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ON SOME AKIVIS–GOLDBERG TYPE METRICS

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Dedicated to Professor Dr. Mileva Prvanović

ABSTRACT. We investigate Akivis–Goldberg type metrics satisfying some additional assumptions.

1. Introduction

Let M be a manifold of dimension $n = pq$, and let $SC(p, q)$ be a differentiable field of Segre cones $SC_x(p, q) \subset T_xM$, $x \in M$. The pair $(M, SC(p, q))$ is called an *almost Grassmann structure* and is denoted by $AG(p-1, p+q-1)$. The manifold M endowed with such structure is said to be an *almost Grassmann manifold* (e.g., see [**1**, Definition 1.1]). Some additional conditions lead to so-called semiintegrable almost Grassmann structures [**1**, Definition 1.2]. The latter were studied in [**1**] and examples of such structures, mainly 4-dimensional, are presented there. Certain semi-Riemannian metrics are related to these structures (see Examples 3.5–3.16 of [**1**]). These metrics are called *Akivis–Goldberg*, in short *AG-metrics* [**20**]. Manifolds admitting *AG*-metrics will be called *AG-manifolds*. Curvature properties and, in particular, curvature properties of pseudosymmetry type of *AG*-manifolds were obtained in [**20**]. For instance, on such manifolds we have [**20**]

$$\operatorname{rank} S \leqslant 2, \tag{1.1}$$

$$(i)\quad S^2 = 0, \qquad (ii)\quad \kappa = 0, \qquad (iii)\quad S \cdot C = 0. \tag{1.2}$$

For precise definitions of the symbols used, we refer to Section **2** of this paper. We note that (1.2)(iii), by making use of (1.2)(i), (1.2)(ii) and the identity

$$S \cdot C = S \cdot R + \frac{4}{n-2}\overline{S} + \frac{2}{n-2} g \wedge S^2 - \frac{2\kappa}{(n-2)(n-1)} g \wedge S, \tag{1.3}$$

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turns into $S \cdot R = -\frac{4}{n-2}\overline{S}$. Moreover, on every AG-manifold (M, g) the following condition of pseudosymmetry type is satisfied [**20**]

$$R \cdot R - Q(S, R) = L_C Q(g, C), \tag{1.4}$$

where L_C is some function on $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$. With respect to the above presentation of curvature properties of AG-manifolds we can define the following extension of this class of manifolds.

Let (M, g), $n \geqslant 4$, be a semi-Riemannian manifold such that $U_C \cap U_S \subset M$ is a nonempty set, where $U_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$. The metric g will be called an *Akivis–Goldberg type metric*, in short an *AG type metric* if on $U_C \cap U_S$ the following three conditions are fulfilled: (1.4),

$$S \cdot R = L_1 \overline{S} + L_2 g \wedge S + L_3 G, \tag{1.5}$$

$$S^2 = L_4 S + L_5 g, \tag{1.6}$$

where $L_1, \ldots, L_5$ are some functions on $U_C \cap U_S$. A manifold admitting an AG type metric will be called an *Akivis–Goldberg type manifold*, in short an *AG type manifold*. Evidently, every AG manifold is an AG type manifold. The converse statement is not true. In Section 3 we present examples of AG type manifolds. In particular, we state that every semi-Riemannian manifold satisfying the Roter type equation [**9**] is an AG type manifold. Some AG type manifolds satisfy also (1.1). In Section **2** we prove (see Corollary 2.1) that if an AG type manifold (M, g) satisfies on $U_C \cap U_S \subset M$ the condition

$$\operatorname{rank} S = 2 \tag{1.7}$$

then (1.6) reduces on $U_C \cap U_S$ to

$$S^2 = \frac{\kappa}{2} S. \tag{1.8}$$

In Remark 3.1 (v) and (vi) we present examples of AG type manifolds satisfying (1.7). These manifolds can be locally realized as hypersurfaces of semi-Euclidean spaces. In the last section we consider hypersurfaces M in semi-Riemannian spaces of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geqslant 4$, or in particular, in semi-Euclidean spaces $\mathbb{E}_s^{n+1}$, with nonempty set $U_C \cap U_S \subset M$, satisfying on this set (1.4), (1.5) and (1.6). It means that the metric g induced on M from the metric of the ambient space is an AG type metric. Hypersurfaces M, with nonempty set $U_C \cap U_S \subset M$, satisfying on this set (1.4), (1.5) and (1.6) will be called *Akivis–Goldberg type hypersurfaces*, in short *AG type hypersurfaces*.

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$. We denote by U_H the set of all points of M at which the tensor H^2 is not a linear combination of H and g. Using (2.18) and Theorem 4.1 of [**19**] we can deduce that $U_H \subset U_C \cap U_S \subset M$. AG type hypersurfaces in $N_s^{n+1}(c)$, $n \geqslant 4$, are also investigated in [**22**] and [**23**]. Among others things in [**22**] it was shown that (1.4), (1.5) and (1.6) hold on $U_C \cap U_S - U_H$. Therefore we restrict our considerations on AG type hypersurfaces M in $N_s^{n+1}(c)$ to the set $U_H \subset M$. We mention that an extension of the class of AG type manifolds was introduced in [**22**] (see also [**23**]).

Our main result states (see Theorem 4.1) that if M is an AG type hypersurface in $\mathbb{E}_s^{n+1}$, $n \geqslant 5$, the set $U_H \subset M$ is nonempty, and (1.7) holds on U_H, then the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent at all points of U_H at which $\kappa \neq 0$. An example of a semisymmetric AG type hypersurface, with $\kappa \neq 0$, is given in Section **3** (see Remark 3.1(v)). That hypersurface satisfies

$$R = \frac{2}{\kappa}\overline{S}. \tag{1.9}$$

2. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M, g) be an n-dimensional, $n \geqslant 3$, semi-Riemannian manifold. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. We define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z), Y$$
$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2}\Big(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}X \wedge_g Y\Big),$$

respectively, where $X, Y, Z \in \Xi(M)$ and A is a symmetric $(0, 2)$-tensor. Now the Riemann-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the $(0, 4)$-tensor G of (M, g) are defined by

$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$
$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$
$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$

respectively, where $X, Y, Z, X_1, X_2, \ldots \in \Xi(M)$. Let $\mathcal{B}(X, Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be a $(0, 4)$-tensor associated with $\mathcal{B}(X, Y)$ by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \tag{2.1}$$

The tensor B is said to be a *generalized curvature tensor* if

$$B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0,$$
$$B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2).$$

For a generalized curvature tensor B we denote by $\mathrm{Ric}(B)$, $\mathrm{Weyl}(B)$ and $\kappa(B)$ the Ricci tensor, the Weyl tensor and the scalar curvature of B, respectively. The subsets U_B, $U_{\mathrm{Ric}(B)}$ and $U_{\mathrm{Weyl}(B)}$ are defined in the same way as the subsets U_R, U_S and U_C, respectively. Clearly, the tensors R, C and G are generalized curvature tensors. For symmetric $(0, 2)$-tensors E and F we denote by $E \wedge F$ their Kulkarni–Nomizu product. The tensor $E \wedge F$ is also a generalized curvature tensor. For a symmetric $(0,2)$-tensor E we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E} = \frac{1}{2}E \wedge E$. In particular, we have $\overline{g} = G = \frac{1}{2}g \wedge g$. Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the tensor defined by (2.1). We extend the endomorphism $\mathcal{B}(X,Y)$ to derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y) \cdot f = 0$ for any smooth function on M. Now for a $(0,k)$-tensor field T, $k \geqslant 1$, and a symmetric $(0,2)$-tensor A we can define the $(0,k+2)$-tensors $B \cdot T$ and $Q(A,T)$ and the $(0,k)$-tensor $A \cdot T$. For the definition of these tensors we refer, for instance, to [**2**] or [**13**]. Setting $T = R$, $T = C$ or $T = S$ and $A = g$ or $A = S$ we obtain the tensors: $S \cdot R$, $S \cdot C$, $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $C \cdot S$, $Q(g,R)$, $Q(g,C)$, $Q(g,S)$, $Q(S,R)$, and $Q(S,C)$. The tensors $C \cdot R$, $C \cdot C$ and $C \cdot S$ are defined in the same manner as the tensors $R \cdot R$ and $R \cdot S$, respectively.

A semi-Riemannian manifold (M,g), $n \geqslant 3$, is called a *quasi-Einstein manifold* if its Ricci tensor S has the form

$$S = \alpha g + \epsilon w \otimes w, \qquad \epsilon = \pm 1, \tag{2.2}$$

for some function α and 1-form w on M. We refer to [**2**] for a review of results on quasi-Einstein manifolds. AG type quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature are investigated in [**23**].

A semi-Riemannian manifold (M,g), $n \geqslant 3$, is said to be *pseudosymmetric* if at every point of M the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent. This is equivalent to

$$R \cdot R = L_R Q(g,R) \tag{2.3}$$

on $U_R = \{x \in M \mid R - \frac{\kappa}{(n-1)n}G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. We note that $U_C \subset U_R$ and $U_S \subset M$. The class of pseudosymmetric manifolds is an extension of the class of *semisymmetric manifolds* ($R \cdot R = 0$). A semi-Riemannian manifold (M,g), $n \geqslant 3$, is said to be *Ricci-pseudosymmetric* if at every point of M the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent. This is equivalent to

$$R \cdot S = L_S Q(g,S) \tag{2.4}$$

on U_S, where L_S is some function on U_S. We say that (2.3) and (2.4) are certain *conditions of pseudosymmetry type* [**2**], [**12**]. The class of Ricci-pseudosymmetric manifolds is an extension of the class of *Ricci-semisymmetric* manifolds ($R \cdot S = 0$) as well as of the class of pseudosymmetric manifolds. Some geometrical considerations show that (2.3), resp., (2.4), is a more natural curvature condition than the condition $R \cdot R = 0$, resp. $R \cdot S = 0$. For a presentation of facts related to these statements and, in general, on pseudosymmetry type conditions we refer to [**2**] and [**12**].

LEMMA 2.1. *Let (M,g), $n \geqslant 3$, be a semi-Riemannian manifold and let A be a nonzero symmetric $(0,2)$-tensor at $x \in M$.*

(i) If

$$\operatorname{rank} A = 2 \tag{2.5}$$

at x, then at x we have

$$A^3 = \operatorname{tr}(A)A^2 + \frac{\operatorname{tr}(A^2) - (\operatorname{tr}(A))^2}{2}A. \tag{2.6}$$

Moreover, if

$$A^2 = \alpha A + \beta g, \qquad \alpha, \beta \in \mathbb{R}, \tag{2.7}$$

at x, then at x we have

$$A^2 = \frac{\operatorname{tr}(A)}{2}A. \tag{2.8}$$

(ii) If $\operatorname{rank} A \leqslant 2$ *and*

$$A = \alpha g + \epsilon w \otimes w, \qquad \alpha \in \mathbb{R}, \quad \epsilon = \pm 1, \quad w \in T_x^* M, \tag{2.9}$$

at x and w is nonzero, then at x we have $\operatorname{rank} A = 1$.

Proof. (i) It is clear that (2.5) is equivalent to

$$A_{il}(A_{hk}A_{jm} - A_{hm}A_{jk}) + A_{jl}(A_{ik}A_{hm} - A_{im}A_{hk}) + A_{hl}(A_{jk}A_{im} - A_{ik}A_{jm}) = 0.$$

Contracting this with g^{hk} and g^{jl} we obtain

$$\operatorname{tr}(A)(A_{il}A_{jm} - A_{im}A_{jl}) + A_{jl}A^2_{im} + A_{im}A^2_{jl} - A_{il}A^2_{jm} - A_{jm}A^2_{il} = 0 \tag{2.10}$$

and (2.6), respectively. Further, substituting (2.7) into (2.10) we get

$$(\operatorname{tr}(A) - 2\alpha)A \wedge A = 2\beta g \wedge A. \tag{2.11}$$

We suppose that $\operatorname{tr}(A) - 2\alpha \neq 0$ at x. Now (2.11) yields

$$A \wedge A = \frac{2\beta}{\operatorname{tr}(A) - 2\alpha} g \wedge A. \tag{2.12}$$

We note that from (2.5) it follows that A is not proportional to g. Thus (2.12), in view of Lemma 3.1 of [**21**], implies $\beta = 0$ and, in a consequence, $\operatorname{rank} A = 1$, a contradiction. Therefore $2\alpha = \operatorname{tr}(A)$. Now (2.11) reduces to $\beta g \wedge A = 0$ whence $\beta(A - \frac{\operatorname{tr}(A)}{n}g) = 0$, and in a consequence, $\beta = 0$, completing the proof of (i).
(ii) We suppose that (2.5) holds at x. From (2.9) we have

$$A_{ij} = \alpha g_{ij} + \epsilon w_i w_j, \tag{2.13}$$

$$A^2_{ij} = \alpha A_{ij} + \epsilon w^r A_{ri} w_j, \qquad w^r = g^{rs} w_s. \tag{2.14}$$

(2.14) yields $w^r A_{ri} w_j = w^r A_{rj} w_i$ whence

$$w^r A_{ri} = \lambda w_i, \qquad \lambda \in \mathbb{R}. \tag{2.15}$$

Now (2.14) turns into $A^2_{ij} = \alpha A_{ij} + \epsilon\lambda w_i w_j$, which by making use of (2.8) and (2.9) gives $(\alpha + \lambda - \frac{\operatorname{tr}(A)}{2})A = \alpha\lambda g$. This implies $\alpha + \lambda = \frac{\operatorname{tr}(A)}{2}$ and $\alpha\lambda = 0$. We suppose that $\alpha \neq 0$. Now the last two relations yield

$$(a) \quad \lambda = 0, \qquad (b) \quad \alpha = \operatorname{tr}(A)/2. \tag{2.16}$$

Evidently, (2.15) by (2.16)(a) reduces to $w^r A_{ri} = 0$. Now, contracting (2.13) with g^{ij} and transvecting with w^j, respectively, and using (2.16)(b) we obtain $\frac{n-2}{2}\operatorname{tr}(A) + \epsilon w^r w_r = 0$ and $\operatorname{tr}(A) + \epsilon w^r w_r = 0$, respectively. These relations imply

$\operatorname{tr}(A) = 0$, which by (2.16)(b) yields $\alpha = 0$, a contradiction. Since $\alpha = 0$, (2.9) reduces to $A = \epsilon w \otimes w$, completing the proof. $\square$

COROLLARY 2.1. *Let (M, g), $n \geqslant 4$, be a semi-Riemannian manifold.*
(i) If (1.6) and (1.7) are satisfied on $U_S \subset M$, then (1.8) holds on this set.
(ii) If (1.1) and (2.2) are satisfied at every point of $U_S \subset M$, then $\operatorname{rank} S = 1$ *on this set.*

Let M, $n \geqslant 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold (N, g^N). We denote by g the metric tensor induced on M from the metric tensor g^N. Further, we denote by ∇ and ∇^N the Levi-Civita connections corresponding to the metric tensors g and g^N, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = g^N(\xi, \xi) = \pm 1$. We can present the Gauss formula and the Weingarten formula of (M, g) in (N, g^N) in the form: $\nabla^N_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where X, Y are vector fields tangent to M, H is the second fundamental tensor of (M, g) in (N, g^N), $\mathcal{A}$ is the shape operator and $H^k(X, Y) = g(\mathcal{A}^k X, Y)$, $k \geqslant 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and R^N the Riemann–Christoffel curvature tensors of (M, g) and (N, g^N), respectively. The *Gauss equation* of (M, g) in (N, g^N) has the form $R(X_1, \dots, X_4) = R^N(X_1, \dots, X_4) + \varepsilon \overline{H}(X_1, \dots, X_4)$, where $\overline{H} = \frac{1}{2} H \wedge H$ and $X_1, \dots, X_4$ are vector fields tangent to M. Let the equations $x^r = x^r(y^k)$ be the local parametric expression of (M, g) in (N, g^N), where y^k and x^r are the local coordinates of M and N, respectively, and $a, b, h, i, j, k, l, m \in \{1, 2, \dots, n\}$ and $p, r, t, u \in \{1, 2, \dots, n+1\}$.

Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geqslant 4$, $c = \frac{\tau}{n(n+1)}$, where τ denote the scalar curvature of the ambient space. Now the Gauss reads (see e.g. [**14**])

$$R_{hijk} = \varepsilon \overline{H}_{hijk} + \frac{\tau}{n(n+1)} G_{hijk}, \tag{2.17}$$

where R_{hijk}, G_{hijk}, H_{hk} and $\overline{H}_{hijk} = H_{hk}H_{ij} - H_{hj}H_{ik}$ denote the local components of the tensors R, G, H and $\overline{H}$, respectively. Contracting (2.17) with g^{ij} we obtain

$$S_{hk} = \varepsilon\big(\operatorname{tr}(H) H_{hk} - H^2_{hk}\big) + \frac{(n-1)\tau}{n(n+1)} g_{hk}, \tag{2.18}$$

where $\operatorname{tr}(H) = g^{hk}H_{hk}$ and S_{hk} are the local components of the Ricci tensor S of M. From (2.18) we easily get

$$\begin{aligned} S^2_{hk} = g^{ij} S_{hi} S_{kj} = H^4_{hk} - 2 \operatorname{tr}(H) H^3_{hk} + ((\operatorname{tr}(H))^2 - \frac{2(n-1)\varepsilon\tau}{n(n+1)}) H^2_{hk} \\ + \frac{2\varepsilon(n-1)\tau \operatorname{tr}(H)}{n(n+1)} H + \Big(\frac{(n-1)\tau}{n(n+1)}\Big)^2 g_{hk}. \end{aligned} \tag{2.19}$$

Further, on every hypersurface M in $N^{n+1}_s(c)$, $n \geqslant 4$, we have [**19**]

$$R \cdot R - Q(S, R) = -\frac{(n-2)\tau}{n(n+1)} Q(g, C). \tag{2.20}$$

Thus (1.4) is satisfied on every hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$. Evidently, if $x \in U_R - U_H$, then at x we have $H^2 = \alpha H + \beta g$, $\alpha, \beta \in \mathbb{R}$. The last relation leads to (cf. [**17**, Proposition 3.1(ii)])

$$R \cdot R = \Big(\frac{\tau}{n(n+1)} - \varepsilon\beta\Big) Q(g,R). \tag{2.21}$$

Thus (2.3) holds on $U_R - U_H$. Further, if M is a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geqslant 3$, then on $U_H \subset M$ we have [**8**, Theorem 3.1]

$$R \cdot R = \frac{\tau}{n(n+1)} Q(g,R). \tag{2.22}$$

It is also known [**7**, eq. (3.8)] that if M is a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geqslant 3$, then on $U_S \subset M$ we have

$$Q\Big(S - \Big(L_R + \frac{(n-2)\tau}{n(n+1)}\Big) g, R - \frac{\tau}{n(n+1)} G\Big) = 0. \tag{2.23}$$

In particular, applying (2.22) into (2.23) we get on $U_H \subset U_S$

$$Q\Big(S - \frac{(n-1)\tau}{n(n+1)} g, R - \frac{\tau}{n(n+1)} G\Big) = 0.$$

From this, in view of Lemma 3.4 of [**15**] it follows that

$$R - \frac{\tau}{n(n+1)} G = \frac{\phi}{2} \Big(S - \frac{(n-1)\tau}{n(n+1)} g\Big) \wedge \Big(S - \frac{(n-1)\tau}{n(n+1)} g\Big),$$

on the set V of all points of U_H at which S has no a decomposition of the form (2.2) and ϕ is some function on V.

3. Examples

Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold, with nonempty set $U_C \cap U_S \subset M$, and let its curvature tensor R satisfies on $U_C \cap U_S$

$$R = \phi \overline{S} + \mu g \wedge S + \eta G, \tag{3.1}$$

where ϕ, μ and η are some functions on $U_C \cap U_S$. According to [**9**], (3.1) is called the *Roter type equation*. We mention that above decomposition of R on $U_C \cap U_S$ is unique [**16**, Lemma 3.2]. From (3.1) we have [**15**, Theorem 4.2]: (2.3), with $L_R = (n-2)(\frac{\mu}{\phi}(\mu - \frac{1}{n-2}) - \eta)$,

$$R \cdot R - Q(S,R) = \Big(L_R + \frac{\mu}{\phi}\Big) Q(g,C),$$

$$S^2 = \Big(\kappa + \frac{(n-2)\mu - 1}{\phi}\Big) S + \frac{\mu\kappa + (n-1)\eta}{\phi} g.$$

Further, as it was shown in [**15**], (3.1) implies

$$\begin{aligned} S^r_m R_{rijk} &= (\alpha + \mu)(S_{mk}S_{ij} - S_{mj}S_{ik}) + \Big(\frac{\alpha\mu}{\phi} + \eta\Big)(g_{ij}S_{mk} - g_{ik}S_{mj}) \\ &\quad + \beta(g_{mk}S_{ij} - g_{mj}S_{ik}) + \frac{\beta\mu}{\phi} G_{mijk}, \end{aligned} \tag{3.2}$$

where $\alpha = \phi\kappa - 1 + (n-2)\mu$, $\beta = \mu\kappa + (n-1)\eta$. Now (3.2) leads to (1.5), where

$$L_1 = -4(\alpha + \mu), \quad L_2 = -2\Big(\frac{\alpha\mu}{\phi} + \eta + \beta\Big), \quad L_3 = -\frac{4\beta\mu}{\phi}.$$

Thus we have

THEOREM 3.1. *Every semi-Riemannian manifold* (M, g), $n \geqslant 4$, *satisfying the Roter type equation is an AG type manifold.*

REMARK 3.1. (i) Semi-Riemannian manifolds satisfying $R = \phi\overline{S}$, i.e. the special case of (3.1), were investigated in [**24**] (see also references therein).

(ii) Examples of warped products satisfying (3.1) are given in [**18**]. In Example 5.1 of that paper a warped product fulfilling (3.1) is given. That warped product can be locally realized on a hypersurface in a semi-Riemannian space of constant curvature.

(iii) Applying Lemma 3.4 of [**15**] to (2.23) we conclude that the curvature tensor R of a pseudosymmetric hypersurface M in $N_s^{n+1}(c)$, $n \geqslant 4$, is of the form (3.1) at all points of $U_S \cap U_C \subset M$ at which its Ricci tensor is not of the form (2.2).

(iv) Let $M_1 \times_F M_2$, $p = n - 1 = \dim M_1 \geqslant 3$, $\dim M_2 = 1$, be the warped product defined in [**13**, Example 4.1]. This manifold satisfies (1.2) and $\operatorname{rank} S = 1$. Furthermore, applying the two last relations to (1.3) we get $S \cdot R = 0$. The manifold $M_1 \times_F M_2$, satisfies $R \cdot R = Q(S, R)$, i.e. (1.4) with $L_C = 0$. Thus we see that the warped product $M_1 \times_F M_2$ is an AG type manifold. This manifold is locally isometric to a hypersurface in a semi-Euclidean space [**13**, Example 5.1]. We mention that warped products satisfying (1.4) were investigated in [**5**]. For instance, in [**5**] it was shown that any warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = 3$, satisfies (1.4).

(v) Let $M_1 \times_F M_2$, $p = \dim M_1 \geqslant 3$, $n - p = \dim M_2 \geqslant 1$, be the warped product defined in Section 4 of [**4**]. This manifold satisfies $R \cdot R = Q(S, R)$, i.e. (1.4) with $L_C = 0$, and $\operatorname{rank} S \geqslant n - p + 1$. Further, if we assume that $n - p = 1$ and the constant $\xi^f \xi_f$, defined in Section 4 of [**4**], is nonzero, then $\operatorname{rank} S = 2$. Moreover, from (44) of [**4**] it follows that in this case the scalar curvature κ of $M_1 \times_F M_2$ is a nonzero constant and (1.7) and (1.8) are satisfied. On such manifolds we also have (1.9) [**26**, Example 3.1]. Thus, in view of Theorem 3.1, $M_1 \times_F M_2$ is an AG type manifold. In addition, this warped product is locally isometric to a hypersurface in a semi-Euclidean space ([**4**]; see also [**26**, Example 4.2]).

(vi) Let $(\overline{M}, \overline{g})$ be a non-flat 2-dimensional Riemannian manifold. It is easy to check that the product manifold $\overline{M} \times \mathbb{E}^{n-2}$, $n \geqslant 4$, satisfies (1.7), (1.8) and (1.9). Moreover, the manifold $\overline{M} \times \mathbb{E}^{n-2}$, $n \geqslant 4$, can be realized as a hypersurface in $\mathbb{E}^{n+1}$.

Let (M, g), $n \geqslant 4$, be a semi-Riemannian manifold. We define on $U_C \cap U_S \subset M$ the tensor $W(R)$ by

$$W(R) = R - \phi\overline{S} - \mu g \wedge S - \eta G,$$

where ϕ, μ and η are some functions on $U_C \cap U_S$. The tensor $W(R)$ will be called the *Roter type tensor*. Manifolds satisfying pseudosymmetry type curvature conditions related to the Roter type tensor will be investigated in subsequent papers.

We present now an extension of the above definition. Namely, for a generalized curvature tensors B and symmetric $(0,2)$-tensors A and D we define on $U_{\mathrm{Ric}(B)} \cap U_{\mathrm{Weyl}(B)} \subset M$ the $(0,4)$-tensor $W(B,A,D)$ by

$$W(B,A,D) = B - \phi\overline{A} - \mu A \wedge D - \eta\overline{D},$$

where ϕ, μ and η are some functions on $U_{\mathrm{Ric}(B)} \cap U_{\mathrm{Weyl}(B)}$. The tensor $W(B,A,D)$ will be also called a Roter type tensor. For instance, we have the following Roter type tensors

$$W(B,A,g) = B - \phi\overline{A} - \mu g \wedge A - \eta G,$$
$$W(B) = W(B,\mathrm{Ric}(B),g) = B - \phi\overline{\mathrm{Ric}(B)} - \mu g \wedge \mathrm{Ric}(B) - \eta G.$$

Some results on Roter type tensors $W(B,A,g)$ and $W(B,\mathrm{Ric}(B),g)$ are given in [**12**] and [**25**]. For instance, we have

Proposition 3.1. [**25**] *Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor B satisfying $W(B,A,g) = 0$ on $U_{\mathrm{Ric}(B)} \cap U_{\mathrm{Weyl}(B)} \subset M$. Then on this set we have*

$$B \cdot B - Q(\mathrm{Ric}(B),B) = LQ(g,\mathrm{Weyl}(B)), \qquad L = (n-2)\Big(\frac{\mu^2}{\phi} - \eta\Big).$$

Moreover, if $A = \mathrm{Ric}(B)$ on $U_{\mathrm{Ric}(B)} \cap U_{\mathrm{Weyl}(B)}$, then on this set we have

$$B \cdot B = L_B Q(g,B), \qquad L_B = (n-2)\Big(\frac{\mu^2}{\phi} - \eta\Big) - \frac{\mu}{\phi}.$$

Proposition 3.2. [**12**] *Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor B and let the conditions $B \cdot B = Q(\mathrm{Ric}(B),B) + LQ(g,\mathrm{Weyl}(B))$ and $B \cdot B = L_B Q(g,B)$ be satisfied on $U_{\mathrm{Ric}(B)} \cap U_{\mathrm{Weyl}(B)} \subset M$. Then on this set we have*

$$Q\Big(\mathrm{Ric}(B) - (L_B - L)g, B - \frac{L}{n-2}G\Big) = 0.$$

Proposition 3.3. [**2**, Corollary 6.1] *Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor B and let*

$$Q(\mathrm{Ric}(B) - L_2 g, B - L_1 G) = 0$$

be satisfied on $U = U_{\mathrm{Ric}(B)} \cap U_{\mathrm{Weyl}(B)} \subset M$. Then $W(B) = \phi\overline{\mathrm{Ric}(B)} + \mu g \wedge \mathrm{Ric}(B) + \eta G$ on the subset $V \subset U$ of all points at which the tensor $\mathrm{Ric}(B)$ has no a decomposition in a metrical term and in a term of rank one, where ϕ, μ and η are some functions on V.

4. *AG* type hypersurfaces satisfying $\operatorname{rank} S = 2$

Let now M be a hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$, We set [**14**, eq. (13)]

$$A = H^3 - \mathrm{tr}(H)H^2 + \frac{\varepsilon\kappa}{n-1}H. \tag{4.1}$$

Further, let B be a $(0,2)$-tensor with the local components B_{hk} defined by $B_{hk} = g^{ij} H_{hi} S_{kj}$. Using (2.17), (2.18) and (4.1) we obtain

$$B = -\varepsilon A + \Big(\frac{(n-1)\tau}{n(n+1)} + \frac{\kappa}{n-1}\Big) H, \tag{4.2}$$

$$S \cdot R = -2\varepsilon H \wedge B - \frac{2\tau}{n(n+1)} g \wedge S, \tag{4.3}$$

respectively. Substituting (4.2) into (4.3) and using (2.17) we get

$$S \cdot R = 2H \wedge A - 4\Big(\frac{(n-1)\tau}{n(n+1)} + \frac{\kappa}{n-1}\Big)\Big(R - \frac{\tau}{n(n+1)} G\Big) - \frac{2\tau}{n(n+1)} g \wedge S. \tag{4.4}$$

Let now M be a Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$. On $U_H \subset M$ we have [**3**, Theorem 3.1 and Proposition 3.2]

$$R \cdot S = \frac{\tau}{n(n+1)} Q(g, S). \tag{4.5}$$

It is known (see Proposition 3.2 and Theorem 3.1 of [**3**]) that (4.5) is equivalent on U_H to

$$H^3 = \operatorname{tr}(H) H^2 + \lambda H, \tag{4.6}$$

where λ is some function on U_H. Now (4.1) turns into

$$A = \Big(\lambda + \frac{\varepsilon\kappa}{n-1}\Big) H. \tag{4.7}$$

Applying (2.17) and (4.7) in (4.4) we obtain (cf. [**11**, Theorem 3.1])

$$S \cdot R = 4\Big(\varepsilon\lambda - \frac{(n-1)\tau}{n(n+1)}\Big)\Big(R - \frac{\tau}{n(n+1)} G\Big) - \frac{2\tau}{n(n+1)} g \wedge S. \tag{4.8}$$

If the ambient space is $\mathbb{E}_s^{n+1}$, then (2.20) reduces to

$$R \cdot R = Q(S, R). \tag{4.9}$$

Similarly, in this case, (2.17) reduces to

$$R_{hijk} = \varepsilon \overline{H}_{hijk}. \tag{4.10}$$

Proposition 4.1. *Let M be a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. If at $x \in U_C \cap U_S - U_H \subset M$ we have $R \cdot S = 0$, then $R \cdot R = 0$ at x.*

Proof. Evidently, (2.21) reduces to $R \cdot R = -\varepsilon\beta Q(g, R)$, which implies $R \cdot S = -\varepsilon\beta Q(g, S)$, and in a consequence, $\beta = 0$ at x. This completes the proof. □

It is clear that every semisymmetric manifold is Ricci-semisymmetric. The converse statement is not true. Under some additional assumptions both conditions are equivalent to each other. This problem, named the *problem of P.J. Ryan*, was considered by several authors, see [**6**], [**10**] and [**11**] and references therein. Among other things, in [**6**] it was proved that the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on hypersurfaces in $N_s^5(c)$.

Proposition 4.2. *Let M be a Ricci-semisymmetric an AG type hypersurface in $\mathbb{E}_s^{n+1}$, $n \geqslant 5$, and let the set $U_H \subset M$ be nonempty. In addition, let* (1.7) *be satisfied on U_H.*

(*i*) *The condition $R \cdot R = 0$ is satisfied at all points of U_H at which $\kappa \neq 0$. Moreover,* (1.9) *holds at such points.*

(*ii*) *The condition $R \cdot R \neq 0$ is satisfied at all points of U_H at which $\kappa = 0$.*

Proof. Let $x \in U_H$. From (2.19), in view of Corollary 2.1(i) and (4.6), we get

$$H^4 = \Big(2(\mathrm{tr}(H))^2 - \frac{\varepsilon\kappa}{2}\Big)H^2 + \Big(2\lambda + \frac{\varepsilon\kappa}{2}\Big)\,\mathrm{tr}(H)H. \tag{4.11}$$

Furthermore, from (4.6) we get

$$H^4 = \big((\mathrm{tr}(H))^2 + \lambda\big)H^2 + \lambda\,\mathrm{tr}(H)H. \tag{4.12}$$

Comparing the right-hand sides of (4.11) and (4.12) we obtain

$$\Big(\lambda + \frac{\varepsilon\kappa}{2} - \mathrm{tr}(H)\Big)H^2 + \Big(\lambda + \frac{\varepsilon\kappa}{2}\Big)\,\mathrm{tr}(H)H = 0,$$

whence $\lambda + \frac{\varepsilon\kappa}{2} = \mathrm{tr}(H)$ and $(\lambda + \frac{\varepsilon\kappa}{2})\,\mathrm{tr}(H) = 0$. These relations yield

$$(a)\quad \lambda = -\frac{\varepsilon\kappa}{2}, \qquad (b)\quad \mathrm{tr}(H) = 0. \tag{4.13}$$

Now (4.6) and (4.8) turn into

$$A = -\frac{\varepsilon(n-3)\kappa}{2(n-1)}H, \tag{4.14}$$

$$S \cdot R = -\frac{\kappa}{2}R. \tag{4.15}$$

respectively. Since M is an AG type manifold, (1.5) holds on U_H. Now (4.15), by (1.5), leads to

$$\begin{aligned} -\frac{\kappa}{2}R_{hijk} &= L_1(S_{hk}S_{ij} - S_{hj}S_{ik}) + L_3(g_{hk}g_{ij} - g_{hj}g_{ik}) \\ &\quad + L_2(g_{ij}S_{hk} + g_{hk}S_{ij} - g_{hj}S_{ik} - g_{ik}S_{hj}). \end{aligned} \tag{4.16}$$

If $\kappa \neq 0$ at x, then from (4.16), in view of Theorem 4.2 of [**15**], it follows that (2.3) holds at x. Evidently, (2.3) implies (2.4), and in a consequence, we obtain $L_R = 0$ and $R \cdot R = 0$ at x. Further, contracting (4.16) with S^l_h and using (1.8) we obtain

$$-\frac{\kappa}{2}S^h_l R_{hijk} = \Big(L_2 + \frac{\kappa L_1}{2}\Big)\overline{S}_{lijk} + \Big(L_3 + \frac{\kappa L_2}{2}\Big)(g_{ij}S_{lk} - g_{ik}S_{lj}).$$

Symmetrizing this in l and i and using the relation $R \cdot S = 0$ we get $(L_3 + \frac{\kappa L_2}{2})Q(g, S) = 0$, whence

$$L_3 = -\frac{\kappa L_2}{2}. \tag{4.17}$$

On the other hand, contracting (4.16) with g^{ij} and using (1.8) we find

$$\Big(\frac{\kappa}{2} + \frac{\kappa L_1}{2} + (n-2)L_2\Big)S = -(\kappa L_2 + (n-1)L_3)g,$$

whence

$$(4.18)\qquad (a)\quad \kappa L_2 = -(n-1)L_3, \qquad (b)\quad \frac{\kappa}{2} + \frac{\kappa L_1}{2} + (n-1)L_2 = 0.$$

From (4.17) and (4.18)(a) we get $L_3 = 0$. Now (4.17) reduces to $L_2 = 0$. Applying this to (4.18)(b) we obtain $\kappa(L_1 + 1) = 0$, whence $L_1 = -1$. Now (4.16) reduces to (1.9). But this completes the proof of (i).

Let now $\kappa = 0$ at $x \in U_H$. Thus (4.14) turns into $A = 0$. This, together with (4.13)(b), reduces (2.18) and (4.2) to

$$(4.19)\qquad S_{jk} = -\varepsilon H^2_{jk}, \qquad B_{hk} = H^j_h S_{jk} = H^3_{jk} = 0,$$

respectively. We suppose that $R \cdot R = 0$ at x. Now (4.9) yields

$$\begin{aligned} &S_{hl}R_{mijk} + S_{il}R_{hmjk} + S_{jl}R_{himk} + S_{kl}R_{hijm} \\ &- S_{hm}R_{lijk} - S_{im}R_{hljk} - S_{jm}R_{hilk} - S_{km}R_{hijl} = 0. \end{aligned}$$

This, by transvection with H^l_a and H^h_b and making use of (4.10) and (4.19), leads

$$(4.20)\qquad S_{im}(S_{bj}S_{lk} - S_{bk}S_{lj}) + S_{il}(S_{bj}S_{km} - S_{bk}S_{jm}) = 0.$$

We set $Y_k = X^j S_{jk}$, where X^j and Y^j are the local components of vectors $X, Y \in T_xM$ such that $Y_1^2 + \cdots + Y_n^2 > 0$, where $Y_k = g_{jk}Y^j$. Transvecting now (4.20) with X^l and X^m we obtain $Y_i(Y_kS_{bj} - Y_jS_{bk}) = 0$, whence it follows that $\operatorname{rank} S = 1$ at x, a contradiction. Thus if $\kappa = 0$ at $x \in U_H$, then $R \cdot R \neq 0$ at x. Our proposition is thus proved. □

The last proposition implies

THEOREM 4.1. *Let M be an AG type hypersurface in $\mathbb{E}^{n+1}_s$, $n \geqslant 5$, satisfying* (1.7) *on nonempty $U_H \subset M$. The conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on the subset of U_H of all points at which $\kappa \neq 0$.*

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