PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 89(103) (2011), 11–17 DOI: 10.2298/PIM1103011D

THE ERDŐS–KAC THEOREM FOR CURVATURES IN INTEGRAL APOLLONIAN CIRCLE PACKINGS

Goran Djanković

Communicated by Žarko Mijajlović

Abstract. The Erdős–Kac type result for curvatures of circles in integral Apollonian packings is proved.

1. Introduction

A classical Descartes configuration consists of four mutually tangent circles in the plane. An Apollonian circle packing $\mathcal{P}$ arises as a result of the following procedure: starting with a given Descartes configuration, one can construct four new circles, each of which is tangent to three of the given ones and then keep repeating the procedure for each subconfiguration. In this note we consider bounded Apollonian packings (an unbounded one arises if two of the circles in the starting configuration are parallel lines whose tangency point is at infinity). The Hausdorff dimension $\alpha = 1.305...$ of the residual set obtained by removing of all interiors of circles in the packing $\mathcal{P}$, does not depend on $\mathcal{P}$, since any two packings can be transformed into each other by Möbius transformations.

The curvature of a circle is the reciprocal of its radius, and for a bounded Apollonian packing $\mathcal{P}$ we denote by $N^{\mathcal{P}}(T)$ the number of circles in the packing $\mathcal{P}$ whose curvature is at most T. Concerning the growth rate of $N^{\mathcal{P}}(T)$, the first result is showed in [**2**]:

$$\lim_{T\to\infty} \frac{\log N^{\mathcal{P}}(T)}{\log T} = \alpha,$$

and more recently and more precisely, Kontorovich and Oh in [**6**], using the Lax–Phillips spectral theory for the Laplacian on the infinite volume hyperbolic three manifold and the corresponding Patterson–Sullivan base eigenfunction theory, obtained the purely polynomial asymptotic

$$N^{\mathcal{P}}(T) \sim cT^{\alpha}, \qquad T \to \infty,$$

2010 *Mathematics Subject Classification:* Primary 11N60; Secondary 11N36.

for some constant $c = c(\mathcal{P}) > 0$.

The curvatures (a, b, c, d) of four circles in a Descartes configuration satisfy the quadratic equation

$$a^2 + b^2 + c^2 + d^2 = \frac{1}{2}(a + b + c + d)^2,$$

from which easily follows that if the curvatures in the initial Descartes configuration of $\mathcal{P}$ are integers, then all others are in $\mathbb{Z}$ as well, and in that case we say that the packing $\mathcal{P}$ is integral. Moreover, if we assume that $\gcd(a, b, c, d) = 1$ for the initial curvatures, then we call $\mathcal{P}$ a primitive integral. One such configuration is for example $(-6, 11, 14, 23)$, where -6 indicates different orientation of the outer circle.

Such integral Apollonian circle packings are the source of intriguing arithmetic questions, some of them described in [**3**]. For example, if $\pi^{\mathcal{P}}(T)$ denotes the number of circles with *prime* curvature at most T in bounded, primitive integral Apollonian packing $\mathcal{P}$, Sarnak showed in [**7**] that

$$\pi^{\mathcal{P}}(T) \gg \frac{T}{(\log T)^{3/2}},$$

by reduction to Iwaniec's result in [**5**], obtained by half-dimensional sieve. Conjecturally,

$$\pi^{\mathcal{P}}(T) \sim c\frac{T^{\alpha}}{\log T},$$

and an upper bound of correct order of magnitude is proved in [**6**]: $\pi^{\mathcal{P}}(T) \ll \frac{T^{\alpha}}{\log T}$.

The classical Erdős–Kac theorem establishes normal distribution of the sequence $\left\{\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\right\}$, where $\omega(n)$ is the number of distinct prime divisors of n and n runs over all natural numbers. One can ask a similar question for any other arithmetically interesting sequence.

In this note we demonstrate the following Erdős–Kac type result concerning the distribution of numbers of distinct prime divisors of curvatures in integral Apollonian circle packings:

Theorem 1.1. *Let $\mathcal{P}$ be any bounded, integral, primitive Apollonian circle packing, and let $\mathcal{AP}(T)$ be the multiset of curvatures of the circles in $\mathcal{P}$ of radius at least T^{-1}. Then for all $\alpha \leqslant \beta$, we have:*

$$\lim_{T\to\infty} \frac{1}{|\mathcal{AP}(T)|} \left| \left\{ a \in \mathcal{AP}(T) \mid \alpha \leqslant \frac{\omega(a) - \log\log T}{\sqrt{\log\log T}} \leqslant \beta \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt.$$

Notation is standard: $f = O(g)$ or $f \ll g$ means that for some constant $C > 0$, $|f| \leqslant Cg$ and $f \asymp g$ means that both $f \ll g$ and $g \ll f$ hold.

2. Sieving data: counting on orbits of a Kleinian group in a cone

In order to gain access to the arithmetic properties of the set of curvatures of an integral Apollonian packing that are required for the main result, one has to be able to perform the sieving procedure that we describe in this section. Such a sieve, on orbits of non-commutative groups is initiated and to the considerable extent developed by Bourgain, Gamburd and Sarnak in [**1**]. The existence of the underlying group is exactly the reason why extraction of such an arithmetic information is possible.

The orthogonal group corresponding to the Descartes quadratic form

$$Q(a,b,c,d) = a^2 + b^2 + c^2 + d^2 - \frac{1}{2}(a+b+c+d)^2$$

is given by

$$O_Q = \{g \in \mathrm{GL}_4 \mid Q(xg^t) = Q(x) \text{ for all } x \in \mathbb{R}^4\}.$$

Also, denote by $A = \langle S_1, S_2, S_3, S_4 \rangle$ the Apollonian group generated by reflections:

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

It is a subgroup of infinite index in $O_Q(\mathbb{Z}) = O_Q \cap GL_4(\mathbb{Z})$.

All Descartes' quadruples (a,b,c,d) lie on the cone $Q(x) = 0$ and moreover, all unordered Descartes quadruples in a fixed integral packing $\mathcal{P}$ can be obtained by action of A. It is proved in [**3**, theorems 3.2 and 3.3] that every bounded integral Apollonian circle packing $\mathcal{P}$ contains the unique *root quadruple* $\xi = (a,b,c,d)$ such that $a < 0 \leqslant b \leqslant c \leqslant d$ and $a+b+c \geqslant d$. This root quadruple ξ enables description of the set of curvatures occurring in $\mathcal{P}$ and counted with multiplicity, in terms of the orbit $\xi.A^t \subset \mathbb{Z}^4$: it consists precisely of the four entries of ξ and the largest entry in each $\xi\gamma^t$, where $\gamma \in A \smallsetminus \{I\}$.

The description of curvatures in $\mathcal{P}$ in terms of the orbit $\xi.A^t$ translates the problem of counting such circles with curvatures at most T to counting points on the orbit inside the cone:

$$B(T) = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid Q(x) = 0,\ \|x\| \leqslant T\},$$

while the curvature itself can be recovered as the value $x_1(\xi\gamma)$ of the polynomial x_1 at the orbit point.

Because of this setup and symmetry between a, b, c, d, we will investigate Erdős–Kac statistic on the multiset (counting with multiplicities)

$$\mathcal{AP}(T) = \{a \mid (a,b,c,d) \text{ a Descartes config. in } \mathcal{P} \text{ with } \max(|a|,|b|,|c|,|d|) \leqslant T\}$$

of curvatures at most T in the packing $\mathcal{P}$.

Now if we denote by $\iota : \mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Spin}_Q(\mathbb{R}) \to \mathrm{SO}_Q(\mathbb{R})$ the spin double cover, let $\Gamma = \iota^{-1}(\mathrm{SO}_Q(\mathbb{R})^\circ \cap A)$. Since ι factors through $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$, the group Γ is recognized as a Kleinian group, that is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, where $\mathbb{H}^3 = \{(x_1, x_2, y) \in \mathbb{R}^3 \mid y > 0\}$ is the hyperbolic upper half-space with the metric $\frac{dx_1^2+dx_2^2+dy^2}{y^3}$. The limit set $\Lambda(\Gamma)$ of a Kleinian group Γ is defined as a set of limit points of an orbit Γz, $z \in \mathbb{H}^3$ in the boundary $\partial_\infty(\mathbb{H}^3) = \mathbb{C} \cup \{\infty\}$. The Hausdorff dimension of $\Lambda(\Gamma)$ is denoted with δ_Γ, and in the case of Γ coming from the Apollonian group A, $\delta_\Gamma = \alpha$. For more information about these identifications, cf. [**6**].

The base eigenvalue in $\mathrm{Spec}(\Gamma \setminus \mathbb{H}^3)$, the spectrum of the Laplace operator Δ on $L^2(\Gamma \setminus \mathbb{H}^3)$, is given by $\lambda_0 = \delta_\Gamma(2-\delta_\Gamma) = \alpha(2-\alpha)$. The principal congruence subgroups of Γ are defined by

$$\Gamma(d) = \{\gamma \in \Gamma \mid \gamma \equiv I \pmod d\},$$

and for them $\mathrm{Spec}(\Gamma(d) \setminus \mathbb{H}^3) \supseteq \mathrm{Spec}(\Gamma \setminus \mathbb{H}^3)$. We will encounter for any square-free d also the following congruence subgroups $\Gamma_\xi(d) = \{\gamma \in \Gamma \mid \xi\gamma \equiv \xi \pmod d\} \supseteq \Gamma(d)$; obviously $\mathrm{Stab}_\Gamma(\xi) = \mathrm{Stab}_{\Gamma_\xi(d)}(\xi)$.

The cornerstone of the required counting results is the following *spectral gap* theorem (because of which it was necessary to pass from A to Γ):

Theorem (Bourgain, Gamburd, Sarnak). *For any Zariski dense subgroup* Γ *of* $\mathrm{Spin}_Q(\mathbb{Z})$*, with* $\delta_\Gamma > 1$*, there exists* $\theta \in [1, \delta_\Gamma)$*, such that for all square-free integers* d*,*

$$\mathrm{Spec}(\Gamma(d) \setminus \mathbb{H}^3) \cap [\theta(2-\theta), \delta_\Gamma(2-\delta_\Gamma)] = \{\delta_\Gamma(2-\delta_\Gamma)\}.$$

From the initial multiset $\mathcal{AP}(T)$, we define the sifting sequence $\mathcal{A}(T) = \{a_n(T)\}$, where

$$a_n(T)_{\text{prelim.}} = \sum_{\substack{\gamma \in A^t \\ \|\xi\gamma\| \leqslant T \\ x_1(\xi\gamma) = n}} 1.$$

In fact, because of the counting results in [**6**], we will switch to Γ and moreover we will count with a smoothed weight w_T (from definition 8.3 in [**6**], which controls the norm condition $\|\xi\gamma\| \leqslant T$ by its support):

$$a_n(T) = \sum_{\substack{\gamma \in \mathrm{Stab}_\Gamma(\xi) \setminus \Gamma \\ x_1(\xi\gamma) = n}} w_T(\gamma).$$

The crucial ingredient that we need is the asymptotic formula for square-free d, uniformly in some range $d \leqslant D$, for the quantity

$$\begin{aligned} |\mathcal{A}_d(T)| &= \sum_{n \equiv 0\ (d)} a_n(T) = \sum_{\substack{\gamma \in \mathrm{Stab}_\Gamma(\xi) \setminus \Gamma \\ x_1(\xi\gamma) \equiv 0\ (d)}} w_T(\gamma) \\ &= \sum_{\substack{\gamma_1 \in \Gamma_\xi(d) \setminus \Gamma \\ x_1(\xi\gamma_1) \equiv 0\ (d)}} \sum_{\gamma \in \mathrm{Stab}_\Gamma(\xi) \setminus \Gamma_\xi(d)} w_T(\gamma\gamma_1). \end{aligned}$$

Now the inner sum can be evaluated using the spectral gap theorem given above [**6**, Proposition 8.7]: there exists $\varepsilon_0 > 0$, *uniform* over all square-free integers d, such that for any congruence subgroup $\Gamma_\xi(d) < \Gamma$ and for any $\gamma_1 \in \Gamma$,

$$\sum_{\gamma \in \mathrm{Stab}_\Gamma(\xi) \backslash \Gamma_\xi(d)} w_T(\gamma\gamma_1) = \frac{c\|\xi\|^\alpha}{\alpha[\Gamma:\Gamma_\xi(d)]} T^\alpha + O(T^{\alpha-\varepsilon_0}),$$

where $c > 0$ is a constant depending on Γ and the smoothing parameters involved in w, but of course, independent of d. Together we have

$$|\mathcal{A}_d(T)| = \mathcal{O}_{x_1}(d) \left(\frac{1}{[\Gamma:\Gamma_\xi(d)]} X + O(X^{1-\varepsilon_0}) \right), \tag{2.1}$$

where

$$\mathcal{O}_{x_1}(d) = \sum_{\substack{\gamma_1 \in \Gamma_\xi(d) \backslash \Gamma \\ x_1(\xi\gamma_1) \equiv 0 \ (d)}} 1 \quad \text{and} \quad X = \alpha^{-1} c \|\xi\|^\alpha T^\alpha \quad \text{is a total mass.}$$

Moreover, if we set $g(d) := \frac{\mathcal{O}_{x_1}(d)}{[\Gamma:\Gamma_\xi(d)]}$, then g is multiplicative in the following sense: there is a finite set of bad primes $\mathcal{B}$ such that $g(d_1 d_2) = g(d_1) g(d_2)$ for any square-free $d_1 d_2$ with no prime factors in $\mathcal{B}$. Moreover, for any prime p not in $\mathcal{B}$, $0 < g(p) < 1$ and

$$g(p) = \frac{1}{p} + O\Big(\frac{1}{p^{3/2}}\Big). \tag{2.2}$$

For the proof of this, we refer to [**1**] or [**6**].

3. The Erdős–Kac for small prime divisors

We assume that for a finite multiset $\tilde{\mathcal{A}}$ of X natural numbers (or associated sifting sequence $\mathcal{A} = (a_n = |\{a \in \tilde{\mathcal{A}} : a = n\}|)_n$) we can find a real valued, non-negative multiplicative function $g(d)$, by means of which we can model the sequence $\mathcal{A}$ in the following sense: for a square-free d,

$$|\mathcal{A}_d| = \sum_{\substack{a \in \tilde{\mathcal{A}} \\ d|a}} 1 = \sum_{n \equiv 0 \ (d)} a_n = g(d)X + r_d,$$

where $g(d) \in [0,1]$ represents "probability" of finding elements in $\mathcal{A}$ with $d|n$ and r_d has a role of remainder term and should be small for each d or at least small on average in some range of d. In this general sieving environment Granville and Soundararajan proved the following moment result which describes the distribution of the number of "small" prime divisors of elements in $\tilde{\mathcal{A}}$. For a fixed parameter $z > 0$, let $\omega_z(n)$ denote the number of different primes $p \leqslant z$ which divide n.

Proposition (Granville, Soundararajan [**4**]). *Let*

$$\mu_z = \sum_{p \leqslant z} g(p) \quad \textit{and} \quad \sigma_z^2 = \sum_{p \leqslant z} g(p)(1 - g(p));$$

then uniformly for all natural numbers $k \leqslant \sigma_z^{2/3}$ we have

$$\sum_{a \in \tilde{\mathcal{A}}} (\omega_z(a) - \mu_z)^k = C_k X \sigma_z^k \left(1 + O\left(\frac{k^3}{\sigma_z^2}\right)\right) + O\Big(\mu_z^k \sum_{d \in D_k(z)} |r_d|\Big), \tag{3.1}$$

for k even, and

$$\sum_{a\in\tilde{\mathcal{A}}}(\omega_z(a)-\mu_z)^k \ll C_k X\sigma_z^k\,\frac{k^{3/2}}{\sigma_z}+\mu_z^k\sum_{d\in D_k(z)}|r_d|, \tag{3.2}$$

for k odd. Here $D_k(z)$ denotes the set of square-free integers which are the product of at most k primes $\leqslant z$ and $C_k=\frac{\Gamma(k+1)}{2^{k/2}\Gamma(k/2+1)}$.

4. Proof of Theorem 1.1

It will be enough to calculate asymptotically the moments

$$\sum_{a\in\mathcal{AP}(T)}(\omega(a)-\log\log T)^k, \tag{4.1}$$

for any fixed k. In fact asymptotic will hold uniformly for $k\leqslant(\log\log T)^{1/3}$. Information about the distribution of small prime divisors provided by Proposition from Section 3 applied on multiset $\mathcal{AP}(T)$ is sufficient to extract the correct leading term. Therefore, using (2.2) we find

$$\mu_z=\sigma_z^2=\log\log z+O(1).$$

Moreover, since $|\mathcal{O}_{x_1}(p)|=p^2+O(p^{3/2})$ for all p not in $\mathcal{B}$, we have $|\mathcal{A}_d|=g(d)X+r_d$ for all square-free d without prime factors in $\mathcal{B}$, $X\asymp T^\alpha$ defined as in Section 2 and from (2.1):

$$|r_d|\ll d^2X^{1-\varepsilon_0}.$$

Set $z=T^{1/s}$, for some parameter $s>0$, to be chosen later. Hence, we approximate

$$\omega(a)-\log\log T=\omega_z(a)-\mu_z+(\omega(a)-\omega_z(a))+(\mu_z-\log\log T)=\omega_z(a)-\mu_z+O(s),$$

and consequently, for some positive absolute constant c, the moment (4.1) is

$$=\sum_{a\in\mathcal{AP}(T)}(\omega_z(a)-\mu_z)^k+O\Bigg(\sum_{l=0}^{k-1}\binom{k}{l}(cs)^{k-l}\sum_{a\in\mathcal{AP}(T)}|\omega_z(a)-\mu_z|^l\Bigg). \tag{4.2}$$

Since

$$\mu_z^l\sum_{d\in D_l(z)}|r_d|\ll(\log\log z)^l\left(\frac{z}{\log z}\right)^l z^{2l}X^{1-\varepsilon_0},$$

as soon as $z\leqslant X^{\varepsilon_0/3k}=T^{\alpha\varepsilon_0/3k}$, the rightmost error terms in (3.1) and (3.2) are negligible. Now for the main term one can use directly the proposition, while in the error term for $0\leqslant l\leqslant k-1$ and l even we use (3.1), and for l odd, by Cauchy–Schwarz

$$\begin{aligned}\sum_{a\in\mathcal{AP}(T)}|\omega_z(a)-\mu_z|^l &\leqslant\Bigg(\sum_{a\in\mathcal{AP}(T)}(\omega_z(a)-\mu_z)^{l-1}\Bigg)^{1/2}\Bigg(\sum_{a\in\mathcal{AP}(T)}(\omega_z(a)-\mu_z)^{l+1}\Bigg)^{1/2}\\ &\ll(C_{l-1}C_{l+1})^{1/2}X\sigma_z^l.\end{aligned}$$

Taking $s = \frac{3k}{\alpha\varepsilon_0}$ and since we assume that $k \leqslant \sigma_z^{2/3}$ (range of uniformity required for the proposition) the error term in (4.2) is bounded by

$$\ll C_k X \sigma_z^k \sum_{l=0}^{k-1} \binom{k}{l} \frac{\max(C_l, \sqrt{C_{l-1}C_{l+1}})}{C_k} \left(\frac{3c}{\alpha\varepsilon_0} \frac{k}{\sigma_z}\right)^{k-l} \ll C_k X \sigma_z^{k-\frac{1}{3}}.$$

Hence, for all $k \in \mathbb{Z}_{>0}$ we have as $T \to \infty$

$$\frac{1}{|\mathcal{AP}(T)|} \sum_{a \in \mathcal{AP}(T)} \left(\frac{\omega(a) - \log\log T}{\sqrt{\log\log T}}\right)^k \to \begin{cases} \frac{k!}{2^{k/2}(k/2)!}, & k \text{ even;} \\ 0, & k \text{ odd.} \end{cases}$$

Since on the right hand side are the moments of a random variable with the normal distribution $\mathcal{N}(0,1)$, and the moments uniquely determine the distribution, the proof is finished.

References

1. J. Bourgain, A. Gamburd, P. Sarnak, *Affine linear sieve, expanders, and sum-product*, Invent. Math. **179** (2010), 559–644
2. D. Boyd, *The sequence of radii of the Apollonian packing*, Math. Comp. **39**(159) (1982), 249–254
3. R. Graham, J. Lagarias, C. Mallows, A. Wilks, C. Yan, *Apollonian circle packings: number theory*, J. Number Theory **100** (2003), 1–45
4. A. Granville, K. Soundararajan, *Sieving and the Erdős–Kac theorem*: in: Andrew Granville, and Zeév Rudnick (eds), *Proc. NATO Adv. Study Inst. on Equidistribution in Number Theory, Montreal*, NATO Sci. Ser. II, Math. Phys. Chem. 237, Springer, pp. 15Ű-27.
5. H. Iwaniec, *Primes represented by quadratic polynomials in two variables*, Acta Arith. **24** (1974), 435–459
6. A. Kontorovich, H. Oh *Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds* (with Appendix by H. Oh, N. Shah), arXiv: 0811.2236v4
7. P. Sarnak, *Letter to J. Lagarias about Integral Apollonian Packings*, 2007; available at www.math.princeton.edu/sarnak

University of Belgrade
Faculty of Mathematics
11000 Belgrade
Serbia
djankovic@matf.bg.ac.rs

(Received 09 07 2010)