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ON SEQUENCE-COVERING mssc-IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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ABSTRACT. We characterize sequence-covering (resp., 1-sequence-covering, 2-sequence-covering) mssc-images of locally separable metric spaces by means of σ-locally finite *cs*-networks (resp., *sn*-networks, *so*-networks) consisting of $\aleph_0$-spaces (resp., *sn*-second countable spaces, *so*-second countable spaces). As the applications, we get characterizations of certain sequence-covering, quotient mssc-images of locally separable metric spaces.

1. Introduction

A study of some images of metric spaces under certain mappings is an important task on general topology. In [**12**], Li characterized sequence-covering (pseudo-sequence-covering) mssc-images of metric spaces by means of $\aleph$-spaces as follows.

THEOREM 1.1. [**12**, Theorem 4] *The following are equivalent for a space X.*
(1) *X is an $\aleph$-space.*
(2) *X is a sequence-covering mssc-image of a metric space.*
(3) *X is a pseudo-sequence-covering mssc-image of a metric space.*

In [**18**], Lin and Yan characterized compact-covering, quotient π- and mssc-images of metric spaces by means of g-metrizable spaces, and this result has been proved by a quick and systematic proof in [**25**].

THEOREM 1.2. [**18**, Corollary 18] *The following are equivalent for a space X.*
(1) *X is a g-metrizable space.*
(2) *X is a compact-covering, quotient compact and mssc-image of a metric space.*
(3) *X is a compact-covering, quotient π- and mssc-image of a metric space.*
(4) *X is a compact-covering, quotient π- and σ-image of a metric space.*

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Related to the characterizations of images of metric spaces, many topologists were engaged in characterizing images of locally separable metric spaces, and some noteworthy results have been shown. In [**16**], Lin, Liu, and Dai characterized quotient s-images of locally separable metric spaces. After that, Lin and Yan characterized sequence-covering s-images of locally separable metric spaces [**17**]; Ikeda, Liu and Tanaka characterized quotient compact images of locally separable metric spaces [**11**]; Ge characterized pseudo-sequence-covering compact images of locally separable metric spaces [**8**]; An and Dung characterized quotient π-images of locally separable metric spaces [**1**]. In general, it is difficult to obtain nice characterizations of images of locally separable metric spaces (under covering-mappings) instead of metric domains.

Take the above into account, note that $\aleph$-spaces and g-metrizable spaces are spaces having certain σ-locally finite networks, the following question arises naturally.

QUESTION. *How are sequence-covering* (1*-sequence-covering,* 2*-sequence-covering*) *mssc-images of locally separable metric spaces characterized by means of σ-locally finite networks?*

In this paper, we characterize sequence-covering (resp., 1-sequence-covering, 2-sequence-covering) mssc-images of locally separable metric spaces by means of σ-locally finite *cs*-networks (resp., *sn*-networks, *so*-networks) consisting of $\aleph_0$-spaces (resp., *sn*-second countable spaces, *so*-second countable spaces). As the applications, we get characterizations of certain sequence-covering, quotient mssc-images of locally separable metric spaces. These results make the study of images of locally separable metric spaces more completely.

Throughout this paper, all spaces are regular and T_1, all mappings are continuous and onto, a convergent sequence includes its limit point, and $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \to Y$ be a mapping, and $\mathcal{P}$ be a family of subsets of X, we denote $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ converging to x is *eventually* in A if $\{x_n : n \geqslant n_0\} \cup \{x\} \subset A$ for some $n_0 \in \mathbb{N}$, and it is *frequently* in A if $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset A$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$.

DEFINITION 1.1. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is a *network* for X [**19**] if, $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$, where $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with U open in X, then there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$ for every $x \in X$. Here, $\mathcal{P}_x$ is a *network* at x in X.

(2) $\mathcal{P}$ is a *cs-network* for X [**10**] if, for each convergent sequence S converging to $x \in U$ with U open in X, S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is a *cs*-network* for X [**7**] if, for each convergent sequence S converging to $x \in U$ with U open in X, S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

(4) $\mathcal{P}$ is a *cfp-network* for X [**26**] if, for each compact subset $H \subset U$ with U open in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\} \subset U$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$.

DEFINITION 1.2. [**6**] Let X be a space and P be a subset of X.

(1) P is a *sequential neighborhood* of x in X, if whenever S is a convergent sequence converging to x, then S is eventually in P.

(2) P is a *sequentially open* subset of X, if P is a sequential neighborhood of x in X for every $x \in P$.

DEFINITION 1.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that, for each $x \in X$, $\mathcal{P}_x$ is a network at x in X, and if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base* for X [**23**], if $G \subset X$ such that for each $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then G is open in X. Here, $\mathcal{P}_x$ is a *weak base* at x in X.

(2) $\mathcal{P}$ is an *sn-network* for X [**15**], if each member of $\mathcal{P}_x$ is a sequential neighborhood of x in X. Here, $\mathcal{P}_x$ is an *sn-network* at x in X.

(3) $\mathcal{P}$ is an *so-network* for X [**15**], if each member of $\mathcal{P}_x$ is sequentially open in X. Here, $\mathcal{P}_x$ is an *so-network* at x in X.

DEFINITION 1.4. Let X be a space.

(1) X is a *cosmic* space [**20**] (resp., $\aleph_0$*-space* [**20**], *sn-second countable* space [**9**], *so-second countable* space, *second countable* space [**5**], $\aleph$*-space* [**21**], *g-metrizable* space [**23**]), if X has a countable network (resp., countable *cs*-network, countable *sn*-network, countable *so*-network, countable base, σ-locally finite *cs*-network, σ-locally finite weak base).

(2) X is a *sequential* space [**6**], if each sequentially open subset of X is open.

REMARK 1.1. [**17**] (1) For a space, weak base $\Rightarrow$ *sn*-network $\Rightarrow$ *cs*-network.
(2) An *sn*-network for a sequential space is a weak base.

DEFINITION 1.5. Let $f : X \to Y$ be a mapping.

(1) f is an *mssc-mapping* [**14**], if X is a subspace of the product space $\prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$ of metric spaces, and for each $y \in Y$, there exists a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of y in Y such that each $\overline{p_n(f^{-1}(V_{y,n}))}$ is a compact subset of X_n, where $p_n : \prod_{i\in\mathbb{N}} X_i \to X_n$ is the projection.

(2) f is an 1-*sequence-covering* mapping [**15**] if, for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to y in Y there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$.

(3) f is a 2-*sequence-covering* mapping [**15**] if, for each $y \in Y$, $x_y \in f^{-1}(y)$, and sequence $\{y_n : n \in \mathbb{N}\}$ converging to y in Y, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$.

(4) f is a *sequence-covering* mapping [**22**] if, for each convergent sequence S of Y, there exists a convergent sequence L of X such that $f(L) = S$. Note that a sequence-covering mapping is a *strong sequence-covering* mapping in the sense of [**12**].

(5) f is a *pseudo-sequence-covering* mapping [**11**] if, for each convergent sequence S of Y, there exists a compact subset K of X such that $f(K) = S$.

(6) f is a *sequentially-quotient* mapping [**3**] if, for each convergent sequence S of Y, there exists a convergent sequence L of X so that $f(L)$ is a subsequence of S.

(7) f is a *compact-covering* mapping [**20**] if, for each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.

(8) f is a π-*mapping* [**2**], if for each $y \in Y$ and for each neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.

(9) f is a σ-*mapping* [**18**], if there exists a base $\mathcal{B}$ of X such that $f(\mathcal{B})$ is a σ-locally finite family in Y.

DEFINITION 1.6. [**4**] A space X is *sequentially separable*, if X has a countable subset D such that for each $x \in X$, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ in D converging to x. Here, the subset D is a *sequentially dense* subset of X.

For undefined terms, refer to [**5**] and [**24**].

2. Results

First, we characterize sequence-covering mssc-images of locally separable metric spaces by means of σ-locally finite *cs*-networks.

THEOREM 2.1. *The following are equivalent for a space X.*

(1) *X is a sequence-covering mssc-image of a locally separable metric space.*
(2) *X has a σ-locally finite cs-network consisting of cosmic spaces.*
(3) *X has a σ-locally finite cs-network consisting of $\aleph_0$-spaces.*

PROOF. (1)$\Rightarrow$(2). Let $f : M \to X$ be a sequence-covering mssc-mapping from a locally separable metric space M onto X, and $\{X_n : n \in \mathbb{N}\}$ be the family of metric spaces satisfying that M is a subspace of $\prod_{n\in\mathbb{N}} X_n$, and for each $x \in X$, there exists a sequence $\{V_{x,n} : n \in \mathbb{N}\}$ of open neighborhoods of x in X such that each $\overline{p_n(f^{-1}(V_{x,n}))}$ is a compact subset of X_n, where $p_n : \prod_{i\in\mathbb{N}} X_i \to X_n$ is the projection. Since M is locally separable metric, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where each M_λ is a separable metric space by [**5**, 4.4.F]. Since each X_n is a metric space, X_n has a σ-locally finite base $\mathcal{C}_n = \bigcup\{\mathcal{C}_{n,i} : i \in \mathbb{N}\}$, where each $\mathcal{C}_{n,i}$ is locally finite. Assume, if necessary, that $\mathcal{C}_{n,i} \subset \mathcal{C}_{n,i+1}$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, set

$$\mathcal{B}_n = \Big\{M \cap \bigcap_{i\leqslant n} p_i^{-1}(C_i) : C_i \in \bigcup_{j\leqslant n} \mathcal{C}_{i,j},\ i \leqslant n,\ M \cap \bigcap_{i\leqslant n} p_i^{-1}(C_i) \subset M_\lambda \text{ for some } \lambda \in \Lambda\Big\},$$

set $\mathcal{P}_n = f(\mathcal{B}_n)$, and set $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$, $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$. Then $\mathcal{B}$ is a base for M consisting of separable subsets. Assume, if necessary, that $\mathcal{B}$ is closed under finite intersections. We shall show that $\mathcal{P}$ is a σ-locally finite *cs*-network for X consisting of cosmic spaces by the following facts (a), (b), and (c).

(a) $\mathcal{P}$ is a *cs*-network for X.

Let S be a convergent sequence being eventually in U with U open in X. Since f is sequence-covering, there exists a convergent sequence L in M such that

$f(L)=S$. Since L is eventually in $B\subset f^{-1}(U)$ for some $B\in\mathcal{B}$, S is eventually in $f(B)\subset U$. It implies that S is eventually in $P\subset U$ with $P=f(B)\in\mathcal{P}$. Therefore, $\mathcal{P}$ is a cs-network for X.

(b) $\mathcal{P}$ is σ-locally finite.

For each $x\in X$ and $n\in\mathbb{N}$, set $V_x=\bigcap_{i\leqslant n}V_{x,i}$, then V_x is an open neighborhood of x in X. For each $i\in\mathbb{N}$, since $\overline{p_i(f^{-1}(V_{x,i}))}$ is a compact subset of X_i and $\mathcal{C}_{i,j}$ is locally finite, $p_i(f^{-1}(V_{x,i}))$ meets only finitely many members of $\mathcal{C}_{i,j}$ for every $j\in\mathbb{N}$. Then $f^{-1}(V_{x,i})$ meets only finitely many members of $\{p_i^{-1}(C_i):C_i\in\bigcup_{j\leqslant n}\mathcal{C}_{i,j}\}$. Therefore, $f^{-1}(V_x)$ meets only finitely many members of $\big\{\bigcap_{i\leqslant n}p_i^{-1}(C_i):C_i\in\bigcup_{j\leqslant n}\mathcal{C}_{i,j},\ i\leqslant n\big\}$. It implies that $f^{-1}(V_x)$ meets only finitely many members of $\mathcal{B}_n$. Hence V_x meets only finitely many members of $f(\mathcal{B}_n)$, i.e., $\mathcal{P}_n$ is locally finite. It follows that $\mathcal{P}$ is σ-locally finite.

(c) Each $P\in\mathcal{P}$ is a cosmic space.

Set $P=f(B)$ for some $B\in\mathcal{B}$. Since B is separable, P is cosmic.

(2) $\Rightarrow$(3). Let $\mathcal{P}=\bigcup\{\mathcal{P}_n:n\in\mathbb{N}\}$ be a σ-locally finite cs-network for X consisting of cosmic spaces. Every locally finite family in a Lindelöf space is countable. Hence for each $P\in\mathcal{P}$, $\{P\cap P':P'\in\mathcal{P}\}$ is countable, and obviously it is a cs-network for P.

(3) $\Rightarrow$(1). Let $\mathcal{P}=\bigcup\{\mathcal{P}_n:n\in\mathbb{N}\}$ be a σ-locally finite cs-network for X consisting of $\aleph_0$-spaces, where each $\mathcal{P}_n=\{P_{\alpha_n}:\alpha_n\in A_n\}$ is a locally finite family. For each $n\in\mathbb{N}$, since each P_{α_n} is an $\aleph_0$-space, P_{α_n} has a countable cs-network $\mathcal{P}_{\alpha_n}=\{P_{\alpha_{n,i}}:i\geqslant n\}$. For each $i\geqslant n$, set

$$\mathcal{Q}_{\alpha_{n,i}}=\{P_{\alpha_n}\}\cup\{P_{\alpha_{n,j}}:n\leqslant j\leqslant i\}=\{Q_\beta:\beta\in B_{\alpha_{n,i}}\},$$

where $B_{\alpha_{n,i}}$ is finite, and set

$$\mathcal{Q}_i=\{X\}\cup\big(\bigcup\{\mathcal{Q}_{\alpha_{j,i}}:\alpha_j\in A_j,j\leqslant i\}\big)=\{Q_\beta:\beta\in B_i\},$$

where $B_i=\{\beta_0\}\cup\big(\bigcup\{B_{\alpha_{j,i}}:\alpha_j\in A_j,j\leqslant i\}\big)$ with $Q_{\beta_0}=X$. Since each $\mathcal{P}_i$ is locally finite and each $\mathcal{Q}_{\alpha_{j,i}}$ is finite, $\mathcal{Q}_i$ is locally finite. Endow B_i with the discrete topology, then B_i is a metric space. Set

$$\begin{aligned}M=\Big\{b=(\beta_i)\in\prod_{i\in\mathbb{N}}B_i:{}&\text{there exists } n\in\mathbb{N}\text{ and }\alpha_n\in A_n\text{ such that}\\&Q_{\beta_i}=X\text{ if } i<n,\ Q_{\beta_i}\in\mathcal{Q}_{\alpha_{n,i}}\text{ if } i\geqslant n,\text{ and}\\&\{Q_{\beta_i}:i\geqslant n\}\text{ forms a network at a point }x_b\text{ in }P_{\alpha_n}\Big\}.\end{aligned}$$

Then M, which is a subspace of the product space $\prod_{i\in\mathbb{N}}B_i$, is a metric space. Since X is T_1 and regular, x_b is unique for every $b\in M$. We define $f:M\to X$ by $f(b)=x_b$ for every $b\in M$.

(a) f is onto.

For each $x\in X$, there exists $n\in\mathbb{N}$ and $\alpha_n\in A_n$ such that $x\in P_{\alpha_n}$. Since $\mathcal{P}_{\alpha_n}$ is a countable cs-network for P_{α_n}, $(\mathcal{P}_{\alpha_n})_x=\{Q_\beta\in\mathcal{P}_{\alpha_n}:x\in Q_\beta\}$ is a countable network at x in P_{α_n}. We may assume that $(\mathcal{P}_{\alpha_n})_x=\{P_{x,j}:j\in\mathbb{N}\}$, where $P_{x,j}\in\mathcal{Q}_{\alpha_{n,i(j)}}$ with some $i(j)\in\mathbb{N}$ satisfying $i(j)<i(j+1)$. For each $i\in\mathbb{N}$, take Q_{β_i} as follows.

(i) $i < n$: $Q_{\beta_i} = X$,

(ii) $i \geqslant n$: $Q_{\beta_i} = P_{x,j}$ if $i = i(j)$ for some $j \in \mathbb{N}$, and otherwise, $Q_{\beta_i} = P_{\alpha_n}$.

Then $\{Q_{\beta_i} : i \geqslant n\} - \{P_{\alpha_n}\} = (\mathcal{P}_{\alpha_n})_x - \{P_{\alpha_n}\}$. Therefore, $\{Q_{\beta_i} : i \geqslant n\}$ forms a network at x in P_{α_n}. It implies that $b = (\beta_i) \in M$ satisfying $x = f(b)$, i.e., f is onto.

(b) f is continuous.

For each $b = (\beta_i) \in M$ and $x = f(b) \in U$ with U open in X. Then $x = f(b) \in Q_{\beta_k} \subset U$ for some $k \in \mathbb{N}$. Set $U_b = \{c = (\gamma_i) \in M : \gamma_k = \beta_k\}$. Then U_b is open in M, and $b \in U_b$. For each $c \in U_b$, we find $f(c) \in Q_{\gamma_k} = Q_{\beta_k} \subset U$. It implies that $f(U_b) \subset U$, i.e., f is continuous.

(c) M is locally separable.

Let $b = (\beta_i) \in M$. Then there exists $n \in \mathbb{N}$ and $\alpha_n \in A_n$ such that $Q_{\beta_i} = X$ if $i < n$, $Q_{\beta_i} \in \mathcal{Q}_{\alpha_{n,i}}$ if $i \geqslant n$, and $\{Q_{\beta_i} : i \geqslant n\}$ forms a network at a point x_b in P_{α_n}. Set $M_b = \{c = (\gamma_i) \in M : \gamma_n = \beta_n\}$. Then M_b is open in M, and $b \in M_b$. For each $c = (\gamma_i) \in M_b$, there exists $m \in \mathbb{N}$ and $\alpha_m \in A_m$ such that $Q_{\gamma_i} = X$ if $i < m$, $Q_{\gamma_i} \in \mathcal{Q}_{\alpha_{m,i}}$ if $i \geqslant m$, and $\{Q_{\gamma_i} : i \geqslant m\}$ forms a network at a point x_c in P_{α_m}. It follows from $Q_{\gamma_n} = Q_{\beta_n}$ that $P_{\alpha_m} \cap P_{\alpha_n} \neq \emptyset$. Since P_{α_n} is an $\aleph_0$-space and $\mathcal{P}_m$ is locally finite, $C_m = \{\alpha_m \in A_m : P_{\alpha_m} \cap P_{\alpha_n} \neq \emptyset\}$ is countable for every $m \in \mathbb{N}$. Then $E_i = \{\beta_0\} \cup \big(\bigcup\{B_{\alpha_{j,i}} : \alpha_j \in C_j, j \leqslant i\}\big)$ is countable. It implies that $\{\beta_1\} \times \cdots \times \{\beta_{n-1}\} \times \prod_{i \geqslant n} E_i$ is hereditarily separable. Since $M_b \subset \{\beta_1\} \times \cdots \times \{\beta_{n-1}\} \times \prod_{i \geqslant n} E_i$, M_b is separable. Therefore, M is locally separable.

(d) f is an mssc-mapping.

For each $x \in X$ and each $i \in \mathbb{N}$, since $\mathcal{P}_i$ is locally finite, there exists an open neighborhood $V_{x,i}$ of x in X such that $D_i = \{\alpha_i \in A_i : P_{\alpha_i} \cap V_{x,i} \neq \emptyset\}$ is finite. Then $F_i = \{\beta_0\} \cup \big(\bigcup\{B_{\alpha_{j,i}} : \alpha_j \in D_j, j \leqslant i\}\big)$ is finite. Since $p_i(f^{-1}(V_{x,i})) \subset F_i$, $\overline{p_i(f^{-1}(V_{x,i}))}$ is compact. It implies that f is an mssc-mapping.

(e) f is sequence-covering.

For each convergent sequence S in X, since $\mathcal{P}$ is a σ-locally finite *cs*-network for X, there exists $n \in \mathbb{N}$ and $\alpha_n \in A_n$ such that S is eventually in $P_{\alpha_n} \in \mathcal{P}_n$. Then $L_{\alpha_n} = S \cap P_{\alpha_n}$ is a convergent sequence in P_{α_n}. For each $i \geqslant n$, we find that $\bigcup\{\mathcal{Q}_{\alpha_{n,i}} : i \geqslant n\}$ is a σ-locally finite *cs*-network for P_{α_n} satisfying $P_{\alpha_n} \in \mathcal{Q}_{\alpha_{n,i}} \subset \mathcal{Q}_{\alpha_{n,i+1}}$. It follows from the proof (3)⇒(2) of [**13**, Theorem 5.1] that there exists a convergent sequence H_{α_n} in M_{α_n} such that $f_{\alpha_n}(H_{\alpha_n}) = L_{\alpha_n}$, where

$$M_{\alpha_n} = \Big\{c = (\gamma_i)_{i \geqslant n} \in \prod_{i \geqslant n} B_{\alpha_{n,i}} : \{Q_{\gamma_i} : i \geqslant n\} \text{ forms a network at a point } x_c \text{ in } P_{\alpha_n}\Big\},$$

and $f_{\alpha_n} : M_{\alpha_n} \to P_{\alpha_n}$ defined by $f_{\alpha_n}(c) = x_c$ for every $c \in M_{\alpha_n}$. For each $c = (\gamma_i)_{i \geqslant n} \in H_{\alpha_n}$, set $b_c = (\beta_i)_{i \in \mathbb{N}}$, where $Q_{\beta_i} = X$ if $i < n$ and $\beta_i = \gamma_i$ if $i \geqslant n$, and set $H = \{b_c : c \in H_{\alpha_n}\}$. Then H is a convergent sequence in M and $f(H) = L_{\alpha_n}$. Since S is eventually in P_{α_n}, $S - P_{\alpha_n}$ is finite. Then $S - P_{\alpha_n} = f(F)$ with some finite subset F of M. Set $L = H \cup F$, then L is a convergent sequence in M satisfying $f(L) = S$. It implies that f is sequence-covering. □

REMARK 2.1. The argument for cs-networks in the proof(2) $\Rightarrow$(3) of Theorem 2.1 can not apply to cs^*-networks or cfp-networks.

COROLLARY 2.1. *The following are equivalent for a space X.*

(1) *X is a sequence-covering, quotient mssc-image of a locally separable metric space.*
(2) *X is a sequential space having a σ-locally finite cs-network consisting of cosmic spaces.*
(3) *X is a sequential space having a σ-locally finite cs-network consisting of $\aleph_0$-spaces.*

PROOF. (1) $\Rightarrow$ (2). Since X is a quotient image of a locally separable metric space, X is a sequential space by [**6**, Proposition 1.2]. Then X is a sequential space having a σ-locally finite cs-network consisting of cosmic spaces by Theorem 2.1.

(2) $\Rightarrow$ (3). As in the proof(2) $\Rightarrow$(3) of Theorem 2.1.

(3) $\Rightarrow$(1). It follows from Theorem 2.1 that X is a sequence-covering mssc-image of a locally separable metric space under some mapping f. Since f is a sequence-covering mapping onto a sequential space, f is a quotient mapping by [**17**, Lemma 3.5]. It implies that X is a sequence-covering, quotient mssc-image of a locally separable metric space. □

Next, we characterize 1-sequence-covering mssc-images of locally separable metric spaces by means of σ-locally finite sn-networks.

THEOREM 2.2. *The following are equivalent for a space X.*

(1) *X is an 1-sequence-covering mssc-image of a locally separable metric space.*
(2) *X has a σ-locally finite sn-network consisting of cosmic spaces.*
(3) *X has a σ-locally finite sn-network consisting of sn-second countable spaces.*

PROOF. (1) $\Rightarrow$ (2). Let $f : M \to X$ be an 1-sequence-covering mssc-mapping from a locally separable metric space M onto X. For each $x \in X$, let $a_x \in f^{-1}(x)$ satisfying that whenever $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X there exists a sequence $\{a_n : n \in \mathbb{N}\}$ converging to a_x in M with each $a_n \in f^{-1}(x_n)$. By using notations in the proof (1) $\Rightarrow$ (2) of Theorem 2.1 again, let $\mathcal{Q}_x = \{P \in \mathcal{P} : P = f(B) \text{ with } a_x \in B \in \mathcal{B}\}$, and let $\mathcal{Q} = \bigcup\{\mathcal{Q}_x : x \in X\}$. We shall prove that $\mathcal{Q}$ is a σ-locally finite sn-network for X consisting of cosmic spaces by the following facts (a), (b), (c) for every $x \in X$, and (d), (e).

(a) $\mathcal{Q}_x$ is a network at x in X.

It is clear that $x \in \bigcap \mathcal{Q}_x$. Let $x \in U$ with U open in X, then $a_x \in f^{-1}(U)$. Since $\mathcal{B}$ is a base for M, $a_x \in B \subset f^{-1}(U)$ for some $B \in \mathcal{B}$. Set $Q = f(B)$, then $Q \in \mathcal{Q}_x$ and $x \in Q \subset U$. It implies that $\mathcal{Q}_x$ is a network at x in X.

(b) If $Q_1, Q_2 \in \mathcal{Q}_x$, then $Q \subset Q_1 \cap Q_2$ for some $Q \in \mathcal{Q}_x$.

Set $Q_1 = f(B_1), Q_2 = f(B_2)$, where $B_1, B_2 \in \mathcal{B}$ with $a_x \in B_1$ and $a_x \in B_2$. Since $\mathcal{B}$ is a base for M, $a_x \in B \subset B_1 \cap B_2$ for some $B \in \mathcal{B}$. Set $Q = f(B)$, then $Q \in \mathcal{Q}_x$ and $Q \subset Q_1 \cap Q_2$.

(c) Each $Q \in \mathcal{Q}_x$ is a sequential neighborhood of x.

Set $Q = f(B)$ with $a_x \in B \in \mathcal{B}$. For each convergent sequence S converging to x, there exists a convergent sequence L converging to a_x in M such that $f(L) = S$. Since L is eventually in B, S is eventually in Q. It implies that Q is a sequential neighborhood of x.

(d) $\mathcal{Q}$ is σ-locally finite.

Since $\mathcal{Q} \subset \mathcal{P}$ and $\mathcal{P}$ is σ-locally finite, $\mathcal{Q}$ is σ-locally finite.

(e) Each $Q \in \mathcal{Q}$ is a cosmic space.

Set $Q = f(B)$ for some $B \in \mathcal{B}$. Since B is separable, Q is cosmic.

(2) $\Rightarrow$ (3). As in the proof(2) $\Rightarrow$ (3) of Theorem 2.1.

(3) $\Rightarrow$ (1). Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally finite sn-network for X consisting of $\aleph_0$-spaces. By using notations and arguments in the proof (3) $\Rightarrow$ (1) of Theorem 2.1 again, since each sn-network is also a cs-network, it suffices to prove that the mapping f is 1-sequence-covering.

For each $x \in X$, since $\mathcal{P}$ is a σ-locally finite sn-network for X, there exists $n \in \mathbb{N}$ and $\alpha_n \in A_n$ such that P_{α_n} is a sequential neighborhood of x. Then $\bigcup\{\mathcal{Q}_{\alpha_{n,i}} : i \geqslant n\}$ is a σ-locally finite sn-network for P_{α_n}. It implies that f_{α_n} is 1-sequence-covering by [**13**, Theorem 2.1]. Hence, there exists $c_x = (\gamma_{x,i})_{i \geqslant n} \in f_{\alpha_n}^{-1}(x)$ such that whenever $\{x_m : m \in \mathbb{N}\}$ is a sequence converging to x in P_{α_n} there exists a sequence $\{c_m : m \in \mathbb{N}\}$ converging to c_x in M_{α_n} with each $c_m \in f_{\alpha_n}^{-1}(x_m)$. Set $b_x = (\beta_{x,i})$, where $Q_{\beta_{x,i}} = X$ if $i < n$ and $\beta_{x,i} = \gamma_{x,i}$ if $i \geqslant n$, then $b_x \in f^{-1}(x)$. Let $\{y_m : m \in \mathbb{N}\}$ be a sequence in X converging to x. Since P_{α_n} is a sequential neighborhood of x, there exists $m_0 \in \mathbb{N}$ such that $\{y_m : m \geqslant m_0\} \subset P_{\alpha_n}$ is a sequence converging to x in P_{α_n}. Then there exists a sequence $\{c_m : m \geqslant m_0\}$ in M_{α_n} converging to c_x and $c_m \in f_{\alpha_n}^{-1}(y_m)$ for each $m \geqslant m_0$. For each $c_m = (\gamma_{m,i})_{i \geqslant n}$, set $b_m = (\beta_{m,i})$, where $Q_{\beta_{m,i}} = X$ if $i < n$ and $\beta_{m,i} = \gamma_{m,i}$ if $i \geqslant n$. Then $b_m \in M$ and $f(b_m) = y_m$ for each $m \geqslant m_0$. For each $m < m_0$, take some $b_m \in f^{-1}(y_m)$. Then $\{b_m : m \in \mathbb{N}\}$ is a sequence in M converging to b_x and $b_m \in f^{-1}(y_m)$ for each $m \in \mathbb{N}$. It implies that f is 1-sequence-covering. □

COROLLARY 2.2. *The following are equivalent for a space X.*

1. *X is an 1-sequence-covering, quotient mssc-image of a locally separable metric space.*
2. *X has a σ-locally finite weak base consisting of cosmic spaces.*
3. *X has a σ-locally finite weak base consisting of sn-second countable spaces.*

PROOF. (1) $\Rightarrow$ (2). Since X is a quotient image of a locally separable metric space, X is a sequential space by [**6**, Proposition 1.2]. Then X is a sequential space having a σ-locally finite sn-network $\mathcal{P}$ consisting of cosmic spaces by Theorem 2.2. It follows from Remark 1.1 that $\mathcal{P}$ is a weak base for X. Therefore, X has a σ-locally finite weak base consisting of cosmic spaces.

(2) $\Rightarrow$ (3). Since X has a σ-locally weak base, X is a sequential space. It follows from Theorem 2.2 that X is a sequential space having a σ-locally finite sn-network $\mathcal{P}$ consisting of sn-second countable spaces. By Remark 1.1, $\mathcal{P}$ is a weak base for X. It implies that X has a σ-locally finite weak base consisting of sn-second countable spaces.

(3) ⇒ (1). It follows from Theorem 2.2 that X is an 1-sequence-covering mssc-image of a locally separable metric space under some mapping f. Since X has a σ-locally finite weak base, X is a sequential space. Then f is an 1-sequence-covering mapping onto a sequential space, and so f is a quotient mapping by [**17**, Lemma 3.5]. It implies that X is an 1-sequence-covering, quotient mssc-image of a locally separable metric space. □

REMARK 2.2. We can replace "*cosmic spaces*" in Theorem 2.2 and Corollary 2.2 by "$\aleph_0$*-spaces*".

In the following, we characterize 2-sequence-covering mssc-images of locally separable metric spaces by means of σ-locally finite *so*-networks.

THEOREM 2.3. *The following are equivalent for a space X.*

(1) *X is a 2-sequence-covering mssc-image of a locally separable metric space.*
(2) *X has a σ-locally finite so-network consisting of cosmic spaces.*
(3) *X has a σ-locally finite so-network consisting of so-second countable spaces.*

PROOF. (1) ⇒ (2). Let $f : M \to X$ be a 2-sequence-covering mssc-mapping from a locally separable metric space M onto X. For each $x \in X$, by using notations in the proof(1) ⇒ (2) of Theorem 2.1 again, let $\mathcal{B}_x = \{B \in \mathcal{B} : f^{-1}(x) \cap B \neq \emptyset\}$, and let $\mathcal{R}_x$ be the family of all finite intersections of members of $f(\mathcal{B}_x)$. We shall prove that $\mathcal{R} = \bigcup\{\mathcal{R}_x : x \in X\}$ is a σ-locally finite *so*-network for X consisting of cosmic spaces by the following facts (a), (b), (c) for every $x \in X$ and (d), (e).

(a) $\mathcal{R}_x$ is a network at x in X.

This is obvious because $\mathcal{B}_x$ is a base for $f^{-1}(x)$.

(b) If $R_1, R_2 \in \mathcal{R}_x$, then $R \subset R_1 \cap R_2$ for some $R \in \mathcal{R}_x$.

This is obvious by choosing $R = R_1 \cap R_2$.

(c) Each $R \in \mathcal{R}_x$ is sequentially open.

Let $B \in \mathcal{B}_x$, $y \in f(B)$, and S be a convergent sequence converging to y. Since $y \in f(B)$, $f^{-1}(y) \cap B \neq \emptyset$. Take some $a_y \in f^{-1}(y) \cap B$. Then there exists a convergent sequence L converging to a_y in M such that $f(L) = S$. Since L is eventually in B, S is eventually in $f(B)$. It implies that $f(B)$ is sequentially open, i.e., every member of $f(\mathcal{B}_x)$ is sequentially open. Because R is some finite intersection of members of $f(\mathcal{B}_x)$, we find that R is sequentially open.

(d) $\mathcal{R}$ is σ-locally finite.

Since $\bigcup\{f(\mathcal{B}_x) : x \in X\} \subset \mathcal{P}$ and $\mathcal{P}$ is σ-locally finite, $\bigcup\{f(\mathcal{B}_x) : x \in X\}$ is σ-locally finite. It implies that $\mathcal{R}$ is σ-locally finite.

(e) Each $R \in \mathcal{R}$ is a cosmic space.

For each $B \in \mathcal{B}_x$, since B is separable, $f(B)$ is cosmic, i.e., every member of $f(\mathcal{B}_x)$ is cosmic. It implies that R is cosmic.

(2) ⇒ (3). As in the proof(2) ⇒(3) of Theorem 2.1.

(3) ⇒ (1). Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally finite *so*-network for X consisting of $\aleph_0$-spaces. By using notations and arguments in the proof (3) ⇒ (1) of Theorem 2.1 again, since each *so*-network is also a *cs*-network, it suffices to prove that the mapping f is 2-sequence-covering.

For each $x \in X$ and each $b_x \in f^{-1}(x)$, let $b_x = (\beta_{x,i})$. Then there exists some $n \in \mathbb{N}$ and $\alpha_n \in A_n$ such that $Q_{\beta_{x,i}} = X$ if $i < n$, $Q_{\beta_{x,i}} \in \mathcal{Q}_{\alpha_{n,i}}$ if $i \geqslant n$, and $\{Q_{\beta_{x,i}} : i \geqslant n\}$ forms a network at x in P_{α_n}. Set $c_x = (\beta_{x,i})_{i \geqslant n}$, then $c_x \in f^{-1}_{\alpha_n}(x)$. Since $\{\mathcal{Q}_{\alpha_{n,i}} : i \geqslant n\}$ is a σ-locally finite *so*-network for P_{α_n}, f_{α_n} is a 2-sequence-covering by [**13**, Theorem 3.1]. Let $\{x_m : m \in \mathbb{N}\}$ be a sequence converging to x in X. Since P_{α_n} is sequentially open, there exists $m_0 \in \mathbb{N}$ such that $\{x_m : m \geqslant m_0\}$ is a sequence converging to x in P_{α_n}. Then there exists a sequence $\{c_m : m \geqslant m_0\}$ in M_{α_n} converging to c_x and $c_m \in f^{-1}_{\alpha_n}(x_m)$ for each $m \geqslant m_0$. For each $c_m = (\gamma_{m,i})_{i \geqslant n}$, set $b_m = (\beta_{m,i})$, where $Q_{\beta_{m,i}} = X$ if $i < n$, and $\beta_{m,i} = \gamma_{m,i}$ if $i \geqslant n$. Then $b_m \in M$ and $f(b_m) = x_m$ for each $m \geqslant m_0$. For each $m < m_0$, take some $b_m \in f^{-1}(x_m)$. Then $\{b_m : m \in \mathbb{N}\}$ is a sequence in M converging to b_x and $b_m \in f^{-1}(x_m)$ for each $m \in \mathbb{N}$. It implies that f is 2-sequence-covering. □

Corollary 2.3. *The following are equivalent for a space X.*

(1) *X is a 2-sequence-covering, quotient mssc-image of a locally separable metric space.*
(2) *X has a σ-locally finite base consisting of cosmic spaces.*
(3) *X has a σ-locally finite base consisting of second countable spaces.*

Proof. (1) $\Rightarrow$ (2). Since X is a quotient image of a locally separable metric space, X is a sequential space by [**6**, Proposition 1.2]. It follows from Theorem 2.3 that X is a sequential space having a σ-locally finite *so*-network $\mathcal{P}$ consisting of cosmic spaces. For each $P \in \mathcal{P}$, since X is sequential and P is sequential open, P is open in X. Hence $\mathcal{P}$ is a σ-locally finite base for X consisting of cosmic spaces.

(2) $\Rightarrow$(3). It follows from Theorem 2.3 that X has a σ-locally finite *so*-network $\mathcal{P}$ consisting of *so*-second countable spaces. Since X has a σ-locally finite base, X is sequential. It implies that every $P \in \mathcal{P}$ is open. Then $\mathcal{P}$ is a σ-locally finite base consisting of *so*-second countable spaces.

Let $P \in \mathcal{P}$ and $\mathcal{Q}$ be a countable *so*-network for P. Since P is open, P is a sequential space by [**6**, Proposition 1.9]. Then every $Q \in \mathcal{Q}$ is open in P. Hence $\mathcal{Q}$ is a countable base for P. It implies that P is a second countable space.

By the above, X has a σ-locally finite base consisting of second countable spaces.

(3) $\Rightarrow$ (1). It follows from Theorem 2.3 that X is a 2-sequence-covering mssc-image of a locally separable metric space under some mapping f. Since X has a σ-locally finite base, X is sequential. Then f is a 2-sequence-covering mapping onto a sequential space, and so f is a quotient mapping by [**17**, Lemma 3.5]. It implies that X is a 2-sequence-covering, quotient mssc-image of a locally separable metric space. □

Remark 2.3. We can replace "*cosmic spaces*" in Theorem 2.3 and Corollary 2.3 by "*$\aleph_0$-spaces*", or "*sn-second countable spaces*".

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