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CONVOLUTED C-GROUPS

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Abstract. We introduce and systematically analyze the class of convoluted C-groups unifying the well known classes of fractionally integrated groups and C-regularized groups. We relate convoluted C-groups to analytic convoluted C-semigroups and present illustrative examples of differential operators which generate exponentially bounded convoluted groups.

1. Introduction and preliminaries

The theory of convoluted C-semigroups is an attractive field of research of many authors and becomes inevitable in the analysis of various kinds of ill-posed abstract Cauchy problems in a Banach space setting.

Local convoluted C-semigroups were introduced and studied in the papers of Ciorănescu and Lumer [**11**]–[**13**] as a unification concept for local integrated semigroups and local C-semigroups ([**1**], [**48**]). The class of exponentially bounded convoluted semigroups was introduced independently by Keyantuo, Müller and Vieten [**22**] and the author [**27**] while global convoluted semigroups which are not necessarily exponentially bounded have been recently analyzed in [**27**] and [**29**]. The important researches of Bäumer, Lumer and Neubrander ([**7**]–[**8**], [**40**]) are related to the use of asymptotic Laplace transform techniques in the theory of convoluted semigroups. Of importance is also to stress that K-regularized resolvent families, analyzed by Lizama and his collaborators ([**37**]–[**39**]), allow one to consider in a unified treatment the notions of convoluted semigroups and cosine functions as well as to enquire into the abstract Volterra equations of convolution type.

On the other hand, global integrated groups (cf. [**2**], [**4**]–[**6**], [**15**]–[**16**], [**20**]–[**21**], [**28**] and [**42**]) were introduced and investigated by El-Mennaoui in his doctoral dissertation [**16**]. We especially refer the reader to the paper [**21**] where Keyantuo briefly considered an abstract Laplacian in $L^p(\mathbb{R}^n)$-type spaces as well as the relations between exponentially bounded integrated cosine functions and global integrated groups. Further study of global α-times integrated groups with

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corresponding growth order and smooth distribution groups ([**4**]–[**5**]) was obtained by Miana in [**42**] by the use of fractional calculus. More generally, one-parameter groups of regular quasimultipliers, introduced recently by Galé and Miana [**20**] in the framework of the Esterle quasimultipliers theory [**17**], present a relevant tool in the analysis of regularized and integrated groups. It is also worthwhile to accent that many authors related global integrated groups to functional calculi and proved, in such a way, different generalizations of Stone's theorem. Some references on this subject are [**6**], [**14**]–[**15**], [**18**] and [**20**]. Local fractionally integrated groups and distribution groups have been recently investigated in [**28**].

In this paper, we deal with the notion of (local) convoluted C-groups and prove several generalizations of results known for integrated groups and global regularized groups (cf. [**14**], [**20**], [**28**] and [**42**]). In Section 1, we recall the definitions of a local convoluted C-semigroup and an analytic convoluted C-semigroup which are necessary in our further work. Several interesting properties of subgenerators of convoluted C-semigroups are also proven in this section. Section 2 is devoted to the study of structural properties of (exponentially bounded) convoluted C-groups and their relations with analytic convoluted C-semigroups. Motivated by the analysis of the backward heat equation given in [**7**] and [**29**]–[**31**], we discuss in Section 3 the polyharmonic operator Δ^{2n}, $n \in \mathbb{N}$ acting on $L^2[0,\pi]$ with appropriate boundary conditions. We also connect (local) fractionally integrated cosine functions to *global* differentiable regularized groups and prove that derivatives of constructed groups possess some properties of vector-valued ultradifferentiable functions of the Beurling type (cf. also [**26**] and [**49**]).

Finally, let us point out that convoluted C-groups can be used in the analysis of different kinds of abstract Cauchy problems within the theory of generalized function spaces.

NOTATION. In this paper, E and $L(E)$ denote a complex Banach space and Banach algebra of bounded linear operators on E. For a closed linear operator A acting on E, $D(A)$, $N(A)$, $R(A)$ and $\rho(A)$ denote its domain, kernel, range and resolvent set, respectively. We assume $C \in L(E)$ and C is injective; recall, the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda - A) \text{ and } \lambda - A \text{ is injective}\}.$$

Suppose Y is a subspace of E and $Y \subseteq D(A)$. Let us recall that Y is a core for $D(A)$ if for every $x \in D(A)$ there exists a sequence (x_n) in Y satisfying $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = Ax$.

Henceforth we assume that K is not identical to zero and put

$$\Theta(t) := \int_0^t K(s)\,ds, \quad t \in [0,\tau).$$

DEFINITION 1.1. [**31**] Let A be a closed operator and let K be a locally integrable, complex-valued function on $[0,\tau)$, $0 < \tau \leqslant \infty$. If there exists a strongly continuous operator family $(S_K(t))_{t\in[0,\tau)}$ such that, for every $t \in [0,\tau)$,

$S_K(t)C = CS_K(t)$, $S_K(t)A \subseteq AS_K(t)$, $\int_0^t S_K(s)x\,ds \in D(A)$, $x \in E$ and

$$A\int_0^t S_K(s)x\,ds = S_K(t)x - \Theta(t)Cx, \quad x \in E,$$

then $(S_K(t))_{t\in[0,\tau)}$ is called a (local) K-convoluted C-semigroup having A as a subgenerator. If $\tau = \infty$, then it is said that $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-semigroup with a subgenerator A if, in addition, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.

We refer the reader to [**31**] for the notion of (local) K-convoluted C-cosine functions.

Notice that the function Θ is absolutely continuous in $[0,\tau)$ and that $\Theta'(t) = K(t)$, for a.e. $t \in [0,\tau)$. The following fact is easy to prove ([**29**]–[**31**]): If A is a subgenerator of a K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, then $S_K(0) = 0$, $CA \subseteq AC$ and $S_K(t)x \in \overline{D(A)}$, $t \in [0,\tau)$, $x \in E$. Since $K \neq 0$ in $L^1_{\mathrm{loc}}([0,\tau))$, we have that the semigroup $(S_K(t))_{t\in[0,\tau)}$ is non-degenerate, i.e.,

$$\text{if } S_K(t)x = 0 \text{ for all } t \in [0,\tau), \text{ then } x = 0.$$

The integral generator $\hat{A}$ of $(S_K(t))_{t\in[0,\tau)}$ is defined by

$$\left\{(x,y) \in E^2 : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y\,ds,\ t \in [0,\tau)\right\}.$$

It is straightforward to see that $\hat{A}$ is a closed linear operator which is an extension of any subgenerator of $(S_K(t))_{t\in[0,\tau)}$. In what follows, we denote by $\wp(S_K)$ the set of all subgenerators of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. We know that $\hat{A} \in \wp(S_K)$ and that $C^{-1}\hat{A}C = \hat{A}$ ([**31**]). In general, $\wp(S_K)$ need not be a singleton, and moreover, $\wp(S_K)$ can possess infinitely many elements ([**31**]).

Furthermore, it is well known that the semigroup $(S_K(\sigma))_{\sigma\in[0,\tau)}$ fulfills the next composition property (cf. [**27**, Proposition 5.4] and [**35**, Proposition 2.4]):

$$S_K(t)\,S_K(s)\,x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)\,S_K(r)\,Cx\,dr,$$

for every $x \in E$ and $t,s \in [0,\tau)$ with $t+s < \tau$. The strong continuity of $(S_K(\sigma))_{\sigma\in[0,\tau)}$ implies $S_K(t)S_K(s) = S_K(s)S_K(t)$, for $t,s \in [0,\tau)$ with $t+s \leqslant \tau$ (cf. also [**22**, p. 400] and the assertion (a) given below).

The following proposition generalizes [**27**, Prop. 5.4 (1), Prop. 5.5 (3)–(4)], [**51**, Coroll. 2.9, Prop. 3.3], some statements given in [**31**, Section 2] and has the natural analog in the theory of convoluted C-cosine functions.

PROPOSITION 1.1. *Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$ and $\{A, B\} \subseteq \wp(S_K)$. Then:*

(i) *The integral generator of $(S_K(t))_{t\in[0,\tau)}$ is $C^{-1}AC$.*
(ii) *$C^{-1}AC = C^{-1}BC$, $C(D(A)) \subseteq D(B)$ and $A \subseteq B \Leftrightarrow D(A) \subseteq D(B)$.*
(iii) *If $A \neq \hat{A}$, then $\rho(A) = \emptyset$.*

(iv) *For every* $\lambda \in \rho_C(A)$*:*

$$(\lambda - A)^{-1} C S_K(t) = S_K(t)(\lambda - A)^{-1} C, \quad t \in [0, \tau). \tag{1.1}$$

(v) *A and B have the same eigenvalues.*

(vi) *If $A \subseteq B$, then $\rho_C(A) \subseteq \rho_C(B)$.*

(vii) $|\wp(S_K)| = 1$*, if $C(D(\hat{A}))$ is a core for $D(\hat{A})$.*

Proof. Obviously, $CA \subseteq AC$, $A \subseteq C^{-1}AC$ and $C^{-1}AC$ is closed. Assume $(x, y) \in \hat{A}$, i.e.,

$$S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y\,ds,\ t \in [0, \tau).$$

Consequently,

$$A \int_0^t S_K(s)x ds = \int_0^t S_K(s)y\,ds,\ t \in [0, \tau),$$

which simply implies $S_K(t)x \in D(A)$, $AS_K(t)x = S_K(t)y$ and

$$A\left[\Theta(t)Cx + \int_0^t S_K(s)y\,ds\right] = S_K(t)y,\ t \in [0, \tau).$$

Since $\int_0^t S_K(s)yds \in D(A),\ t \in [0, \tau)$ and $\Theta \neq 0$ in $C([0, \tau))$, one gets $Cx \in D(A)$ and $\Theta(t)ACx + S_K(t)y - \Theta(t)Cy = S_K(t)y,\ t \in [0, \tau)$. This implies $ACx = Cy$, $(x, y) \in C^{-1}AC$ and $\hat{A} \subseteq C^{-1}AC$. Further,

$$\int_0^t S_K(s)x\,ds \in D(A) \subseteq D(C^{-1}AC)$$

and

$$C^{-1}AC \int_0^t S_K(s)x\,ds = A \int_0^t S_K(s)x\,ds = S_K(t)x - \Theta(t)Cx,\ t \in [0, \tau),\ x \in E.$$

Suppose now $x \in D(C^{-1}AC)$ and $t \in [0, \tau)$. Since $Cx \in D(A)$ and $S_K(t)A \subseteq AS_K(t)$, one obtains $CS_K(t)x = S_K(t)Cx \in D(A)$ and

$$\begin{aligned} ACS_K(t)x &= AS_K(t)Cx = S_K(t)ACx \\ &= S_K(t)C[C^{-1}AC]x = CS_K(t)[C^{-1}AC]x \in R(C) \end{aligned}$$

and $[C^{-1}AC]S_K(t)x = S_K(t)[C^{-1}AC]x$. So, $S_K(t)[C^{-1}AC] \subseteq [C^{-1}AC]S_K(t)$, $C^{-1}AC$ is a subgenerator of $(S_K(t))_{t \in [0,\tau)}$ and $C^{-1}AC \subseteq \hat{A}$. Therefore, $\hat{A} = C^{-1}AC$ and the proof of (i) is completed.

(ii) and (iii) follow automatically from (i).

To prove (iv), assume $\lambda \in \rho_C(A)$, $t \in [0, \tau)$ and $x \in E$. Then

$$(\lambda - A)^{-1}Cx \in D(A), \quad S_K(t)(\lambda - A)^{-1}Cx \in D(A),$$

$$(\lambda - A)S_K(t)(\lambda - A)^{-1}Cx = S_K(t)(\lambda - A)(\lambda - A)^{-1}Cx = S_K(t)Cx = CS_K(t)x.$$

This gives (1.1).

To prove (v) and (vi), observe only that $N(\lambda - A) \subseteq N(\lambda - \hat{A})$ and that $C^{-1}BC = \hat{A}$ implies $C(N(\lambda - \hat{A})) \subseteq N(\lambda - B)$, $\lambda \in \mathbb{C}$. Suppose now $A \in \wp(S_K)$,

$x \in D(\hat{A})$ and $C(D(\hat{A}))$ is a core of $D(\hat{A})$. Let (x_n) be a sequence in $D(\hat{A})$ such that $\lim_{n\to\infty} Cx_n = x$ and that $\lim_{n\to\infty} \hat{A}Cx_n = \hat{A}x$. Since $C(D(\hat{A})) \subseteq D(A)$, we obtain that $\lim_{n\to\infty} Cx_n = x$ and that $\lim_{n\to\infty} ACx_n = \hat{A}x$. The closedness of A implies $x \in D(A)$, $D(\hat{A}) \subseteq D(A)$ and $\hat{A} = A$. □

Remark 1.1. Even if $K(t) = t$, there exists an example of a local twice integrated C-semigroup whose integral generator has the empty C-resolvent set [**35**].

Let us recall that a function $K \in L^1_{\mathrm{loc}}([0,\tau))$ is called a kernel if, for every $\phi \in C([0,\tau))$, the assumption $\int_0^t K(t-s)\,\phi(s)\,ds = 0$, $t \in [0,\tau)$ implies $\phi \equiv 0$; owing to Titchmarsh's theorem, K is a kernel if $0 \in \operatorname{supp} K$. Suppose now that $(S_K(t))_{t\in[0,\tau)}$ is a (local) K-convoluted C-semigroup and that K is a kernel. Then we have:

(a) $S_K(t)S_K(s) = S_K(s)S_K(t),\ 0 \leqslant t,\ s < \tau.$
(b) $(S_K(t))_{t\in[0,\tau)}$ is uniquely determined by one of its subgenerators.

Remark 1.2. (i) Define the operator A_1 by:

$$D(A_1) := \left\{ \sum_{k=1}^{m} \int_0^{t_k} S_K(s)x_k\,ds : x_k \in E,\ t_k \in [0,\tau),\ k = 1,\dots,m \right\}$$
$$A_1\left[\sum_{k=1}^{m} \int_0^{t_k} S_K(s)x_k\,ds\right] := \sum_{k=1}^{m} [S(t_k)x_k - \Theta(t_k)Cx_k].$$

It is straightforward to verify that A_1 is well-defined and closable. Suppose, additionally, $\tau = \infty$ or K is a kernel. Then we know $S_K(t)S_K(s) = S_K(s)S_K(t)$, $t, s \in [0,\tau)$ and this enables one to see that:

$$S_K(t)(D(A_1)) \subseteq D(A_1), \quad S_K(t)A_1 \subseteq A_1S_K(t), \quad S_K(t)\overline{A_1} \subseteq \overline{A_1}S_K(t), \quad t \in [0,\tau)$$

and $\overline{A_1} \in \wp(S_K)$. Obviously, $\overline{A_1} \subseteq A$, if $A \in \wp(S_K)$.

(ii) Suppose $|\wp(S_K)| < \infty$. Proceeding as in [**51**, Section 2], one can prove the existence of a non-negative integer n satisfying $|\wp(S_K)| = 2^n$.

We use occasionally the following condition for K :

(P1) K is Laplace transformable, i.e., $K \in L^1_{\mathrm{loc}}([0,\infty))$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{K}(\lambda) = \mathcal{L}(K)(\lambda) := \lim\limits_{b\to\infty} \int_0^b e^{-\lambda t}K(t)\,dt := \int_0^\infty e^{-\lambda t}K(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$. Put $\operatorname{abs}(K) := \inf\{\operatorname{Re}\lambda : \tilde{K}(\lambda) \text{ exists}\}$.

Suppose K satisfies (P1) and A is a closed linear operator. We know the following [**31**]:

(α) A is a subgenerator of an exponentially bounded, Θ-convoluted C-semigroup $(S_\Theta(t))_{t\geqslant 0}$ satisfying the condition

$$\|S_\Theta(t+h) - S_\Theta(t)\| \leqslant Che^{\omega(t+h)},\ t \geqslant 0,\ h \geqslant 0,\ \text{for some } C > 0 \text{ and } \omega \geqslant 0,$$

if and only if there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(A), \tag{1.2}$$

(1.3) $\lambda \mapsto \tilde{K}(\lambda)(\lambda - A)^{-1}C,\ \lambda > a,\ \tilde{K}(\lambda) \neq 0$ is infinitely differentiable and

$$\left\| \frac{d^k}{d\lambda^k}\big[\tilde{K}(\lambda)(\lambda - A)^{-1}C\big]\right\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}},\ k \in \mathbb{N}_0,\ \ \lambda > a,\ \tilde{K}(\lambda) \neq 0. \tag{1.4}$$

(β) Assume additionally that A is densely defined. Then A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, $\omega \geqslant 0$, if and only if there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that (1.2), (1.3) and (1.4) are fulfilled.

(γ) Suppose that A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, $\omega \geqslant 0$. Put $a = \max(\omega, \operatorname{abs}(K))$. Then:

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(A) \text{ and} \tag{1.5}$$

$$(\lambda - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)}\int_0^\infty e^{-\lambda t} S_K(t)\,dt,\ \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0. \tag{1.6}$$

(δ) Let $(S_K(t))_{t\geqslant 0}$ be a strongly continuous operator family and $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, $\omega \geqslant 0$. Put $a = \max(\omega, \operatorname{abs}(K))$. If (1.5) and (1.6) are fulfilled, then $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-semigroup with a subgenerator A.

If $\theta \in (0, \pi]$, put $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg\lambda| < \theta\}$.

Definition 1.2. [**29**]–[**31**] Suppose $\alpha \in (0, \frac{\pi}{2}]$ and $(S_K(t))_{t\geqslant 0}$ is a K-convoluted C-semigroup with a subgenerator A. Then we say that $(S_K(t))_{t\geqslant 0}$ is an analytic K-convoluted C-semigroup of angle α having A as a subgenerator, if there exists an analytic function $\mathbf{S}_K : \Sigma_\alpha \to L(E)$ which satisfies:

(i) $\mathbf{S}_K(t) = S_K(t)$, $t > 0$ and
(ii) $\lim\limits_{z\to 0,\, z\in\Sigma_\gamma} \mathbf{S}_K(z)x = 0$, for every $\gamma \in (0, \alpha)$ and $x \in E$.

It is said that A is a subgenerator of an exponentially bounded, analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ of angle α, if for every $\gamma \in (0, \alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$ such that $\|\mathbf{S}_K(z)\| \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$.

2. Convoluted C-groups

Definition 2.1. Let A and B be closed operators and let K be a locally integrable, complex-valued function on $[0, \tau)$, $0 < \tau \leqslant \infty$. A strongly continuous operator family $(S_K(t))_{t\in(-\tau,\tau)}$ is called a (local, if $\tau < \infty$) K-convoluted C-group with a subgenerator A if:

(i) $(S_{K,+}(t) := S_K(t))_{t\in[0,\tau)}$, resp. $(S_{K,-}(t) := S_K(-t))_{t\in[0,\tau)}$, is a (local) K-convoluted C-semigroup with a subgenerator A, resp. B, and
(ii) for every $t, s \in (-\tau, \tau)$ with $t < 0 < s$ and $x \in E$:

$$S_K(t)S_K(s)x = S_K(s)S_K(t)x$$
$$= \begin{cases} \int_{t+s}^{s} K(r-t-s)S_K(r)Cx\,dr + \int_t^0 K(t+s-r)S_K(r)Cx\,dr, & t+s \geqslant 0,\\ \int_t^{t+s} K(t+s-r)S_K(r)Cx\,dr + \int_0^s K(r-t-s)S_K(r)Cx\,dr, & t+s < 0.\end{cases}$$

It is said that $(S_K(t))_{t\in\mathbb{R}}$ is exponentially bounded if there exist $M>0$ and $\omega\geqslant 0$ such that $\|S_K(t)\|\leqslant Me^{\omega|t|}$, $t\in\mathbb{R}$.

A closed linear operator $\hat{A}$ is the integral generator of $(S_K(t))_{t\in(-\tau,\tau)}$ if $\hat{A}$ is the integral generator of $(S_K(t))_{t\in[0,\tau)}$. Put in Definition 2.1 $C=I$ and $K(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t\in[0,\tau)$, where $\alpha>0$. Then $(S_K(t))_{t\in(-\tau,\tau)}$ is an α-times integrated group generated by A (cf. [**20**, Definition 3.6], [**28**, Definition 4.1] and [**42**, Definition 5]).

Suppose $(S_K(t))_{t\in(-\tau,\tau)}$ is a (local) K-convoluted C-group. Then $\wp(S_K)$ designates the set of all subgenerators of $(S_K(t))_{t\in(-\tau,\tau)}$, i.e., $\wp(S_K)=\wp(S_{K,+})$. Let us observe that there exists an exponentially bounded, K-convoluted C-group $(S_K(t))_{t\in\mathbb{R}}$ such that $\wp(S_K)$ has a continuum many elements ([**31**]).

The proof of the following proposition is omitted.

Proposition 2.1. *Suppose $(S_K(t))_{t\in(-\tau,\tau)}$ is a (local) K-convoluted C-group and $A\in\wp(S_K)$. Put $\check{S}_K(t):=S_K(-t)$, $t\in(-\tau,\tau)$. Then $(\check{S}_K(t))_{t\in(-\tau,\tau)}$ is a K-convoluted C-group, $B\in\wp(\check{S}_K)$ and the integral generator of $(\check{S}_K(t))_{t\in(-\tau,\tau)}$ coincides with that of $(S_{K,-}(t))_{t\in[0,\tau)}$.*

Proposition 2.2. *Suppose $\tau\in(0,\infty]$, $K_1\in C([0,\tau))$, $\hat{A}$ is the integral generator of a K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$ and $A\in\wp(S_K)$. Put*

$$S_{K*_0K_1}(t)x=\int_0^t K_1(t-s)S_K(s)x\,ds,\quad t\in[0,\tau),\ x\in E$$
$$S_{K*_0K_1}(t)x=\int_0^{-t} K_1(-t-s)S_K(-s)x\,ds,\quad t\in(-\tau,0),\ x\in E.$$

Then $(S_{K*_0K_1}(t))_{t\in(-\tau,\tau)}$ is a $(K*_0K_1)$-convoluted C-group, $A\in\wp(S_{K*_0K_1})$ and the integral generator of $(S_{K*_0K_1}(t))_{t\in(-\tau,\tau)}$ is $\hat{A}$.

Proof. It can be easily seen that $(S_{K*_0K_1,+}(t)=S_{K*_0K_1}(t))_{t\in[0,\tau)}$ and that $(S_{K*_0K_1,-}(t)=S_{K*_0K_1}(-t))_{t\in[0,\tau)}$ are $(K*_0K_1)$-convoluted C-semigroups whose integral generators are $\hat{A}$ and $\hat{B}$ respectively. Furthermore, $A\in\wp(S_{K*_0K_1,+})$, $B\in\wp(S_{K*_0K_1,-})$ and $S_{K*_0K_1}(t)S_{K*_0K_1}(s)=S_{K*_0K_1}(s)S_{K*_0K_1}(t)$, $\tau<t<0<s<\tau$. So, it is enough to prove the composition property for $S_{K*_0K_1}(t)S_{K*_0K_1}(s)$, $\tau<t<0<s<\tau$. We will do that only in the case $t+s\geqslant 0$ since the proof of composition property in the case $t+s<0$ can be derived similarly. Fix an $x\in E$. Then:

$$\begin{aligned}
S_{K*_0K_1}(t)S_{K*_0K_1}(s)x&=\int_0^{-t}K_1(-t-v)S_K(-v)S_{K*_0K_1}(s)x\,dv\\
&=\int_0^{-t}\int_0^{s}K_1(-t-v)K_1(s-u)S_K(-v)S_K(u)x\,du\,dv\\
&=\int_0^{-t}K_1(-t-v)\bigg[\int_0^{v}K_1(s-u)S_K(-v)S_K(u)x\,du\bigg]dv\\
&\quad+\int_0^{-t}K_1(-t-v)\bigg[\int_v^{s}K_1(s-u)S_K(-v)S_K(u)x\,du\bigg]dv
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{-t} K_1(-t-v) \int_0^v K_1(s-u) \\
&\quad \left[\int_{-v}^{u-v} K(u-v-r)S_K(r)Cx\,dr + \int_0^u K(r-u+v)S_K(r)Cx\,dr\right] du\,dv \\
&\quad + \int_0^{-t} K_1(-t-v) \int_v^s K_1(s-u) \\
&\quad \left[\int_{u-v}^{u} K(r-u+v)S_K(r)Cx\,dr + \int_{-v}^0 K(u-v-r)S_K(r)Cx\,dr\right] du\,dv \\
&:= S_1 + S_2,
\end{aligned}
$$

where

$$
\begin{aligned}
S_1 &:= \int_0^{-t} K_1(-t-v) \int_0^v K_1(s-u) \int_0^u K(r-u+v)S_K(r)Cx\,dr\,du\,dv \\
&\quad + \int_0^{-t} K_1(-t-v) \int_v^s K_1(s-u) \int_{u-v}^u K(r-u+v)S_K(r)Cx\,dr\,du\,dv, \\
S_2 &:= \int_0^{-t} K_1(-t-v) \int_0^v K_1(s-u) \int_{-v}^{u-v} K(u-v-r)S_K(r)Cx\,dr\,du\,dv \\
&\quad + \int_0^{-t} K_1(-t-v) \int_v^s K_1(s-u) \int_{-v}^0 K(u-v-r)S_K(r)Cx\,dr\,du\,dv.
\end{aligned}
$$

The proof is completed if one shows:

$$
S_1 = \int_{t+s}^s (K *_0 K_1)(\xi - t - s) \int_0^\xi K_1(\xi - z)S_K(z)Cx\,dz\,d\xi, \tag{2.1}
$$

$$
S_2 = \int_t^0 (K *_0 K_1)(t + s - \xi) \int_0^{-\xi} K_1(-\xi - z)S_K(-z)Cx\,dz\,d\xi. \tag{2.2}
$$

To prove (2.1), one can use the equality

$$
\begin{aligned}
&\int_{t+s}^s (K *_0 K_1)(\xi - t - s) \int_0^\xi K_1(\xi - z)S_K(z)Cx\,dz\,d\xi \\
&\qquad = \int_{t+s}^s \left[\int_0^{\xi-t-s} K_1(\xi - t - s - \sigma)K(\sigma)\,d\sigma\right] \int_0^\xi K_1(\xi - z)S_K(z)Cx\,dz\,d\xi
\end{aligned}
$$

and the substitution $v = s + \sigma - \xi$, $u = s + z - \xi$ and $r = z$; the proof of (2.2) can be obtained along the same lines. □

Proposition 2.3. *Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$, $A \in \wp(S_K)$, $B \in \wp(S_{K,-})$ and $\hat{B}$ is the integral generator of $(S_{K,-}(t))_{t\in[0,\tau)}$. Then:*

(i) *$\hat{A}S_K(t)x = S_K(t)Ax$, $x \in D(A)$, $t \in (-\tau, 0]$ and $\hat{B}S_K(s)x = S_K(s)Bx$, $x \in D(B)$, $s \in [0, \tau)$.*

(ii) *$S_K(t)\hat{A} \subseteq \hat{A}S_K(t)$, $t \in (-\tau, 0]$ and $S_K(s)\hat{B} \subseteq \hat{B}S_K(s)$, $s \in [0, \tau)$.*

PROOF. Put $\Theta_1(t) = \int_0^t \Theta(s)\,ds$, $t \in [0,\tau)$, $S_{\Theta_1}(t)x = \int_0^t (t-s)S_K(s)x\,ds$, $t \in [0,\tau)$, $x \in E$ and $S_{\Theta_1}(t)x = \int_0^{-t}(-t-s)S_K(-s)x\,ds$, $t \in (-\tau,0)$, $x \in E$. By Proposition 2.2, $(S_{\Theta_1}(t))_{t\in(-\tau,\tau)}$ is a Θ_1-convoluted C-group, $A \in \wp(S_{\Theta_1})$, the integral generator of $(S_{\Theta_1}(t))_{t\in(-\tau,\tau)}$ is $\hat{A}$ and the integral generator of $(S_{\Theta_1,-}(t))_{t\in[0,\tau)}$ is $\hat{B}$. Clearly,

$$\begin{aligned}
S_{\Theta_1}(t)A\int_0^s S_{\Theta_1}(r)x\,dr &= S_{\Theta_1}(t)\biggl(S_{\Theta_1}(s)x - \int_0^s \Theta_1(r)\,drCx\biggr)\\
&= S_{\Theta_1}(s)S_{\Theta_1}(t)x - \int_0^s \Theta_1(r)\,drCS_{\Theta_1}(t)x\\
&= A\int_0^s S_{\Theta_1}(r)S_{\Theta_1}(t)x\,dr + \int_0^s \Theta_1(r)\,drCS_{\Theta_1}(t)x - \int_0^s \Theta_1(r)\,drCS_{\Theta_1}(t)x\\
&= A\int_0^s S_{\Theta_1}(r)S_{\Theta_1}(t)x\,dr = AS_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)x\,dr,\\
&\qquad t \in (-\tau,0),\ s \in [0,\tau),\ x \in E.
\end{aligned}$$

Suppose now $x \in D(A)$. We obtain $S_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)Ax\,dr = AS_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)x\,dr$, $t \in (-\tau,0)$, $s \in [0,\tau)$. The previous equality and closedness of A imply

$$\begin{gathered}
S_{\Theta_1}(t)S_{\Theta_1}(s)x \in D(A), \quad t \in (-\tau,0),\ s \in [0,\tau),\\
AS_{\Theta_1}(t)S_{\Theta_1}(s)x = S_{\Theta_1}(t)S_{\Theta_1}(s)Ax, \quad t \in (-\tau,0),\ s \in [0,\tau).
\end{gathered}$$

Suppose, for a moment, $t \in (-\tau,0)$, $s \in [0,\tau)$ and $t+s \geqslant 0$. The composition property of $S_{\Theta_1}(\cdot)$ allows one to establish the following equality:

$$\begin{aligned}
&\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)CAx\,dr + \int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)CAx\,dr\\
&\qquad = A\biggl[\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr + \int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr\biggr].
\end{aligned}$$

Since $S_{\Theta_1}(r)A \subseteq AS_{\Theta_1}(r)$, $r \in [0,\tau)$ and $CA \subseteq AC$, one gets

$$\begin{gathered}
\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr \in D(A),\\
A\int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr = \int_{t+s}^s \Theta_1(r-t-s)S_{\Theta_1}(r)CAx\,dr.
\end{gathered}$$

Hence, $\int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr \in D(A)$ and:

$$A\int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr = \int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)CAx\,dr. \tag{2.3}$$

Put now $\Omega = \{(t,s) \in (-\tau,0)\times(0,\tau) : t+s > 0\}$ and

$$f_y(t,s) = \int_t^0 \Theta_1(t+s-r)S_{\Theta_1}(r)y\,dr, \quad (t,s) \in \Omega,\ y \in E.$$

The dominated convergence theorem implies:

$$\frac{\partial}{\partial t} f_y(t,s) = \int_t^0 \Theta(t+s-r) S_{\Theta_1}(r) y\, dr - \Theta_1(s) S_{\Theta_1}(t) y$$
$$\frac{\partial}{\partial s} f_y(t,s) = \int_t^0 \Theta(t+s-r) S_{\Theta_1}(r) y\, dr, \quad (t,s) \in \Omega,\ y \in E.$$

By the closedness of A and (2.3), one gets $A \frac{\partial}{\partial s} f_{Cx}(t,s) = \frac{\partial}{\partial s} f_{CAx}(t,s)$, $(t,s) \in \Omega$. In other words,

$$A \int_t^0 \Theta(t+s-r) S_{\Theta_1}(r) Cx\, dr = \int_t^0 \Theta(t+s-r) S_{\Theta_1}(r) CAx\, dr, \quad (t,s) \in \Omega. \tag{2.4}$$

Analogously, $A \frac{\partial}{\partial t} f_{Cx}(t,s) = \frac{\partial}{\partial t} f_{CAx}(t,s)$, $(t,s) \in \Omega$, i.e., for every $(t,s) \in \Omega$,

$$\begin{aligned} A\Big[\int_t^0 \Theta(t+s-r) S_{\Theta_1}(r) Cx\, dr - \Theta_1(s) S_{\Theta_1}(t) Cx\Big] \\ = \int_t^0 \Theta(t+s-r) S_{\Theta_1}(r) CAx\, dr - \Theta_1(s) S_{\Theta_1}(t) CAx\, dr. \end{aligned} \tag{2.5}$$

An employment of (2.4) and (2.5) gives $\Theta_1(s) S_{\Theta_1}(t) Cx \in D(A)$, $(t,s) \in \Omega$ and $A\big(\Theta_1(s) S_{\Theta_1}(t) Cx\big) = \Theta_1(s) S_{\Theta_1}(t) CAx$, $(t,s) \in \Omega$. Similarly, $A\big(\Theta_1(s) S_{\Theta_1}(t) Cx\big) = \Theta_1(s) S_{\Theta_1}(t) CAx$, if $(t,s) \in (-\tau,0) \times (0,\tau)$ and $t+s \leqslant 0$. Thus,

$$A\big(\Theta_1(s) S_{\Theta_1}(t) Cx\big) = \Theta_1(s) S_{\Theta_1}(t) CAx, \quad t \in (-\tau,0),\ s \in [0,\tau). \tag{2.6}$$

It is evident that there exists $s \in [0,\tau)$ with $\Theta_1(s) \neq 0$ and one can apply (2.6) in order to conclude that $A(S_{\Theta_1}(t)Cx) = S_{\Theta_1}(t) CAx$, $t \in (-\tau,0)$. Differentiate the last equality twice with respect to t to obtain that $S_K(t)Cx \in D(A)$ and that $AS_K(t)Cx = S_K(t)CAx$, $t \in (-\tau,0)$. The last equality gives $ACS_K(t)x = CS_K(t)Ax$, $S_K(t)x \in D(C^{-1}AC)$ and $[C^{-1}AC]S_K(t)x = S_K(t)Ax$, $t \in (-\tau,0]$. On the other hand, Proposition 1.1 implies $\hat{A} = C^{-1}AC$, and consequently, $S_K(t)x \in D(\hat{A})$, $x \in D(A)$, $t \in (-\tau,0]$. Since $\hat{A} \in \wp(S_K)$ and $C^{-1}\hat{A}C = \hat{A}$, one obtains that $S_K(t)\hat{A}x = [C^{-1}\hat{A}C]S_K(t)x = \hat{A}S_K(t)x$, $t \in (-\tau,0]$, $x \in D(\hat{A})$. The remnant of the proof follows by the use of Proposition 2.1. □

Assume $r > 0$, $(S_r(t))_{t\in\mathbb{R}}$ is an exponentially bounded, r-times integrated group generated by A and B is the generator of $(S_r(-t))_{t\geqslant 0}$. Let us evoke that El-Mennaoui proved in [**16**] that $B = -A$; the author has recently established in [**28**] the validity of this assertion in the general case of a (local) r-times integrated group; now we state:

Theorem 2.1. *Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$, $\hat{B}$ is the integral generator of $(S_{K,-}(t))_{t\in[0,\tau)}$, $A \in \wp(S_K)$ and $B \in \wp(S_{K,-})$. Then:*

(i) *$S_K(t)x \in D(B)$ and $BS_K(t)x = -S_K(t)\hat{A}x$, $x \in D(\hat{A})$, $t \in (-\tau,0]$; $S_K(s)x \in D(A)$ and $AS_K(s)x = -S_K(s)\hat{B}x$, $x \in D(\hat{B})$, $s \in [0,\tau)$,*

(ii) *$\hat{B} = -\hat{A}$,*

(iii) *$BCx = -C\hat{A}x$, $x \in D(\hat{A})$; $ACx = -C\hat{B}x$, $x \in D(\hat{B})$,*

(iv) $\int_0^t S_K(r)Cx\,dr \in D(A)$, $t \in (-\tau, 0]$ *and*
$\int_0^s S_K(r)Cx\,dr \in D(B)$, $s \in [0, \tau)$.

PROOF. Let

$$\Theta_i(t) = \int_0^t (t-s)^{i-1}\Theta(s)\,ds, \quad i = 1, 2,\ t \in [0, \tau),$$

$$S_{\Theta_1}(t)x = \int_0^t (t-s)S_K(s)x\,ds, \quad t \in [0, \tau),\ x \in E,$$

$$S_{\Theta_1}(t)x = \int_0^{-t} (-t-s)S_K(-s)x\,ds, \quad t \in (-\tau, 0),\ x \in E.$$

Suppose now $t < 0 < s$, $t + s \leqslant 0$ and $x \in E$. Proposition 2.3 and the composition property of $S_{\Theta_1}(\cdot)$ imply:

$$\begin{aligned}
(2.7)\quad S_{\Theta_1}(t)\Big(S_{\Theta_1}(s)x - \int_0^s \Theta_1(r)\,dr Cx\Big) &= S_{\Theta_1}(t)\hat{A}\int_0^s S_{\Theta_1}(r)x\,dr \\
&= \hat{A}S_{\Theta_1}(t)\int_0^s S_{\Theta_1}(r)x\,dr = \hat{A}\int_0^s S_{\Theta_1}(t)S_{\Theta_1}(r)x\,dr \\
= \hat{A}\int_0^s \bigg[\int_t^{t+r} \Theta_1(t+r-v)S_{\Theta_1}(v)Cx\,dv &+ \int_0^r \Theta_1(v-t-r)S_{\Theta_1}(v)Cx\,dv\bigg]dr \\
= \int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr + \int_0^s \Theta_1(r-t-s)&S_{\Theta_1}(r)Cx\,dr - \int_0^s \Theta_1(r)\,dr S_{\Theta_1}(t)Cx.
\end{aligned}$$

Differentiate (2.7) with respect to s in order to conclude that:

$$\begin{aligned}
&\hat{A}\bigg[\int_t^{t+s} \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr + \int_0^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr\bigg] \\
&= \int_t^{t+s} \Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr - \int_0^s \Theta(r-t-s)S_{\Theta_1}(r)Cx\,dr \\
(2.8)\quad &\qquad + \Theta_1(-t)S_{\Theta_1}(s)Cx - \Theta_1(s)S_{\Theta_1}(t)Cx.
\end{aligned}$$

Further on, it is clear that $\int_0^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr \in D(\hat{A})$ and that

$$\begin{aligned}
\hat{A}\int_0^s \Theta_1(r-t-s)S_{\Theta_1}(r)Cx\,dr &= \int_0^s \Theta_1(r-t-s)\hat{A}\int_0^r S_\Theta(v)Cx\,dv\,dr \\
&= \int_0^s \Theta_1(r-t-s)\big(S_\Theta(r)Cx - \Theta_1(r)C^2x\big)\,dr \\
&= \int_0^s \Theta_1(r-t-s)S_\Theta(r)Cx\,dr - \int_0^s \Theta_1(r-t-s)\Theta_1(r)C^2x\,dr.
\end{aligned}$$

This equality and (2.8) imply $\int_t^{t+s} \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr \in D(\hat{A})$ and:

$$\begin{aligned}
(2.9)\quad \hat{A}\int_t^{t+s} \Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr &= \int_t^{t+s} \Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr \\
- \int_0^s \Theta(r-t-s)S_{\Theta_1}(r)Cx\,dr &- \int_0^s \Theta_1(r-t-s)S_\Theta(r)Cx\,dr
\end{aligned}$$

$$+\,\Theta_1(-t)S_{\Theta_1}(s)Cx-\Theta_1(s)S_{\Theta_1}(t)Cx+\int_0^s\Theta_1(r-t-s)\Theta_1(r)\,drC^2x.$$

The partial integration gives

$$-\int_0^s\Theta(r-t-s)S_{\Theta_1}(r)Cx\,dr-\int_0^s\Theta_1(r-t-s)S_{\Theta}(r)Cx\,dr=-\Theta_1(-t)S_{\Theta_1}(s)Cx$$

and, due to (2.9), one gets:

$$\hat{A}\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr \tag{2.10}$$
$$=\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr+\int_0^s\Theta_1(r-t-s)\Theta_1(r)\,drC^2x-\Theta_1(s)S_{\Theta_1}(t)Cx.$$

Next,

$$\begin{aligned}B\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr&=B\int_t^{t+s}\Theta_1(t+s-r)\int_0^{-r}S_{\Theta}(-v)Cx\,dv\,dr\\&=\int_t^{t+s}\Theta_1(t+s-r)\big[S_{\Theta}(r)Cx-\Theta_1(-r)C^2x\big]\,dr\\&=\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta}(r)Cx\,dr-\int_t^{t+s}\Theta_1(t+s-r)\Theta_1(-r)C^2x\,dr\end{aligned}$$
$$=\Theta_1(s)S_{\Theta_1}(t)Cx-\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr-\int_t^{t+s}\Theta_1(t+s-r)\Theta_1(-r)C^2x\,dr,$$

where the last equality follows from integration by parts. Hence,

$$B\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr \tag{2.11}$$
$$=\Theta_1(s)S_{\Theta_1}(t)Cx-\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr-\int_0^s\Theta_1(r-t-s)\Theta_1(r)C^2x\,dr.$$

By (2.10) and (2.11), we have:

$$\hat{A}\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr=-B\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr. \tag{2.12}$$

Suppose $x\in D(\hat{A})$; then $Cx\in D(\hat{A})$. Proposition 2.3 and (2.12) yield:

$$\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)\hat{A}Cx\,dr=-B\int_t^{t+s}\Theta_1(t+s-r)S_{\Theta_1}(r)Cx\,dr. \tag{2.13}$$

Differentiate the previous equality with respect to s to conclude that

$$\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr\in D(B),$$
$$\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)\hat{A}Cx\,dr=-B\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr. \tag{2.14}$$

On the other hand, differentiation of (2.13) with respect to t leads us to the following:

$$
\begin{aligned}
&\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr+\Theta_1(s)S_{\Theta_1}(t)Cx\in D(B),\\
&\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)\hat{A}Cx\,dr+\Theta_1(s)S_{\Theta_1}(t)\hat{A}Cx\\
&\quad=-B\bigg[\int_t^{t+s}\Theta(t+s-r)S_{\Theta_1}(r)Cx\,dr+\Theta_1(s)S_{\Theta_1}(t)Cx\bigg].
\end{aligned}\tag{2.15}
$$

One can employ (2.14) and (2.15) to see that $\Theta_1(s)S_{\Theta_1}(t)Cx\in D(B)$ and that $-B(\Theta_1(s)S_{\Theta_1}(t)Cx)=\Theta_1(s)S_{\Theta_1}(t)\hat{A}Cx$. Using the similar arguments, one obtains that the last equality remains true in the case: $t+s\geqslant 0$ and $x\in D(\hat{A})$. So, $\Theta_1(s)S_{\Theta_1}(t)Cx\in D(B)$ and

$$
-B\big(\Theta_1(s)S_{\Theta_1}(t)Cx\big)=\Theta_1(s)S_{\Theta_1}(t)\hat{A}Cx,\quad t\in(-\tau,0],\ s\in[0,\tau),\ x\in D(\hat{A}).\tag{2.16}
$$

Choose a number $s\in[0,\tau)$ with $\Theta_1(s)\neq 0$; (2.16) implies $S_{\Theta_1}(t)Cx\in D(B)$ and

$$
-B(S_{\Theta_1}(t)Cx)=S_{\Theta_1}(t)\hat{A}Cx,\quad t\in(-\tau,0],\ x\in D(\hat{A}).\tag{2.17}
$$

A consequence of (2.17) is

$$
\begin{aligned}
S_{\Theta_1}(t)Cx-\Theta_2(-t)C^2x&=B\int_0^{-t}S_{\Theta_1}(-v)Cx\,dv=-\int_0^{-t}S_{\Theta_1}(-v)\hat{A}Cx\,dv\\
&=-C\int_0^{-t}S_{\Theta_1}(-v)\hat{A}x\,dv,\ t\in(-\tau,0],\ x\in D(\hat{A}).
\end{aligned}
$$

Therefore,

$$
S_{\Theta_1}(t)x-\Theta_2(-t)Cx=-\int_0^{-t}S_{\Theta_1}(-v)\hat{A}x\,dv,\ t\in(-\tau,0],\ x\in D(\hat{A}),\tag{2.18}
$$

which clearly implies

$$
B\int_0^{-t}S_{\Theta_1}(-v)x\,dv=-\int_0^{-t}S_{\Theta_1}(-v)\hat{A}x\,dv,\quad t\in(-\tau,0],\ x\in D(\hat{A}).
$$

The closedness of B enables one to see that $S_{\Theta_1}(t)x\in D(B)$ and that $BS_{\Theta_1}(t)x=-S_{\Theta_1}(t)\hat{A}x$, $t\in(-\tau,0]$, $x\in D(\hat{A})$. Differentiate the last equality twice with respect to t in order to conclude that $S_K(t)x\in D(B)$ and that $BS_K(t)x=-S_K(t)\hat{A}x$, $t\in(-\tau,0]$, $x\in D(\hat{A})$. This and Proposition 2.1 imply: $\check{S}_K(-s)x\in D(A)$ and $A\check{S}_K(-s)x=-\check{S}_K(-s)\hat{B}x$, $s\in[0,\tau)$, $x\in D(\hat{B})$, i.e., $S_K(s)x\in D(A)$ and $AS_K(s)x=-S_K(s)\hat{B}x$, $x\in D(\hat{B})$, $s\in[0,\tau)$. The proof of (i) is completed.

Further on, it is evident that (2.18) implies $-\hat{A}\subseteq\hat{B}$. Now one can apply Proposition 2.1 and the first part of the proof to obtain that $-\hat{B}\subseteq\hat{A}$; hence, $\hat{B}=-\hat{A}$ and this ends the proof of (ii).

Finally, (iii) and (iv) are simple consequences of the assertion (ii) of this theorem and Proposition 1.1(i)–(ii). □

PROPOSITION 2.4. *Suppose K satisfies* (P1) *and $\hat{A}$ is the integral generator of an exponentially bounded, K-convoluted C-group $(S_K(t))_{t\in\mathbb{R}}$. If there exist $M>0$ and $\beta>0$ such that $|K(t)|\leqslant Me^{\beta t}$, $t\geqslant 0$, then $C^{-1}\hat{A}^2C$ is the integral generator of an exponentially bounded, analytic K_1-convoluted C-semigroup $(S_{K_1}(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$, where*

$$K_1(t):=\int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}}K(s)\,ds,\ t>0,$$
$$S_{K_1}(t)x:=\frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-s^2/4t}\big(S_K(s)x+S_K(-s)x\big)\,ds,\quad t>0,\ x\in E.$$

PROOF. By Theorem 2.1, $\pm\hat{A}$ are the integral generators of exponentially bounded, K-convoluted C-semigroups $(S_{K,\pm}(t))_{t\geqslant 0}$. Now one can apply [**31**, Proposition 8] to conclude that $\hat{A}^2$ is a subgenerator of an exponentially bounded, K-convoluted C-cosine function $(C_K(t))_{t\in\mathbb{R}}$, where $C_K(t)=\frac{1}{2}(S_K(t)+S_K(-t))$, $t\geqslant 0$. By [**31**, Theorem 11], $\hat{A}^2$ is a subgenerator of an exponentially bounded, analytic K_1-convoluted C-semigroup $(S_{K_1}(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$. The proof ends an application of Proposition 1.1(i). □

REMARK 2.1. It could be worth noting the following facts: K_1 fulfills (P1), $\operatorname{abs}(K_1)\leqslant\beta^2$ and $\tilde{K_1}(\lambda)=\tilde{K}(\sqrt{\lambda})$, $\operatorname{Re}\lambda>\beta^2$.

THEOREM 2.2. *Suppose $\tau\in(0,\infty]$ and $\pm\hat{A}$ are the integral generators of K-convoluted C-semigroups $(S_{K,\pm}(t))_{t\in[0,\tau)}$. Put $S_K(t):=S_{K,+}(t)$, $t\in[0,\tau)$ and $S_K(t):=S_{K,-}(-t)$, $t\in(-\tau,0)$. Then $(S_K(t))_{t\in(-\tau,\tau)}$ is a K-convoluted C-group whose integral generator is $\hat{A}$.*

PROOF. Suppose $-\tau<t<0<s<\tau$ and $t+s\geqslant 0$. We will prove the composition property for $S_K(t)S_K(s)$. Fix an $x\in E$ and define

$$f(r)=S_K(t+s-r)\int_0^r S_K(\sigma)x\,d\sigma,\ r\in[t+s,s].$$

Evidently, $\hat{A}S_K(\sigma)\subseteq S_K(\sigma)\hat{A}$, $\sigma\in(-\tau,\tau)$ and the semigroup property of a K-convoluted C-semigroup implies:

$$\begin{aligned}\frac{d}{dr}f(r)&=S_K(t+s-r)S_K(r)x\\&\quad-\hat{A}S_K(t+s-r)\int_0^r S_K(\sigma)x\,d\sigma+K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma\\&=\Theta(r)S_K(t+s-r)Cx+K(r-s-t)C\int_0^r S_K(\sigma)xd\sigma,\end{aligned}$$

for a.e. $r\in(t+s,s)$. Integrate the last equality with respect to r from $t+s$ to s to obtain:

$$\begin{aligned}S_K(t)\int_0^s S_K(\sigma)xd\sigma&=\int_{t+s}^s\Theta(r)S_K(t+s-r)Cx\,dr\\&\quad+\int_{t+s}^s K(r-s-t)C\int_0^r S_K(\sigma)xd\sigma\,dr.\end{aligned}$$

Since $\hat{A} \in \wp(S_{K,+})$, the last equality allows one to conclude that:

$$(2.19)\quad S_K(t)S_K(s)x = S_K(t)\Big[\hat{A}\int_0^s S_K(\sigma)x\,d\sigma + \Theta(s)Cx\Big]$$
$$= \hat{A}\Big[\int_{t+s}^s \Theta(r)S_K(t+s-r)Cx\,dr + \int_{t+s}^s K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma\,dr\Big] + \Theta(s)S_K(t)Cx$$
$$= \hat{A}\int_{t+s}^s \Theta(r)S_{K,-}(r-t-s)Cx\,dr + \int_{t+s}^s K(r-s-t)\big[S_K(r)Cx - \Theta(r)C^2x\big]dr$$
$$+ \Theta(s)S_K(t)Cx.$$

Furthermore,

$$(2.20)\quad \hat{A}\int_{t+s}^s \Theta(r)S_{K,-}(r-t-s)Cx\,dr = \hat{A}\int_0^{-t} \Theta(v+t+s)S_{K,-}(v)Cx\,dv$$
$$= \hat{A}\Big[\Theta(s)\int_0^{-t} S_{K,-}(r)Cx\,dr - \int_0^{-t} K(t+s+r)\int_0^r S_{K,-}(v)Cx\,dv\,dr\Big]$$
$$= -\Theta(s)\big[S_K(t)Cx - \Theta(-t)C^2x\big] + \int_0^{-t} K(t+s+r)\big[S_K(-r)Cx - \Theta(r)C^2x\big]dr.$$

A consequence of (2.19) and (2.20) is:

$$S_K(t)S_K(s)x$$
$$= -\Theta(s)\big[S_K(t)Cx - \Theta(-t)C^2x\big] + \int_0^{-t} K(t+s+r)\big[S_K(-r)Cx - \Theta(r)C^2x\big]dr$$
$$+ \int_{t+s}^s K(r-s-t)\big[S_K(r)Cx - \Theta(r)C^2x\big]dr + \Theta(s)S_K(t)Cx$$
$$= \int_{t+s}^s K(r-t-s)S_K(r)Cx\,dr + \int_t^0 K(t+s-r)S_K(r)Cx\,dr$$
$$+ \Big[\Theta(s)\Theta(-t)C^2x + \int_{-t}^0 K(t+s+r)\Theta(r)C^2x\,dr - \int_{t+s}^s K(r-s-t)\Theta(r)C^2x\,dr\Big],$$

and the composition property for $S_K(t)S_K(s)$ follows from the next computation:

$$\Theta(s)\Theta(-t)C^2x + \int_{-t}^0 K(t+s+r)\Theta(r)C^2x\,dr - \int_{t+s}^s K(r-s-t)\Theta(r)C^2x\,dr$$
$$= \Theta(s)\Theta(-t)C^2x - \int_{t+s}^s K(r)\Theta(r-t-s)C^2x\,dr - \int_{t+s}^s K(r-s-t)\Theta(r)C^2x\,dr$$
$$= \Theta(s)\Theta(-t)C^2x - \Big[\Theta(s)\Theta(-t)C^2x - \int_{t+s}^s K(r-s-t)\Theta(r)C^2x\,dr\Big]$$
$$- \int_{t+s}^s K(r-s-t)\Theta(r)C^2x\,dr = 0.$$

The proof of composition property in the case $t+s<0$ can be derived as follows. Since $\hat{A}\int_0^r S_K(\sigma)x d\sigma = S_K(r)x - \Theta(-r)Cx,\ r \in (-\tau, 0]$, we obtain

$$\begin{aligned}\frac{d}{dr}f(r) &= S_K(t+s-r)S_K(r)x - \hat{A}S_K(t+s-r)\int_0^r S_K(\sigma)x\,d\sigma \\ &\qquad + K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma \\ &= \Theta(|r|)S_K(t+s-r)Cx + K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma,\end{aligned}$$

for a.e. $r \in (t+s, s)$. Integrate the last equality with respect to r from $t+s$ to s to obtain

$$\begin{aligned}(2.21)\quad & S_K(t)\int_0^s S_K(\sigma)x d\sigma \\ &= \int_{t+s}^0 \Theta(-r)S_K(t+s-r)Cx\,dr + \int_{t+s}^0 K(r-t-s)\int_0^r S_K(\sigma)Cx\,d\sigma\,dr \\ &\quad + \int_0^s \Theta(r)S_K(t+s-r)Cx\,dr + \int_0^s K(r-s-t)\int_0^r S_K(\sigma)Cx\,d\sigma\,dr.\end{aligned}$$

Clearly,

$$\begin{aligned}S_K(t)S_K(s)x &= S_K(t)\left[\hat{A}\int_0^s S_K(\sigma)x\,d\sigma + \Theta(s)Cx\right] \\ &= \hat{A}S_K(t)\int_0^s S_K(\sigma)x\,d\sigma + \Theta(s)S_K(t)Cx,\end{aligned}$$

and a tedious computation involving (2.21) leads us to the next equality:

$$\begin{aligned}(2.22)\quad S_K(t)S_K(s)x &= \int_t^{t+s} K(t+s-r)S(r)Cx\,dr + \int_0^s K(r-t-s)S(r)Cx\,dr \\ + \Bigg[\Theta(s)\Theta(-t) &- \int_t^{t+s} K(t+s-r)\Theta(-r)\,dr - \int_0^s K(r-s-t)\Theta(r)\,dr\Bigg]C^2x.\end{aligned}$$

Since

$$\begin{aligned}&\Theta(s)\Theta(-t) - \int_t^{t+s} K(t+s-r)\Theta(-r)\,dr - \int_0^s K(r-s-t)\Theta(r)\,dr \\ &= \Theta(s)\Theta(-t) + \int_s^0 K(r)\Theta(r-t-s)\,dr - \left[\Theta(-t)\Theta(s) - \int_0^s \Theta(r-t-s)K(r)\,dr\right] = 0,\end{aligned}$$

(2.22) implies the composition property for $S_K(t)S_K(s)$. By the foregoing,
$S_K(s)S_K(t)x = \check{S}_K(-s)\check{S}_K(-t)x$

$$= \begin{cases}\int_{-t-s}^{-t} K(r+t+s)\check{S}_K(r)Cx\,dr + \int_{-s}^0 K(-t-s-r)\check{S}_K(r)Cx\,dr, & t+s<0, \\ \int_{-s}^{-t-s} K(-t-s-r)\check{S}_K(r)Cx\,dr + \int_0^{-t} K(r+t+s)\check{S}_K(r)Cx\,dr, & t+s \geqslant 0,\end{cases}$$

$$= \begin{cases}\int_t^{t+s} K(t+s-r)S_K(r)Cx\,dr + \int_0^s K(r-t-s)S_K(r)Cx\,dr, & t+s<0, \\ \int_{t+s}^s K(r-t-s)S_K(r)Cx\,dr + \int_t^0 K(t+s-r)S_K(r)Cx\,dr, & t+s \geqslant 0,\end{cases}$$

for every $x \in E$. The composition property for $S_K(t)S_K(s)$ and previous equality imply $S_K(t)S_K(s) = S_K(s)S_K(t)$, $t < 0 < s$, finishing the proof of the theorem. $\square$

QUESTIONS. (i) Suppose $\hat{A}$ is the integral generator of a (local) K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$, $A \in \wp(S_K)$ and $A \neq \hat{A}$. Is $-A$ a subgenerator of $(S_{K,-}(t))_{t\in[0,\tau)}$?

(ii) Suppose A is the integral generator of a (local) K-convoluted group $(S_K(t))_{t\in(-\tau,\tau)}$. Does there exist an injective operator $C \in L(E)$ such that A generates a *global* C-group?

COROLLARY 2.1. *Let $\tau \in (0,\infty]$, $\hat{A}$ be a closed linear operator and $(S_K(t))_{t\in(-\tau,\tau)}$ a strongly continuous operator family. Then $\hat{A}$ is the integral generator of a K-convoluted C-group $(S_K(t))_{t\in(-\tau,\tau)}$ iff $\pm\hat{A}$ are the integral generators of K-convoluted C-semigroups $(S_{K,\pm}(t))_{t\in[0,\tau)}$.*

The following theorem is a straightforward application of Corollary 2.1 and the assertions (α)–(δ) quoted in the first section.

THEOREM 2.3. *Let K satisfies* (P1) *and let $\hat{A}$ be a closed linear operator. Then:*

$(\alpha 1)$ *$\hat{A}$ is the integral generator of an exponentially bounded, Θ-convoluted C-group $(S_\Theta(t))_{t\in\mathbb{R}}$ satisfying the condition*

$$\|S_\Theta(\pm t \pm h) - S_\Theta(\pm t)\| \leqslant Che^{\omega(t+h)},\ t \geqslant 0,\ h \geqslant 0,\ \text{for some } C > 0 \text{ and } \omega \geqslant 0,$$

iff there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(\pm\hat{A}), \tag{2.23}$$

$$\lambda \mapsto \tilde{K}(\lambda)(\lambda \pm \hat{A})^{-1}C,\ \lambda > a,\ \tilde{K}(\lambda) \neq 0 \text{ is infinitely differentiable,} \tag{2.24}$$

$$\left\|\frac{d^k}{d\lambda^k}\big[\tilde{K}(\lambda)(\lambda \pm \hat{A})^{-1}C\big]\right\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}},\ k \in \mathbb{N}_0,\ \lambda > a,\ \tilde{K}(\lambda) \neq 0. \tag{2.25}$$

$(\beta 1)$ *Suppose, in addition, that $\hat{A}$ is densely defined. Then $\hat{A}$ is the integral generator of an exponentially bounded, K-convoluted C-group $(S_K(t))_{t\in\mathbb{R}}$ satisfying $\|S_K(t)\| \leqslant Me^{\omega|t|}$, $t \in \mathbb{R}$, $\omega \geqslant 0$ iff there exists $a \geqslant \max(\omega, \operatorname{abs}(K))$ such that* (2.23)*,* (2.24) *and* (2.25) *are fulfilled.*

$(\gamma 1)$ *Suppose that $\hat{A}$ is the integral generator of an exponentially bounded, K-convoluted C-group $(S_K(t))_{t\in\mathbb{R}}$ satisfying $\|S_K(t)\| \leqslant Me^{\omega|t|}$, $t \in \mathbb{R}$, $\omega \geqslant 0$. Put $a = \max(\omega, \operatorname{abs}(K))$. Then:*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0\} \subseteq \rho_C(\pm\hat{A}), \tag{2.26}$$

$$(\lambda \pm \hat{A})^{-1}Cx = \frac{1}{\tilde{K}(\lambda)}\int_0^\infty e^{-\lambda t}S_K(\mp t)\,dt,\ \ \operatorname{Re}\lambda > a,\ \tilde{K}(\lambda) \neq 0. \tag{2.27}$$

$(\delta 1)$ *Suppose $(S_K(t))_{t\in\mathbb{R}}$ is a strongly continuous operator family and $\|S_K(t)\| \leqslant Me^{\omega|t|}$, $t \in \mathbb{R}$, $\omega \geqslant 0$. Put $a = \max(\omega, \operatorname{abs}(K))$. If* (2.26) *and* (2.27) *are fulfilled, then $\hat{A}$ is the integral generator of an exponentially bounded, K-convoluted C-group $(S_K(t))_{t\in\mathbb{R}}$.*

Finally, let us also observe that Proposition 1.1 can be easily transferred to convoluted C-groups.

3. Applications

We employ the next auxiliary notations.

1. Let $a > 0$ and $b > 0$. The exponential region $E(a,b)$ is primarily defined as $E(a,b) := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geqslant b,\ |\operatorname{Im}\lambda| \leqslant e^{a\operatorname{Re}\lambda}\}$ by Arendt, El-Mennaoui and Keyantuo in [**1**]. Put $E^2(a,b) := \{\lambda^2 \mid \lambda \in E(a,b)\}$.

2. Suppose $s > 1$. Following Chazarain [**9**] (cf. also [**10**], [**25**] and [**41**, Section 2.3]), we use the ultra-logarithmic regions of type l:

$$\Lambda_{\alpha,\beta,l} := \bigl\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \alpha M(l|\operatorname{Im}\lambda|) + \beta\bigr\},\ \ l > 0,\ \alpha > 0,\ \beta \in \mathbb{R},$$

where $M(t) := \sup_{p\in\mathbb{N}_0} \ln(t^p/p!^s)$, $t > 0$ and $M(0) = 0$.

3. If $\theta \in (0,\pi]$ and $d \in (0,1]$, put $B_d := \{\lambda \in \mathbb{C} : |\lambda| \leqslant d\}$ and $\Omega_{\theta,d} := \Sigma_\theta \cup B_d$.

4. We recall a family of continuous kernels (cf. [**2**, p. 107]):

$$K_\delta(t) := \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{\lambda t-\lambda^\delta}\,d\lambda,\quad t \geqslant 0,\ \delta \in (0,1),\ r > 0,\ \text{where } 1^\delta = 1.$$

Put, for $c > 0$ and $\delta \in (0,1)$, $K_{\delta,c}(t) := K_\delta(ct)$, $t \geqslant 0$. It is well known that, for every $\delta \in (0,1)$, $c > 0$ and $s = 1/\delta$:

$$\begin{aligned}|\widetilde{K_{\delta,c}}(\lambda)| &= \left|\frac{1}{c}\widetilde{K_\delta}\Bigl(\frac{\lambda}{c}\Bigr)\right| = \frac{1}{c}\bigl|e^{-(\lambda/c)^\delta}\bigr| = \frac{1}{c}\,e^{-\cos(\delta\arg(\lambda/c))|\lambda/c|^\delta}\\ &\leqslant \frac{1}{c}\,e^{-(\cos(\pi/2s)c^{-1/s})|\lambda|^{1/s}},\ \ \operatorname{Re}\lambda > 0.\end{aligned}$$

For the sake of simplicity, in the following theorem, we consider only Gevrey type sequences $p!^s$, $s \in (1,2)$ and the functions $K_{1/s,\,c}$, $c > 0$. Actually, the argumentation given in [**26**] and [**41**, Section 1.3] can be applied in proving a more general result.

THEOREM 3.1. *Suppose $\alpha > 0$ and A generates a local α-times integrated cosine function. Then:*

(i) *For every $b \in (\frac{1}{2},1)$ and $\gamma \in \bigl(0,\arctan(\cos(\frac{b\pi}{2}))\bigr)$, there exist two analytic operator families $(T_{b,+}(t))_{t\in\Sigma_\gamma} \subseteq L(E)$ and $(T_{b,-}(t))_{t\in\Sigma_\gamma} \subseteq L(E)$ which satisfies:*

- (i1) *For every $t \in \Sigma_\gamma$, $T_{b,+}(t)$ and $T_{b,-}(t)$ are injective operators.*
- (i2) $\bigl\|t^{\frac{\alpha+1}{2b}}T_{b,\pm}(t)\bigr\| = O(1)$, $t \to 0+$.
- (i3) *For every $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, iA is the generator of a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-group $(S_{b,t_1,t_2}(r))_{r\in\mathbb{R}}$.*
- (i4) *For every $x \in E$, $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, the mapping $r \mapsto S_{b,\,t_1,\,t_2}(r)x$, $r \in \mathbb{R}$ is infinitely differentiable in $(-\infty,0)\cup(0,\infty)$.*
- (i5) *Suppose K is a compact subset of $\mathbb{R}$ and $0 \notin K$. Then, for every $h > 0$ and $s \in (\frac{1}{b},2)$:*

$$\sup_{p\in\mathbb{N}_0,\,r\in K}\frac{h^p\bigl\|\frac{d^p}{dr^p}S_{b,\,t_1,\,t_2}(r)x\bigr\|}{p!^s} < \infty.$$

(ii) *For every $s \in (1,2)$ and $\tau \in (0,\infty)$, there exists $c_\tau > 0$ such that A generates a local $K_{1/s,\,c_\tau}$-convoluted group $(S_{K_{1/s,\,c_\tau}}(t))_{t\in(-\tau,\tau)}$ which satisfies:*

- (ii1) *The mappings $t \mapsto S_{K_{1/s,\,c_\tau}}(\pm t)$, $t \in [0,\tau)$ are infinitely differentiable.*

(ii2) *There exists* $h > 0$ *such that*

$$\sup_{t \in (-\tau,\tau) \smallsetminus \{0\},\, p \in \mathbb{N}_0} \frac{h^p \left\| \frac{d^p}{dt^p} S_{K_{1/s,c_\tau}}(t) \right\|}{p!^s} < \infty.$$

PROOF. The operator $\mathcal{A} \equiv \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ generates a local $(\alpha+1)$-times integrated semigroup in E^2 (cf. [**30**]) and an application of [**34**, Theorem 2.1] gives the existence of positive real numbers a, b and M satisfying $E(a,b) \subseteq \rho(\mathcal{A})$ and $\|R(\lambda : \mathcal{A})\| \leqslant M|\lambda|^{\alpha+1}$, $\lambda \in E(a,b)$. This implies $E^2(a,b) \subseteq \rho(A)$,

$$R(\lambda : \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R(\lambda^2 : A)(\lambda x + y) \\ AR(\lambda^2 : A)x + \lambda R(\lambda^2 : A)y \end{pmatrix}, \quad x, y \in E,\ \lambda \in E(a,b),$$

and $\|R(\lambda^2 : A)\| \leqslant \|R(\lambda : \mathcal{A})\| \leqslant M|\lambda|^{\alpha+1}$, $\lambda \in E(a,b)$. Hence, $E^2(a,b) \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leqslant M|\lambda|^{\frac{\alpha+1}{2}}$, $\lambda \in E^2(a,b)$. Suppose now $s \in (\frac{1}{b}, 2)$. Proceeding as in the proof of [**30**, Theorem 4.3], we get the existence of numbers $\delta > 0$, $\epsilon \in \mathbb{R}$ and $l \geqslant 1$ (cf. also [**9**] and [**10**, Theorem 1.5]) which fulfill:

$$\Lambda_{\delta,\epsilon,l} \subseteq \rho(\pm iA) \text{ and } \|R(\lambda : \pm iA)\| \leqslant M|\lambda|^{\frac{\alpha+1}{2}}, \ \ \lambda \in \Lambda_{\delta,\epsilon,l}.$$

Further on, it is clear that there exist numbers $a \in (0, \frac{\pi}{2})$, $d \in (0,1]$ and $\omega \in \mathbb{R}$ so that: $b \in (0, \frac{\pi}{2(\pi-a)})$, $\gamma \in (0, \arctan(\cos(b(\pi-a))))$ and $\Omega_{a,d} \subseteq \Lambda_{\delta,\epsilon-\omega,l} \subseteq \rho(\pm iA - \omega)$ (cf. [**26**]). Let the curve $\Gamma_{a,d} = \partial(\Omega_{a,d})$ be upwards oriented. Define $T_{b,\pm}(t)$, $t \in \Sigma_\gamma$ by:

$$T_{b,\pm}(t)x := \frac{1}{2\pi i} \int_{\Gamma_{a,d}} e^{-t(-\lambda)^b} R(\lambda : \pm iA - \omega)x\, d\lambda, \ \ x \in E.$$

The arguments given in [**46**, Section 2] show that $(T_{b,\pm}(t))_{t \in \Sigma_\gamma}$ are analytic operator families which fulfill the claimed properties (i1) and (i2). Assume K is a compact subset of $(0,\infty)$, $t \in \Sigma_\gamma$ and $x \in E$. Arguing as in [**26**, Section 2], we get that $\pm iA$ generate global $T_{b,\pm}(t)$-semigroups $(S_{b,t,\pm}(r))_{r \geqslant 0}$. Furthermore, the mappings $r \mapsto S_{b,t,\pm}(r)x$, $r > 0$ are infinitely differentiable and, for every $h > 0$:

$$\sup_{p \in \mathbb{N}_0,\, r \in K} \frac{h^p \left\| \frac{d^p}{dr^p} S_{b,t,\pm}(r)x \right\|}{p!^s} < \infty. \tag{3.1}$$

Let $t_1 \in \Sigma_\gamma$, $t_2 \in \Sigma_\gamma$ and $x \in E$ be fixed. Evidently, $T_{b,+}(t_1)(\pm iA) \subseteq (\pm iA)T_{b,+}(t_1)$, $T_{b,-}(t_2)(\pm iA) \subseteq (\pm iA)T_{b,-}(t_2)$ and $T_{b,+}(t_1)T_{b,-}(t_2) = T_{b,-}(t_2)T_{b,+}(t_1)$. One obtains

$$\begin{aligned} T_{b,-}(t_2)(S_{b,t_1,+}(r)x - T_{b,+}(t_1)x) &= T_{b,-}(t_2) iA \int_0^r S_{b,t_1,+}(v)x\, dv \\ &= iAT_{b,-}(t_2) \int_0^r S_{b,t_1,+}(v)x\, dv, \end{aligned}$$

and consequently,

$$iA \int_0^r T_{b,-}(t_2) S_{b,t_1,+}(v)x\, dv = T_{b,-}(t_2)S_{b,t_1,+}(r)x - T_{b,+}(t_1)T_{b,-}(t_2)x, \ \ r \geqslant 0.$$

Clearly, we have $[T_{b,-}(t_2)S_{b,t_1,+}(r)]T_{b,+}(t_1) = T_{b,+}(t_1)[T_{b,-}(t_2)S_{b,t_1,+}(r)]$, $r \geqslant 0$, and $[T_{b,-}(t_2)S_{b,t_1,+}(r)]iA \subseteq iA[T_{b,-}(t_2)S_{b,t_1,+}(r)]$, $r \geqslant 0$. The above given arguments simply imply that $(T_{b,-}(t_2)S_{b,t_1,+}(r))_{r\geqslant 0}$ is a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-semigroup generated by iA. Analogously, we have that $(T_{b,+}(t_1)S_{b,t_2,-}(r))_{r\geqslant 0}$ is a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-semigroup generated by $-iA$. Hence, iA generates a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-group $(S_{b,t_1,t_2}(r))_{r\in\mathbb{R}}$ given by: $S_{b,t_1,t_2}(r) = T_{b,-}(t_2)S_{b,t_1,+}(r)$, $r \geqslant 0$ and $S_{b,t_1,t_2}(r) = T_{b,+}(t_1)S_{b,t_2,-}(-r)$, $r < 0$. This yields (i3) and (i4), while the proof of (i5) follows immediately from (i4) and (3.1).

To prove (ii), choose arbitrarily numbers $\tau \in (0,\infty)$ and $s \in (1,2)$. Denote by Γ_l the upwards oriented boundary of $\Lambda_{\delta,\epsilon,l}$ and notice that there exists an appropriate constant $d_1 > 0$ such that $M(\lambda) \leqslant d_1\lambda^{1/s}$, $\lambda \geqslant 0$. Put

$$c_\tau = \frac{1}{2}\left[\frac{1}{\cos(\frac{\pi}{2s})}\tau\delta d_1 l^{1/s}\right]^{-s},$$

$$S_{K_{1/s,\,c_\tau},\pm}(t) = \frac{1}{2\pi i}\int_{\Gamma_l} e^{\lambda t}\widetilde{K_{1/s,\,c_\tau}}(\lambda)R(\lambda : \pm iA)\,d\lambda, \quad t \in [0,\tau),$$

$S_{K_{1/s,\,c_\tau}}(t) = S_{K_{1/s,\,c_\tau},+}(t)$, $t \in [0,\tau)$ and $S_{K_{1/s,\,c_\tau}}(t) = S_{K_{1/s,\,c_\tau},-}(-t)$, $t \in (-\tau,0)$. By [**41**, Theorem 1.3.2, p. 58], one obtains that $S_{K_{1/s,\,c_\tau},\pm}(t) \in L(E)$, $t \in [0,\tau)$ and that $\pm iA$ generate local $K_{1/s,\,c_\tau}$-convoluted semigroups $(S_{K_{1/s,\,c_\tau},\pm}(t))_{t\in[0,\tau)}$. Further, an employment of Theorem 2.2 shows that iA generates a local $K_{1/s,\,c_\tau}$-convoluted group $(S_{K_{1/s,\,c_\tau}}(t))_{t\in(-\tau,\tau)}$. The elementary inequality $|e^{\lambda h} - 1| \leqslant h|\lambda|e^{\operatorname{Re}\lambda h}$, $\lambda \in \mathbb{C}$, $h > 0$ and the dominated convergence theorem imply that the mappings $t \mapsto S_{K_{1/s,\,c_\tau}}(\pm t)$, $t \in [0,\tau)$ are infinitely differentiable and that

$$\frac{d^p}{dt^p}S_{K_{1/s,\,c_\tau}}(\pm t) = \frac{1}{2\pi i}\int_{\Gamma_l} \lambda^p e^{\lambda t}\widetilde{K_{1/s,\,c_\tau}}(\lambda)R(\lambda : \pm iA)\,d\lambda, \quad t \in [0,\tau),\ p \in \mathbb{N}_0. \tag{3.2}$$

Due to the choice of c_τ, there is a number $h \in (0,\infty)$ satisfying:

$$d_1h^{1/s} + \tau\delta d_1 l^{1/s} < \cos(\pi/2s)\,c_\tau^{-1/s}. \tag{3.3}$$

Taking into account (3.2) and (3.3), one gets:

$$\begin{aligned}
&\sup_{t\in(-\tau,\tau)\setminus\{0\},\,p\in\mathbb{N}_0} \frac{h^p\left\|\frac{d^p}{dt^p}S_{K_{1/s,c_\tau}}(t)\right\|}{p!^s}\\
&\leqslant C\sup_{t\in(-\tau,\tau)\setminus\{0\},\,p\in\mathbb{N}_0}\int_{\Gamma_l}\frac{(h|\lambda|)^p}{p!^s}e^{\operatorname{Re}\lambda|t|}\left|\widetilde{K_{1/s},c_\tau}(\lambda)\right|\|R(\lambda : \pm iA)\|\,|d\lambda|\\
&\leqslant C\sup_{t\in(-\tau,\tau)\setminus\{0\}}\int_{\Gamma_l} e^{M(h|\lambda|)}e^{|t|(\delta M(l|\operatorname{Im}\lambda|)+\epsilon)}e^{-\cos(\pi/2s)c_\tau^{-1/s}}|\lambda|^{1/s}|\lambda|^{(\alpha+1)/2}|d\lambda|\\
&\leqslant C\sup_{t\in(-\tau,\tau)\setminus\{0\}}\int_{\Gamma_l} e^{d_1h^{1/s}|\lambda|^{1/s}}e^{|t|(\delta d_1 l^{1/s}|\lambda|^{1/s}+\epsilon)}e^{-\cos(\pi/2s)c_\tau^{-1/s}|\lambda|^{1/s}}|\lambda|^{(\alpha+1)/2}|d\lambda|\\
&\leqslant Ce^{|\epsilon|\tau}\int_{\Gamma_l} e^{(d_1h^{1/s}+\tau\delta d_1 l^{1/s}-\cos(\pi/2s)c_\tau^{-1/s})|\lambda|^{1/s}}|\lambda|^{(\alpha+1)/2}|d\lambda| < \infty.
\end{aligned}$$

where $C = \text{const}$. $\square$

REMARK 3.1. Suppose, additionally, that A is densely defined. Then $-(\omega \mp iA)^b$ are generators of analytic semigroups $(T_{b,\pm}(t))_{t\in\Sigma_\gamma}$ of growth order $\frac{\alpha+1}{2b}$ (cf. the formulation of Theorem 3.1 and [**46**]–[**47**]).

The next example is inspired by [**31**, Example 6.1].

EXAMPLE 3.1. Suppose $E := L^2[0,\pi]$ and $A := -\Delta$ with the Dirichlet boundary conditions (cf. [**2**, Section 7.2]). It is well known that there exists an exponentially bounded kernel $K \in C([0,\infty))$ so that A generates a K-convoluted semigroup $(S_K(t))_{t\geqslant 0}$ with $\|S_K(t)\| = O(t+t^2)$, $t \geqslant 0$ ([**7**], [**29**]). This fact has been essentially utilized in [**31**, Section 6] where the authors proved that, for every $n \in \mathbb{N}$, there is an exponentially bounded kernel $K_n \in C([0,\infty))$ so that the polyharmonic operator Δ^{2^n} generates an exponentially bounded, analytic K_n-convoluted semigroup of angle $\frac{\pi}{2}$. On the other hand, an old result of Goldstein (see [**14**, p. 215] and [**31**, Section 6]) says that $-\Delta^{2n}$ generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$. This enables one to see that there exists an injective operator $C_n \in L(L^2[0,\pi])$ such that Δ^{2n} generates an entire C_n-regularized group (cf. [**14**, Section VII, Theorem 8.2]) and that Δ^{2^n} generates an exponentially bounded K_n-convoluted group. Finally, it is also worth noting that the operator $-\Delta$, considered in the first part of this example, generates an exponentially bounded, convoluted group and that, in the meantime, $-\Delta$ cannot be generator of any exponentially bounded, convoluted cosine function [**31**].

It is an open problem to obtain further properties of polynomials of $-\Delta$ in the framework of the theory of convoluted operator families, even in the case of Hurwitz polynomials.

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