}{33}}}$$

where $z_\pm = 23 \pm 4\sqrt{33}$ and $i \geqslant 0$, $a_\pm = \frac{247 \pm 43\sqrt{33}}{33}$ and $b_\pm = \frac{46 \pm 8\sqrt{33}}{33}$.

Proof. We shall first consider the general positive integral solution of the equation (8) for $\eta\dot{n} = 2$. Clearly, $\dot{n} = 1$ and $\eta = 2$. Then relation (8) is reduced to $x^2 - 33y^2 = 32$. Using a computer program[1] we obtain that $\sqrt{33} = [5; \overline{1, 2, 1, 10}\,]$ and $23 + 4\sqrt{33}$ is the fundamental solution of the equation $x^2 - 33y^2 = 1$. Since $\rho_0 \leqslant 19$ and $\varphi_0 \leqslant 3$ (see (7)), it is easy to verify that there is no fundamental solution of $x^2 - 33y^2 = 32$, which means that (8) does not generate any integral graph with $\beta = 2$.

Consider the general positive integral solution of the equation (9) for $\eta n = 2$. We shall distinguish the following two cases:

Case 1. ($n = 1$ and $\eta = 2$). Then (9) is reduced to (i) $x^2 - 33y^2 = 16$. We now find that $\rho_0 \leqslant 13$ and $\varphi_0 \leqslant 2$, and $4 + 0\sqrt{33}$ and $7 + \sqrt{33}$ are the fundamental

[1]All the results given in Propositions1,2 and3 are obtained by using the program called DIOPHANTUS, written by the author in the programming language C.

solutions of (i). Since $8 \nmid (4+0)$ it follows that the class of solutions of (i) which corresponds to $4+0\sqrt{33}$ does not generate any integral graph with $\beta = 2$. Since $m = \frac{\rho_i+\varphi_i}{8}$ and $k = \varphi_i$, for the fundamental solution $7+\sqrt{33}$, we obtain from (5) and (6) that

$$m = \frac{\sqrt{33}+5}{2\sqrt{33}}\left(23+4\sqrt{33}\right)^i + \frac{\sqrt{33}-5}{2\sqrt{33}}\left(23-4\sqrt{33}\right)^i;$$

$$k = \frac{7+\sqrt{33}}{2\sqrt{33}}\left(23+4\sqrt{33}\right)^i - \frac{7-\sqrt{33}}{2\sqrt{33}}\left(23-4\sqrt{33}\right)^i.$$

Further, making use of (2.1), (2.2), (2.3) and (2.4), from the previous relations we easily get

$$\ell = \frac{3\sqrt{33}+17}{2\sqrt{33}}\left(23+4\sqrt{33}\right)^i + \frac{3\sqrt{33}-17}{2\sqrt{33}}\left(23-4\sqrt{33}\right)^i;$$

$$\dot{t} = \frac{5\sqrt{33}+29}{2\sqrt{33}}\left(23+4\sqrt{33}\right)^i + \frac{5\sqrt{33}-29}{2\sqrt{33}}\left(23-4\sqrt{33}\right)^i,$$

which provides the class of integral graphs represented in Proposition 1, understanding that $\dot{p} = 2p-1$.

Case 2. ($n = 2$ and $\eta = 1$). Then (9) is reduced to (ii) $x^2 - 33y^2 = 8$. We now find that (iii) $\rho_0 \leqslant 9$ and $\varphi_0 \leqslant 1$. Using (iii) it is not difficult to show that there exists no fundamental solution of (ii), which completes the proof. □

PROPOSITION 2. *If* $\overline{\alpha K_a \cup 3K_{b,b}}$ *is integral with* $\overline{\lambda}_1 = a+b$ *and* $a > (b+1)$ *then it belongs to one of the following three classes of integral graphs:*

$$\overline{K_{a_+ z_+^{2i} + a_- z_-^{2i} + \frac{1}{97}} \cup 3K_{b_+ z_+^{2i} + b_- z_-^{2i} + \frac{11}{97},\, b_+ z_+^{2i} + b_- z_-^{2i} + \frac{11}{97}}}$$

where $z_\pm = 62809633 \pm 6377352\sqrt{97}$ *and* $i \geqslant 0$*; and*

(1^0) $a_\pm = \frac{6170687737 \pm 626538413\sqrt{97}}{97}$ *and* $b_\pm = \frac{1309509107 \pm 132960505\sqrt{97}}{194}$*;*

(2^0) $a_\pm = \frac{5188723 \pm 526835\sqrt{97}}{194}$ *and* $b_\pm = \frac{550561 \pm 55901\sqrt{97}}{194}$ *and*

(3^0) $a_\pm = \frac{681412777 \pm 69186985\sqrt{97}}{194}$ *and* $b_\pm = \frac{72302819 \pm 7341239\sqrt{97}}{194}$.

PROOF. We shall first consider the general positive integral solution of the equation (8) for $\eta\dot{n} = 3$.

Case 1.1 ($\dot{n} = 1$ and $\eta = 3$). Then (8) is reduced to (i) $x^2 - 97y^2 = 96$. We now have (ii) $\sqrt{97} = [9; \overline{1,5,1,1,1,1,1,1,5,1,18}]$; (iii) $62809633 + 6377352\sqrt{97}$ is the fundamental solution of the equation $x^2 - 97y^2 = 1$ and (iv) $\rho_0 \leqslant 54907$ and $\varphi_0 \leqslant 5575$. According to (iv) we find that $22+2\sqrt{97}$; $463+47\sqrt{97}$; $2738+278\sqrt{97}$ and $49589+5035\sqrt{97}$ are the fundamental solutions of (i). Since $48 \nmid (22+2)$; $48 \nmid (463+47)$ and $48 \nmid (2738+278)$, these solutions do not generate any integral graph with $\beta = 3$.

Consequently, the general solution of (i) is reduced to the class which corresponds to the fundamental solution $49589+5035\sqrt{97}$. Since $m=\frac{\rho_i+\varphi_i}{48}$ and $k=\varphi_i$, using (iii) and (5), (6), we obtain

$$m=\Big(\frac{1138\sqrt{97}+11208}{2\sqrt{97}}\Big)z_+^i+\Big(\frac{1138\sqrt{97}-11208}{2\sqrt{97}}\Big)z_-^i;$$
$$k=\Big(\frac{49589+5035\sqrt{97}}{2\sqrt{97}}\Big)z_+^i-\Big(\frac{49589-5035\sqrt{97}}{2\sqrt{97}}\Big)z_-^i.$$

Next, making use of (1.1), (1.2), (1.3) and (1.4), by a straight-forward calculation, we get from the last relation that

$$\dot{\ell}=\Big(\frac{11863\sqrt{97}+116837}{2\sqrt{97}}\Big)z_+^i+\Big(\frac{11863\sqrt{97}-116837}{2\sqrt{97}}\Big)z_-^i;$$
$$\dot{t}=\Big(\frac{10725\sqrt{97}+105629}{2\sqrt{97}}\Big)z_+^i+\Big(\frac{10725\sqrt{97}-105629}{2\sqrt{97}}\Big)z_-^i,$$

which provides the class of integral graphs represented in Proposition 2 (1^0).

Case 1.2 ($\dot{n}=3$ and $\eta=1$). Then (8) is reduced to (v) $x^2-97y^2=32$. According to (iii) and (7) we find that (vi) $\rho_0\leqslant 31701$ and $\varphi_0\leqslant 3218$. Using (vi) we get $138+14\sqrt{97}$ and $3063+311\sqrt{97}$ are the fundamental solutions of (v). Since $16\nmid(138+14)$ and $16\nmid(3063+311)$ it follows that (v) generates no integral graph with $\beta=3$.

Consider the general positive integral solution of the equation (9) for $\eta n=3$. We shall also distinguish the following two cases:

Case 2.1 ($n=1$ and $\eta=3$). Then (9) is reduced to (vii) $x^2-97y^2=48$. We now find that $\rho_0\leqslant 38825$ and $\varphi_0\leqslant 3942$; $40+4\sqrt{97}$, $719+73\sqrt{97}$ and $15965+1621\sqrt{97}$ are the fundamental solutions of (vii). Since $24\nmid(40+4)$ and $24\nmid(15965+1621)$ it remains to consider the fundamental solution $719+73\sqrt{97}$. Therefore, by an easy calculation we get $m=(\frac{33\sqrt{97}+325}{2\sqrt{97}})z_+^i+(\frac{33\sqrt{97}-325}{2\sqrt{97}})z_-^i$ and $k=(\frac{719+73\sqrt{97}}{2\sqrt{97}})z_+^i-(\frac{719-73\sqrt{97}}{2\sqrt{97}})z_-^i$, which yields $\ell=(\frac{86\sqrt{97}+847}{\sqrt{97}})z_+^i+(\frac{86\sqrt{97}-847}{\sqrt{97}})z_-^i$ and $\dot{t}=(\frac{311\sqrt{97}+3063}{2\sqrt{97}})z_+^i+(\frac{311\sqrt{97}-3063}{2\sqrt{97}})z_-^i$. So we get the class of integral graphs represented in Proposition 2 (2^0).

Case 2.2 ($n=3$ and $\eta=1$). Then (9) is reduced to (viii) $x^2-97y^2=16$. We now find that $\rho_0\leqslant 22416$ and $\varphi_0\leqslant 2275$; $4+0\sqrt{97}$ and $4757+483\sqrt{97}$ are the fundamental solutions of (viii). Consequently, since $8\nmid(4+0)$ and $8\mid(4757+483)$ we obtain that $m=(\frac{655\sqrt{97}+6451}{2\sqrt{97}})z_+^i+(\frac{655\sqrt{97}-6451}{2\sqrt{97}})z_-^i$; $k=(\frac{4757+483\sqrt{97}}{2\sqrt{97}})z_+^i-(\frac{4757-483\sqrt{97}}{2\sqrt{97}})z_-^i$; $\ell=(\frac{569\sqrt{97}+5604}{\sqrt{97}})z_+^i+(\frac{569\sqrt{97}-5604}{\sqrt{97}})z_-^i$; $\dot{t}=(\frac{6173\sqrt{97}+60797}{2\sqrt{97}})z_+^i+(\frac{6173\sqrt{97}-60797}{2\sqrt{97}})z_-^i$, which provides the class represented in Proposition 2 (3^0). □

PROPOSITION 3. *If* $\overline{\alpha K_a \cup 4K_{b,b}}$ *is integral with* $\overline{\lambda}_1 = a + b$ *and* $a > (b+1)$ *then it belongs to one of the following three classes of integral graphs:*

$$\overline{K_{a_+\, z_+^{2i} + a_-\, z_-^{2i} + \frac{1}{193}} \cup 4K_{b_+\, z_+^{2i} + b_-\, z_-^{2i} + \frac{15}{193},\; b_+\, z_+^{2i} + b_-\, z_-^{2i} + \frac{15}{193}}}$$

where $z_\pm = 6224323426849 \pm 448036604040\sqrt{193}$ *and* $i \geqslant 0$*; and*

(1^0) $a_\pm = \frac{1209056824462393 \pm 87029814579823\sqrt{193}}{193}$ *and*

$b_\pm = \frac{179835915982455 \pm 12944872487449\sqrt{193}}{386}$*;*

(2^0) $a_\pm = \frac{758972 \pm 54632\sqrt{193}}{193}$ *and* $b_\pm = \frac{56445 \pm 4063\sqrt{193}}{193}$*;*

(3^0) $a_\pm = \frac{92695388006569 \pm 6672360030889\sqrt{193}}{386}$ *and* $b_\pm = \frac{6893786823015 \pm 496225633751\sqrt{193}}{386}$.

PROOF. We shall first consider the general positive integral solution of the equation (8) for $\eta\dot{n} = 4$. Clearly, $\dot{n} = 1$ and $\eta = 4$. In this case (8) is reduced to (i) $x^2 - 193y^2 = 192$. We now have (ii) $\sqrt{193} = [13; \overline{1, 8, 3, 2, 1, 3, 3, 1, 2, 3, 8, 1, 26}]$; (iii) $6224323426849 + 448036604040\sqrt{193}$ is the fundamental solution of the Pell equation $x^2 - 193y^2 = 1$ and (iv) $\rho_0 \leqslant 24444530$ and $\varphi_0 \leqslant 1759555$. Using (iv) we find that $112+8\sqrt{193}$; $3362+242\sqrt{193}$; $87703+6313\sqrt{193}$; $871862+62758\sqrt{193}$ and $22743973 + 1637147\sqrt{193}$ are the fundamental solutions of (i). Since $96 \nmid (112+8)$; $96 \nmid (3362+242)$; $96 \nmid (87703+6313)$ and $96 \nmid (871862+62758)$, these solutions do not generate any integral graph with $\beta = 4$.

Thus, the general solution of (i) is reduced to the class which corresponds to the fundamental solution $22743973+1637147\sqrt{193}$. Making use of (iii) and (5), (6), we get implicitly that $m = (\frac{126985\sqrt{193}+1764132}{\sqrt{193}})z_+^i + (\frac{126985\sqrt{193}-1764132}{\sqrt{193}})z_-^i$ and $k = (\frac{22743973+1637147\sqrt{193}}{2\sqrt{193}})z_+^i - (\frac{22743973-1637147\sqrt{193}}{2\sqrt{193}})z_-^i$, which provides that $\dot{\ell} = (\frac{3668907\sqrt{193}+50970085}{2\sqrt{193}})z_+^i + (\frac{3668907\sqrt{193}-50970085}{2\sqrt{193}})z_-^i$; $\dot{t} = (\frac{3414937\sqrt{193}+47441821}{2\sqrt{193}})z_+^i + (\frac{3414937\sqrt{193}-47441821}{2\sqrt{193}})z_-^i$. So we arrive at the class of integral graphs represented in Proposition3 (1^0).

Consider the general positive integral solution of the equation (9) for $\eta n = 4$. We shall distinguish the following three cases:

Case 1. ($n = 1$ and $\eta = 4$). Then (9) is reduced to (v) $x^2 - 193y^2 = 96$. We now find that $\rho_0 \leqslant 17284892$ and $\varphi_0 \leqslant 1244193$; $17 + \sqrt{193}$, $403 + 29\sqrt{193}$, $12142 + 874\sqrt{193}$ and $3148778 + 226654\sqrt{193}$ are the fundamental solutions of (v). Of course, since $48 \nmid (17 + 1)$; $48 \nmid (12142 + 874)$ and $48 \nmid (3148778 + 226654)$, these solutions generate no integral graph with $\beta = 4$. For $403 + 29\sqrt{193}$ we have $m = (\frac{9\sqrt{193}+125}{2\sqrt{193}})z_+^i + (\frac{9\sqrt{193}-125}{2\sqrt{193}})z_-^i$; $k = (\frac{403+29\sqrt{193}}{2\sqrt{193}})z_+^i - (\frac{403-29\sqrt{193}}{2\sqrt{193}})z_-^i$; $\ell = (\frac{65\sqrt{193}+903}{2\sqrt{193}})z_+^i + (\frac{65\sqrt{193}-903}{2\sqrt{193}})z_-^i$ and $\dot{t} = (\frac{121\sqrt{193}+1681}{2\sqrt{193}})z_+^i + (\frac{121\sqrt{193}-1681}{2\sqrt{193}})z_-^i$, which provides the class of integral graphs represented in Proposition 3 (2^0).

Case 2. ($n = 2$ and $\eta = 2$). Then (9) is reduced to (vi) $x^2 - 193y^2 = 48$. We now find that $\rho_0 \leqslant 12222265$ and $\varphi_0 \leqslant 879777$; $56 + 4\sqrt{193}$, $1681 + 121\sqrt{193}$

and $435931+31379\sqrt{193}$ are the fundamental solutions of (vi). Consequently, since $24 \nmid (56+4)$, $24 \nmid (1681+121)$ and $24 \nmid (435931+31379)$, the equation (vi) does not generate any integral graph with $\beta = 4$.

Case 3. ($n = 4$ and $\eta = 1$). Then (9) is reduced to (vii) $x^2 - 193y^2 = 24$. We now find that $\rho_0 \leqslant 8642446$ and $\varphi_0 \leqslant 622096$; $6071 + 437\sqrt{193}$ and $1574389+113327\sqrt{193}$ are the fundamental solutions of (vii). Since $12 \nmid (6071+437)$ and $12 \mid (1574389+113327)$, we obtain for $1574389+113327\sqrt{193}$ that $m = (\frac{140643\sqrt{193}+1953875}{2\sqrt{193}})z_+^i + (\frac{140643\sqrt{193}-1953875}{2\sqrt{193}})z_-^i$ and $k = (\frac{1574389+113327\sqrt{193}}{2\sqrt{193}})z_+^i - (\frac{1574389-113327\sqrt{193}}{2\sqrt{193}})z_-^i$. In this way we obtain that $\ell = (\frac{126985\sqrt{193}+1764132}{\sqrt{193}})z_+^i + (\frac{126985\sqrt{193}-1764132}{\sqrt{193}})z_-^i$ and $\dot{t} = (\frac{1891117\sqrt{193}+26272237}{2\sqrt{193}})z_+^i + (\frac{1891117\sqrt{193}-26272237}{2\sqrt{193}})z_-^i$. Using these relations we obtain Proposition 3 (3^0). □

Table 1 contains the set of all integral graphs[2] from the class $\overline{\alpha K_a \cup \beta K_{b,b}}$, whose order '$o$' does not exceed 30. In this table an integral graph is described by the parameters α, β, a, b and ones presented in the class of integral graphs in Theorem 1. The symbol 'i' denotes the identification number of the corresponding integral graph. In Table 1 (i) graphs with identification numbers $1, 2, \ldots, 18$ belong to the classes represented by (2); (ii) graphs with identification numbers $19, 20, \ldots, 47$ belong to the classes represented by (3); and (iii) graphs with $i = 48, 49, \ldots, 70$ belong to the classes represented by (4). We note that there exist exactly 18, 29 and 23 non-isomorphic integral graphs from the classes described by (2), (3) and (4), respectively. In this table[3] identification number 20 is related to the integral graph with the largest eigenvalue $\overline{\mu}_1 = 2b+1$ and $a > (b+1)$, while identification numbers 4, 19 and 44 are related to the integral graphs with $\overline{\mu}_1 = 2b+1$ and $a \leqslant b$. In Table 1 there exists just one integral graph[4] with $\overline{\mu}_1 = (a+b)$ and $a > (b+1)$ and its identification number is 64 – the first next one has 12545 vertices. Identification numbers 24 and 50 are related to the integral graphs with $\overline{\mu}_1 = (a+b)$ and $a \leqslant b$.

There exist exactly 7556 non-isomorphic integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_{b,b}}$, whose order does not exceed 300. In particular, the total number of such integral graphs (obtained by using (2), (3) and (4)) is $(1433+888)$, $(1265+948)$ and $(1736+1286)$, respectively, where m and n in the expression $(m+n)$ are the numbers of integral graphs with $a > (b+1)$ and $a \leqslant b$, respectively. Table 2 contains a distribution of those graphs with respect to their orders. In Table 2 the number n in the symbol o^n denotes the number of integral graphs of the corresponding order $o = 1, 2, \ldots, 300$. In this table o^n is omitted if the corresponding number $n = 0$.

[2] The data given in Tables 1 and 2 are obtained in two different ways: (i) they are generated by using relations (2), (3) and (4); and (ii) by varying the parameters α, β, a, b in all possible ways in equation (1).

[3] In Tables 1 and 3 the number $\overline{\mu}_2$ denotes the second main eigenvalue of the corresponding integral graph $\overline{\alpha K_a \cup \beta K_{b,b}}$.

[4] For any integral graph $\overline{\alpha K_a \cup \beta K_{b,b}}$ with the largest eigenvalue $\overline{\mu}_1 = a+b$ we have (i) $\overline{\mu}_2 = -\frac{2\beta ab}{a+b}$ and (ii) $(a+b)(a+2b+1) = 2\beta b(2a+b)$ (see the proof of Theorem1).

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	$\overline{\mu}_1$	$\overline{\mu}_2$
1	0	−1	1	10	1	1	8	1	4	1	2	1	1	1	4	−4
2	1	1	0	14	2	2	5	1	1	1	1	1	1	1	10	−3
3	1	0	1	16	10	1	1	3	1	1	1	1	3	1	14	−3
4	1	0	1	18	3	1	2	6	2	1	1	2	2	1	13	−4
5	−1	−1	1	20	2	1	9	1	3	3	1	1	1	1	12	−3
6	7	4	−1	20	2	1	7	3	1	1	1	1	3	1	14	−5
7	1	2	−1	22	1	1	18	2	2	1	5	1	2	1	9	−8
8	0	−1	1	22	1	3	16	1	8	2	2	1	1	2	12	−8
9	0	−1	2	22	2	3	8	1	4	1	2	1	1	1	16	−4
10	1	0	1	22	7	1	2	4	2	1	1	1	4	1	19	−4
11	0	−1	1	24	1	2	20	1	4	1	6	1	1	1	10	−8
12	0	−1	1	24	1	6	12	1	4	1	2	1	1	2	18	−8
13	−1	−1	1	26	3	2	6	2	2	1	1	1	2	1	21	−4
14	1	1	0	28	2	2	12	1	4	2	2	1	1	1	18	−4
15	−7	−3	−1	28	2	1	5	9	1	1	1	2	3	1	20	−7
16	−13	−2	−1	28	6	1	3	5	1	1	1	1	5	1	24	−5
17	3	8	−4	30	2	4	11	1	1	1	3	1	1	1	22	−5
18	1	1	0	30	1	5	10	2	2	1	1	1	2	3	25	−8
19	1	0	1	7	3	1	1	2	1	1	1	1	2	1	5	−2
20	1	1	0	9	1	1	5	2	1	1	1	1	2	1	5	−4
21	0	−1	1	10	1	2	6	1	2	1	2	1	1	1	6	−4
22	1	1	0	14	1	1	8	3	2	1	2	1	3	1	8	−6
23	1	0	1	14	6	1	1	4	1	1	2	1	4	1	11	−3
24	−1	−1	0	15	1	1	3	6	1	1	1	2	2	1	9	−4
25	1	0	2	15	7	2	1	2	1	1	1	1	2	1	13	−2
26	0	−1	1	16	1	3	10	1	2	1	4	1	1	1	10	−6
27	1	0	1	18	4	1	2	5	2	1	2	1	5	1	14	−4
28	3	4	−1	19	1	1	11	4	1	1	3	1	4	1	11	−8
29	0	−1	1	20	1	1	18	1	6	2	4	1	1	1	6	−6
30	1	3	−1	21	1	2	13	2	1	1	5	1	2	1	13	−8
31	1	0	1	21	9	1	1	6	1	1	3	1	6	1	17	−4
32	1	0	1	22	2	1	2	9	2	1	2	2	3	1	14	−4
33	0	−1	1	22	1	4	14	1	2	1	6	1	1	1	14	−8
34	0	−1	2	22	2	5	6	1	2	1	2	1	1	1	18	−4
35	1	0	3	23	11	3	1	2	1	1	1	1	2	1	21	−2
36	3	4	−1	24	1	1	14	5	2	1	4	1	5	1	14	−10
37	0	−1	1	26	1	1	20	3	10	3	2	2	1	1	12	−10
38	−3	−1	0	26	3	1	4	7	2	1	2	1	7	1	20	−6
39	3	5	−1	26	2	3	7	2	1	1	2	1	2	1	21	−5
40	0	−1	1	28	1	5	18	1	2	1	8	1	1	1	18	−10
41	1	0	1	28	12	1	1	8	1	1	4	1	8	1	23	−5
42	5	7	−2	29	1	1	17	6	1	1	5	1	6	1	17	−12
43	1	0	1	29	5	1	1	12	1	1	3	2	4	1	19	−4
44	1	0	1	29	3	1	3	10	3	2	1	3	2	1	21	−6
45	1	0	1	29	21	1	1	4	1	2	1	1	4	1	27	−4

Table 1

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	$\overline{\mu}_1$	$\overline{\mu}_2$
46	−1	−1	1	30	2	1	12	3	6	2	2	1	3	1	20	−6
47	1	0	2	30	14	2	1	4	1	1	2	1	4	1	27	−3
48	1	0	1	8	2	1	1	3	1	1	1	1	3	1	5	−2
49	0	−1	1	13	1	4	5	1	1	1	1	1	1	1	10	−4
50	−1	−1	0	14	1	1	2	6	1	1	1	2	3	1	8	−3
51	0	−1	1	16	1	2	12	1	3	2	2	1	1	1	8	−6
52	1	0	1	16	4	1	1	6	1	1	2	1	6	1	11	−3
53	0	−1	1	17	1	4	9	1	3	2	1	1	1	2	12	−6
54	1	0	2	17	5	2	1	3	1	1	1	1	3	1	14	−2
55	0	−1	1	18	1	1	14	2	7	4	1	2	1	1	8	−7
56	1	2	0	19	1	2	7	3	1	1	1	1	3	1	14	−6
57	0	−1	1	20	1	6	8	1	1	1	2	1	1	1	16	−6
58	−5	−2	−1	22	2	1	3	8	1	1	2	1	8	1	15	−5
59	1	0	1	22	12	1	1	5	1	2	1	1	5	1	19	−4
60	1	0	1	23	3	1	1	10	1	1	2	2	5	1	14	−3
61	1	0	1	24	6	1	1	9	1	1	3	1	9	1	17	−4
62	1	0	1	24	4	2	1	5	1	1	1	1	5	2	19	−2
63	1	0	3	26	8	3	1	3	1	1	1	1	3	1	23	−2
64	0	−1	1	27	1	2	15	3	5	3	1	3	1	1	18	−10
65	0	−1	1	27	1	8	11	1	1	1	3	1	1	1	22	−8
66	0	−1	1	28	1	4	20	1	5	3	2	1	1	2	16	−10
67	0	−1	2	28	2	9	5	1	1	1	1	1	1	1	25	−4
68	−3	−2	0	29	3	2	3	5	1	1	1	1	5	1	24	−4
69	0	−1	1	30	1	2	22	2	11	6	1	2	1	2	16	−11
70	−4	−1	−1	30	3	1	2	12	1	2	1	4	3	1	20	−5

TABLE 1. (continued)

007^{01}	008^{01}	009^{01}	010^{02}	013^{01}	014^{04}	015^{02}	016^{04}	017^{02}	018^{03}	019^{02}
020^{04}	021^{02}	022^{09}	023^{02}	024^{05}	026^{05}	027^{02}	028^{07}	029^{05}	030^{06}	031^{05}
032^{10}	033^{04}	034^{21}	035^{04}	036^{07}	037^{02}	038^{11}	039^{02}	040^{10}	041^{01}	042^{06}
043^{07}	044^{16}	045^{06}	046^{22}	047^{02}	048^{12}	049^{05}	050^{13}	051^{06}	052^{14}	053^{04}
054^{17}	055^{03}	056^{10}	057^{05}	058^{22}	059^{06}	060^{18}	061^{10}	062^{27}	063^{06}	064^{15}
065^{05}	066^{19}	067^{07}	068^{16}	069^{09}	070^{18}	071^{12}	072^{12}	073^{08}	074^{29}	075^{03}
076^{34}	077^{04}	078^{24}	079^{07}	080^{20}	081^{04}	082^{29}	083^{06}	084^{22}	085^{04}	086^{23}
087^{06}	088^{22}	089^{10}	090^{22}	091^{09}	092^{26}	093^{14}	094^{34}	095^{12}	096^{31}	097^{09}
098^{33}	099^{09}	100^{21}	101^{07}	102^{37}	103^{13}	104^{30}	105^{07}	106^{46}	107^{11}	108^{29}
109^{10}	110^{23}	111^{11}	112^{29}	113^{05}	114^{37}	115^{09}	116^{34}	117^{07}	118^{40}	119^{11}

TABLE 2

120^{30}	121^{12}	122^{31}	123^{12}	124^{42}	125^{11}	126^{30}	127^{11}	128^{30}	129^{15}	130^{37}
131^{08}	132^{36}	133^{12}	134^{45}	135^{12}	136^{39}	137^{13}	138^{48}	139^{15}	140^{35}	141^{11}
142^{50}	143^{13}	144^{39}	145^{10}	146^{42}	147^{07}	148^{44}	149^{15}	150^{35}	151^{09}	152^{30}
153^{16}	154^{33}	155^{15}	156^{43}	157^{14}	158^{47}	159^{18}	160^{49}	161^{10}	162^{47}	163^{12}
164^{50}	165^{10}	166^{57}	167^{11}	168^{39}	169^{10}	170^{33}	171^{09}	172^{53}	173^{10}	174^{50}
175^{08}	176^{51}	177^{09}	178^{51}	179^{10}	180^{30}	181^{12}	182^{35}	183^{17}	184^{47}	185^{07}
186^{49}	187^{12}	188^{56}	189^{17}	190^{62}	191^{17}	192^{40}	193^{21}	194^{60}	195^{19}	196^{53}
197^{20}	198^{47}	199^{19}	200^{33}	201^{13}	202^{61}	203^{14}	204^{76}	205^{15}	206^{54}	207^{18}
208^{49}	209^{13}	210^{41}	211^{11}	212^{58}	213^{12}	214^{69}	215^{15}	216^{47}	217^{12}	218^{59}
219^{14}	220^{49}	221^{14}	222^{65}	223^{13}	224^{40}	225^{17}	226^{69}	227^{18}	228^{48}	229^{16}
230^{55}	231^{20}	232^{47}	233^{18}	234^{60}	235^{18}	236^{55}	237^{20}	238^{64}	239^{13}	240^{55}
241^{24}	242^{64}	243^{13}	244^{74}	245^{13}	246^{68}	247^{11}	248^{56}	249^{25}	250^{73}	251^{16}
252^{53}	253^{20}	254^{68}	255^{22}	256^{57}	257^{10}	258^{73}	259^{16}	260^{57}	261^{22}	262^{55}
263^{17}	264^{66}	265^{16}	266^{50}	267^{12}	268^{66}	269^{14}	270^{51}	271^{17}	272^{57}	273^{21}
274^{71}	275^{19}	276^{83}	277^{17}	278^{65}	279^{28}	280^{52}	281^{17}	282^{75}	283^{20}	284^{84}
285^{24}	286^{72}	287^{19}	288^{60}	289^{15}	290^{62}	291^{23}	292^{66}	293^{10}	294^{77}	295^{16}
296^{80}	297^{13}	298^{70}	299^{14}	300^{67}						

TABLE 2. (continued)

a	b	o	$\overline{\mu}_1$	$\overline{\mu}_2$
7865	585	12545	8450	−4356
53492	5676	87548	59168	−30789
7024874	745390	11497214	7770264	−4043315
127230675	13500094	208231239	140730769	−73230300
480286984490	35719102710	766039806170	516006087200	−265972368231
12529086263859	931792310790	19983424750179	13460878574649	−6938332399120

TABLE 3

Table 3 contains the integral graphs $\overline{\alpha K_a \cup \beta K_{b,b}}$ with $\overline{\mu}_1 = a + b$ and $a > (b+1)$, obtained from the classes represented in Propositions2 and3 for $i = 0$. We note that any graph in this list is an integral graph with the minimal number of vertices for the corresponding class. The first, second, ..., sixth integral graph in List 3 belongs to the class described in Proposition m (n^0), where $(m, n) = (3, 2)$, (2,2), (2,3), (2,1), (3,3) and (3,1), respectively.

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