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TOTAL REDUCTION OF LINEAR SYSTEMS OF OPERATOR EQUATIONS WITH THE SYSTEM MATRIX IN THE COMPANION FORM

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Abstract. We consider a total reduction of a nonhomogeneous linear system of operator equations with the system matrix in the companion form. Totally reduced system obtained in this manner is completely decoupled, i.e., it is a system with separated variables. We introduce a method for the total reduction, not by a change of basis, but by finding the adjugate matrix of the characteristic matrix of the system matrix. We also indicate how this technique may be used to connect differential transcendence of the solution with the coefficients of the system.

1. Introduction

The order of a linear operator equation is the highest power of the operator in the equation. The reduction of a nonhomogeneous linear system of the first order operator equations to the *partially reduced system*, i.e., to the system consisting of a higher order linear operator equation having only one variable and the rest of the first order linear operator equations in two variables, was studied in paper [**6**]. In this paper we will be concerned with the reduction of a nonhomogeneous linear system of the first order operator equations with the system matrix in the companion form to the *totally reduced system*, i.e., to the system with completely decoupled equations. The common method for transforming a system into the totally reduced system relies upon the changing of basis in which the system matrix is given in Jordan canonical form. In papers [**1**] and [**2**] we can find a procedure for determining the transformation matrix S such that $C = S^{-1} \cdot J \cdot S$, where C is the matrix in the companion form and J is the matrix in Jordan form. In

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order to establish the form of the totally reduced system we will use the form of the coefficients of the adjugate matrix of the characteristic matrix of the system matrix presented as A polynomial with matrix coefficients. As a consequence of this method we will get a connection between the entries of the system matrix and differential transcendence of the solution of a linear system of the first order differential equations with the system matrix in the companion form and with complex coefficients, where exactly one nonhomogeneous part is a differentially transcendental meromorphic function.

2. Preliminaries

In this section we will review some standard facts of linear algebra. A more complete presentation can be found in [**4, 5**].

Let C be an $n \times n$ matrix with coefficients in a field K. An element $\lambda \in K$ is called an eigenvalue of C with the corresponding eigenvector v if v is a nonzero $n \times 1$ column with coefficients in K such that $\lambda v = Cv$. The set of all eigenvectors with the same eigenvalue λ, together with the zero vector, is a vector space called the eigenspace of the matrix C that corresponds to the eigenvalue λ. The geometric multiplicity of an eigenvalue λ is defined as the dimension of the associated eigenspace, i.e., it is the number of linearly independent eigenvectors corresponding to that eigenvalue. The algebraic multiplicity of an eigenvalue λ is defined as the multiplicity of the corresponding root of the characteristic polynomial. A generalized eigenvector u of C associated to λ is a nonzero $n \times 1$ column with coefficients in K satisfying $(C - \lambda I)^k u = 0$, for some $k \in \mathbb{N}$. The set of all generalized eigenvectors for a given eigenvalue λ, together with the zero vector, form the generalized eigenspace for λ.

The $k \times k$ matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

is called the Jordan block of size k with eigenvalue λ. A matrix is said to be in a Jordan canonical form if it is a block diagonal matrix with Jordan blocks along the diagonal. The number of Jordan blocks corresponding to an eigenvalue λ is equal to its geometric multiplicity and the sum of their sizes is equal to the algebraic multiplicity of λ.

Invariant factors of matrix C are polynomials

$$i_1(\lambda) = \frac{D_1(\lambda)}{D_0(\lambda)}, \quad i_2(\lambda) = \frac{D_2(\lambda)}{D_1(\lambda)}, \ \dots \ i_r(\lambda) = \frac{D_r(\lambda)}{D_{r-1}(\lambda)},$$

where $D_j(\lambda)$ is the greatest common divisor of all the minors of order j in $\lambda I - C$ and $D_0(\lambda) = 1$, $1 \leqslant j \leqslant r$. The companion matrix of a polynomial $\Delta(\lambda) =$

$\lambda^n + d_1\lambda^{n-1} + \cdots + d_{n-1}\lambda + d_n$ is the matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -d_n & -d_{n-1} & -d_{n-2} & \dots & -d_2 & -d_1 \end{bmatrix}$$

It can easily be seen that the characteristic polynomial of the companion matrix C is $\Delta(\lambda)$. The characteristic equation $\Delta(\lambda) = 0$ can also be written in the following matrix form

$$(\lambda I - C) \cdot v(\lambda) = \begin{bmatrix} \lambda & -1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -1 \\ d_n & d_{n-1} & d_{n-2} & \dots & d_2 & \lambda + d_1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Delta(\lambda) \end{bmatrix} = \mathbb{O}.$$

If $\lambda_1, \lambda_2, \dots, \lambda_t$ are different eigenvalues of the matrix C, we can conclude from the previous equation that $v(\lambda_i)$ is the eigenvector corresponding to the eigenvalue λ_i. Since the rank of the matrix $\lambda_i I - C$ is equal to $n - 1$, it follows that for each eigenvalue there is only one eigenvector. Thus, geometric multiplicity of each eigenvalue is equal to 1 and the Jordan canonical form has exactly t blocks. The number of generalized eigenvectors associated to λ_i is equal to algebraic multiplicity k_i of λ_i. It also holds $\Delta(\lambda_i) = \Delta'(\lambda_i) = \cdots = \Delta^{(k_i-1)}(\lambda_i) = 0$ and $\Delta^{(k_i)}(\lambda_i) \neq 0$. Differentiating the corresponding matrix equation $k_i - 1$ times with respect to λ we obtain that

$$v'(\lambda_i),\ \ \frac{1}{2}v''(\lambda_i), \dots, \frac{1}{(k_i-1)!}v^{(k_i-1)}(\lambda_i)$$

are generalized eigenvectors. Let S be the matrix whose columns are these generalized eigenvectors. Then $J = S^{-1} \cdot C \cdot S$ is a Jordan canonical form of the matrix C, (see [**1, 2**] for more details).

In the next section we will derive some properties of a companion matrix that we need for the total reduction by finding the adjugate matrix of the characteristic matrix of the system matrix.

3. Properties of Companion Matrix of a Monic Polynomial

We already mentioned that the characteristic polynomial of the companion matrix C is $\Delta(\lambda)$. The minor of size $n-1$ of the matrix $\lambda I - C$ obtained by deleting the n-th row and the first column is equal to 1. Hence the minimal polynomial of the matrix C is also equal to $\Delta(\lambda)$ and all invariant factors of the matrix $\lambda I - C$ except the last one are 1. Furthermore, the determinant of the matrix C is $(-1)^n d_n$, and consequently if $d_n \neq 0$ the matrix C is invertible. Then the inverse matrix of the matrix C is

$$G=\begin{bmatrix}-\frac{d_{n-1}}{d_n} & -\frac{d_{n-2}}{d_n} & -\frac{d_{n-3}}{d_n} & \dots & -\frac{d_1}{d_n} & -\frac{1}{d_n}\\ 1 & 0 & 0 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 1 & 0\end{bmatrix}.$$

The adjugate matrix of C is

$$\operatorname{adj}(C)=(-1)^{n-1}\begin{bmatrix}d_{n-1} & d_{n-2} & d_{n-3} & \dots & d_1 & 1\\ -d_n & 0 & 0 & \dots & 0 & 0\\ 0 & -d_n & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -d_n & 0\end{bmatrix}.$$

Let the adjugate matrix of the characteristic matrix $\lambda I - C$ be written in the form $\operatorname{adj}(\lambda I - C) = \lambda^{n-1}C_0 + \lambda^{n-2}C_1 + \cdots + \lambda C_{n-2} + C_{n-1}$. Let us determine the coefficients C_k using the recurrences $C_k = C \cdot C_{k-1} + d_k I$, for $1 \leqslant k \leqslant n-1$ and $C_0 = I$. The recurrences are obtained by equating coefficients at the same powers of λ on the both sides of the equality $\operatorname{adj}(\lambda I - C)(\lambda)\cdot(\lambda I - C) = \Delta_C(\lambda)I$, (see [**5**]).

LEMMA 3.1. *Coefficients C_k, $1 \leqslant k \leqslant n-1$, of the matrix* $\operatorname{adj}(\lambda I - C)$ *are matrices of the form*

$$\begin{bmatrix}d_k & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 & 1 & \dots & 0 & 0 & 0\\ 0 & d_k & d_{k-1} & \dots & d_3 & d_2 & d_1 & \dots & 0 & 0 & 0\\ 0 & 0 & d_k & \dots & d_4 & d_3 & d_2 & \dots & 0 & 0 & 0\\ \vdots & & & \ddots & & & & \ddots & & & \vdots\\ 0 & 0 & 0 & \dots & d_k & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 & 1\\ -d_n & -d_{n-1} & -d_{n-2} & \dots & -d_{k+1} & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & -d_n & -d_{n-1} & \dots & -d_{k+2} & -d_{k+1} & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & -d_n & \dots & -d_{k+3} & -d_{k+2} & -d_{k+1} & \dots & 0 & 0 & 0\\ \vdots & & & \ddots & & & & \ddots & & & \vdots\\ 0 & 0 & 0 & \dots & -d_n & -d_{n-1} & -d_{n-2} & \dots & -d_{k+1} & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & -d_n & -d_{n-1} & \dots & -d_{k+2} & -d_{k+1} & 0\end{bmatrix}.$$

PROOF. The proof follows by induction on k. We have $C_0 = I$. For the coefficient C_1 it holds $C_1 = C \cdot I + d_1 I$, i.e.,

$$C_1=\begin{bmatrix}d_1 & 1 & 0 & \dots & 0 & 0\\ 0 & d_1 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & d_1 & 1\\ -d_n & -d_{n-1} & -d_{n-2} & \dots & -d_2 & 0\end{bmatrix}.$$

Suppose that

$$C_{k-1} = \begin{bmatrix} d_{k-1} & d_{k-2} & \dots & d_1 & 1 & \dots & 0 & 0 \\ 0 & d_{k-1} & \dots & d_2 & d_1 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 0 & 0 & \dots & d_{k-1} & d_{k-2} & \dots & d_1 & 1 \\ -d_n & -d_{n-1} & \dots & -d_k & 0 & \dots & 0 & 0 \\ 0 & -d_n & \dots & -d_{k+1} & -d_k & \dots & 0 & 0 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 0 & 0 & \dots & -d_n & -d_{n-1} & \dots & -d_k & 0 \end{bmatrix}.$$

Let $(C_{k-1})_{\to j}$ stand for the j-th row of matrix C_{k-1}, and let $(C \cdot C_{k-1})_{\to j}$ denote the j-th row of the product of the matrices C and C_{k-1}. Then $(C \cdot C_{k-1})_{\to j} = (C_{k-1})_{\to j+1}$, $1 \leqslant j \leqslant n-1$. We also have

$$\begin{aligned} &(C_{k-1})_{\to n} \cdot C \\ &= [\underbrace{0 \dots 0}_{k-2}\ -d_n\ -d_{n-1}\ \dots\ -d_{k+1}\ -d_k\ 0] \cdot \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -d_n & -d_{n-1} & -d_{n-2} & \dots & -d_2 & -d_1 \end{bmatrix} \\ &= [\underbrace{0 \dots 0}_{k-1}\ -d_n\ -d_{n-1}\ \dots\ -d_{k+1}\ -d_k\ 0] \end{aligned}$$

From $C \cdot C_{k-1} = C_{k-1} \cdot C$ we conclude that the above row is the last row of the product $C \cdot C_{k-1}$ and hence we deduce

$$C \cdot C_{k-1} = \begin{bmatrix} 0 & d_{k-1} & \dots & d_2 & d_1 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 0 & 0 & \dots & d_{k-1} & d_{k-2} & \dots & d_1 & 1 \\ -d_n & -d_{n-1} & \dots & -d_k & 0 & \dots & 0 & 0 \\ 0 & -d_n & \dots & -d_{k+1} & -d_k & \dots & 0 & 0 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 0 & 0 & \dots & -d_n & -d_{n-1} & \dots & -d_k & 0 \\ 0 & 0 & \dots & 0 & -d_n & \dots & -d_{k+1} & -d_k \end{bmatrix}.$$

Adding the matrix $d_k I$ to this product we obtain the matrix C_k. $\square$

4. The Main Result

In this section we will derive explicit formulas for the transformation of the following system (4.1) into the totally reduced system.

Let K be a field, V a vector space over K and $A : V \to V$ a linear operator on the vector space V. We will consider a nonhomogeneous linear system of operator equations of the form

$$\begin{aligned} A(x_1) &= x_2 + \varphi_1\\ A(x_2) &= x_3 + \varphi_2\\ &\ \vdots\\ A(x_{n-1}) &= x_n + \varphi_{n-1}\\ A(x_n) &= -d_n x_1 - d_{n-1}x_2 - \cdots - d_1 x_n + \varphi_n, \end{aligned} \tag{4.1}$$

for $d_i \in K$, $\varphi_i \in V$, $1 \leqslant i \leqslant n$. The system can be rewritten in the matrix form

$$\vec{A}(\vec{x}) = C\vec{x} + \vec{\varphi},$$

where $\vec{x} = [x_1\, x_2\, \dots\, x_n]^T \in K^{n\times 1}$ is the column of unknowns, $\vec{A} : V^{n\times 1} \to V^{n\times 1}$ is a vector operator defined componentwise $\vec{A}(\vec{x}) = [A(x_1)\, A(x_2)\, \dots\, A(x_n)]^T$, $\vec{\varphi} = [\varphi_1\, \varphi_2\, \dots\, \varphi_n]^T \in V^{n\times 1}$ is a nonhomogeneous term and the system matrix C is the companion matrix of the polynomial $\Delta(\lambda) = \lambda^n + d_1\lambda^{n-1} + \cdots + d_{n-1}\lambda + d_n$. Since A is a linear operator, the system $\vec{A}(\vec{x}) = C\vec{x} + \vec{\varphi}$ can be transformed into the system $\vec{A}(\vec{y}) = J\vec{y} + \vec{\psi}$ by multiplying on the left by the matrix S^{-1}, where J is the matrix in Jordan canonical form and S is a transformation matrix such that $C = S^{-1} \cdot J \cdot S$. Here $\vec{y} = S^{-1}\vec{x}$ is the column of unknowns and $\vec{\psi} = S^{-1}\vec{\varphi}$ is the nonhomogeneous term. If the matrix J has only one block, then system (4.1) can be transformed into the partially reduced system

$$\begin{aligned} A(y_1) &= \lambda y_1 + y_2 + \psi_1\\ A(y_2) &= \lambda y_2 + y_3 + \psi_2\\ &\ \vdots\\ A(y_{n-1}) &= \lambda y_{n-1} + y_n + \psi_{n-1}\\ A(y_n) &= \lambda y_n + \psi_n. \end{aligned}$$

The totally reduced system is obtained by acting of operators

$$(A-\lambda)^{n-1}, \dots,\ \ (A-\lambda)^2,\ \ A-\lambda$$

successively on the equations of the partially reduced system and by substituting the expressions $(A-\lambda)^{n+1-i}(y_i)$ appearing on the right-hand sides of the equalities with $\sum_{j=i}^{n}(A-\lambda)^{n-j}(\psi_j)$, for $2 \leqslant i \leqslant n$, assuming $(A-\lambda)^0$ is the identity operator. Thus the system is of the form

$$\begin{aligned} (A-\lambda)^n(y_1) &= \psi_n + (A-\lambda)(\psi_{n-1}) + \cdots + (A-\lambda)^{n-1}(\psi_1)\\ (A-\lambda)^{n-1}(y_2) &= \psi_n + (A-\lambda)(\psi_{n-1}) + \cdots + (A-\lambda)^{n-2}(\psi_2)\\ &\ \vdots\\ (A-\lambda)^2(y_{n-1}) &= \psi_n + (A-\lambda)(\psi_{n-1})\\ (A-\lambda)(y_n) &= \psi_n. \end{aligned}$$

If the matrix J has t blocks $J_1, \dots, J_t$, then system (4.1) is equivalent to the system $\bigwedge_{i=1}^{t} \big(\vec{A}(\vec{y}_i) = J_i\vec{y}_i + \vec{\psi}_i\big)$, where $\vec{y}_i = [y_{\ell_i+1} \dots y_{\ell_i+k_i}]^T$ and $\vec{\psi}_i = [\psi_{\ell_i+1} \dots \psi_{\ell_i+k_i}]^T$, for $l_1 = 0$ and $l_i = \sum_{j=1}^{i-1} k_j$, $2 \leqslant i \leqslant t$. Each of the subsystems has the same form as the above partially reduced system, and therefore the corresponding totally reduced system is a conjunction of totally reduced systems.

THEOREM 4.1. *For the linear system of operator equations* $\vec{A}(\vec{x}) = C\vec{x} + \vec{\varphi}$ *it holds* $\Delta(A)(\vec{x}) = \sum_{k=1}^{n} C_{k-1} \cdot \vec{A}^{n-k}(\vec{\varphi})$, *where* $\Delta(\lambda)$ *is the characteristic polynomial of the system matrix* C *and* $C_0, C_1, \dots, C_{n-1}$ *are the coefficients of the matrix polynomial* $\operatorname{adj}(\lambda I - C)$.

For the proof we refer the reader to [**7**].

THEOREM 4.2. *The linear system of operator equations* (4.1) *implies the totally reduced system*

$$
\begin{aligned}
\Delta(A)(x_1) &= \sum_{k=1}^{n}\sum_{j=0}^{k-1} d_j A^{n-k}(\varphi_{k-j})\\
&\ \ \vdots\\
\Delta(A)(x_i) &= \sum_{k=1}^{n+1-i}\sum_{j=0}^{k-1} d_j A^{n-k}(\varphi_{i-1+k-j}) - \sum_{k=n+2-i}^{n}\sum_{j=k}^{n} d_j A^{n-k}(\varphi_{i-1+k-j})\\
&\ \ \vdots\\
\Delta(A)(x_n) &= A^{n-1}(\varphi_n) - \sum_{k=2}^{n}\sum_{j=k}^{n} d_j A^{n-k}(\varphi_{n-1+k-j}),
\end{aligned}
\tag{4.2}
$$

where $d_0 = 1$.

PROOF. According to Theorem 4.1 we have $\Delta(\vec{A})(\vec{x}) = \sum_{k=1}^{n} C_{k-1} \cdot \vec{A}^{n-k}(\vec{\varphi})$. Moreover it holds

$$
\begin{bmatrix} A^{n-k}(\varphi_1)\\ A^{n-k}(\varphi_2)\\ \vdots\\ A^{n-k}(\varphi_{k-1})\\ A^{n-k}(\varphi_k)\\ A^{n-k}(\varphi_{k+1})\\ \vdots\\ A^{n-k}(\varphi_n) \end{bmatrix}
=
\begin{bmatrix} \sum_{j=0}^{k-1} d_j A^{n-k}(\varphi_{k-j})\\ \vdots\\ \sum_{j=0}^{k-1} d_j A^{n-k}(\varphi_{n-j})\\ -\sum_{j=k}^{n} d_j A^{n-k}(\varphi_{n+1-j})\\ \vdots\\ -\sum_{j=k}^{n} d_j A^{n-k}(\varphi_{n+k-1-j}) \end{bmatrix}
$$

Consequently we get (4.2). □

Let B be an arbitrary $n \times n$ matrix with coefficients in a field K and let $C = C_1 \oplus \cdots \oplus C_k$ be the rational canonical form of the matrix B. Each block C_i, $1 \leqslant i \leqslant k$, is the companion matrix of a invariant factor of the matrix B. The system

$$\vec{A}(\vec{x}) = B\vec{x} + \vec{\varphi}$$

can be reduced to the system

$$\bigwedge_{i=1}^{k} \Big(\vec{A}(\vec{z}_i) = C_i \vec{z}_i + \vec{\nu}_i \Big),$$

where $\vec{\nu}_i = [\nu_{\ell_i+1} \dots \nu_{\ell_i+n_i}]^T$ and $\vec{z}_i = [z_{\ell_i+1} \dots z_{\ell_i+n_i}]^T$, for $l_1 = 0$ and $l_i = \sum_{j=1}^{i-1} n_j$, $2 \leqslant i \leqslant k$. According to Theorem 4.2 each subsystem

$$\vec{A}(\vec{z}_i) = C_i \vec{z}_i + \vec{\nu}_i$$

corresponding to the companion matrix C_i of the polynomial

$$\Delta_{C_i}(\lambda) = \lambda^{n_i} + d_{i,1}\lambda^{n_i-1} + \dots + d_{i,n_i-1}\lambda + d_{i,n_i}$$

can be transformed into the totally reduced system

$$\begin{aligned}
\Delta_{C_i}(A)(z_{l_i+1}) &= \sum_{k=1}^{n_i} \sum_{j=0}^{k-1} d_{i,j} A^{n_i-k}(\nu_{l_i+k-j}) \\
&\;\;\vdots \\
\Delta_{C_i}(A)(z_{l_i+t}) &= \sum_{k=1}^{n_i+1-t} \sum_{j=0}^{k-1} d_{i,j} A^{n_i-k}(\nu_{l_i+t-1+k-j}) \\
&\quad - \sum_{k=n_i+2-t}^{n_i} \sum_{j=k}^{n_i} d_{i,j} A^{n_i-k}(\nu_{l_i+t-1+k-j}) \\
&\;\;\vdots \\
\Delta_{C_i}(A)(z_{l_i+n_i}) &= A^{n_i-1}(\nu_{l_i+n_i}) - \sum_{k=2}^{n_i} \sum_{j=k}^{n_i} d_{i,j} A^{n_i-k}(\nu_{l_i+n_i-1+k-j}),
\end{aligned}$$

where $d_{i,0} = 1$. These systems consist of higher order linear operator equations in only one variable. The homogeneous parts of the equations are obtained by replacing λ by A in the invariant factors of the matrix B.

In addition, each of the subsystems $\vec{A}(\vec{z}_i) = C_i \vec{z}_i + \vec{\nu}_i$ can be transformed by a change of basis into a system with the matrix in the Jordan canonical form. Let S_i be the matrix constructed from the eigenvectors of C_i. Then $J_i = S_i^{-1} \cdot C_i \cdot S_i$ is a matrix in the Jordan canonical form and $J_i = J_{i,1} \oplus J_{i,2} \oplus \dots \oplus J_{i,t_i}$. The blocks $J_{i,1}, J_{i,2}, \dots, J_{i,t_i}$ correspond to distinct roots of the polynomial $\Delta_{C_i}(\lambda)$, and their dimensions are equal to the multiplicities of these roots. Let us denote by S the direct sum of the matrices S_i, i.e., $S = S_1 \oplus S_2 \oplus \dots \oplus S_k$. Then $J = S^{-1} \cdot C \cdot S$ is the Jordan canonical form of the matrix B. Therefore the system $\vec{A}(\vec{x}) = B\vec{x} + \vec{\varphi}$ can be reduced to an equivalent system $\vec{A}(\vec{y}) = J\vec{y} + \vec{\psi}$, from which, as we have seen, we can obtain the totally reduced system.

5. Differential transcendence

In this section we restrict our attention to system (4.1) on the assumptions that V is the vector space of meromorphic functions over the complex field $\mathbb{C}$ and that $A(x) = \frac{d}{dz}(x)$ is a differential operator.

Recall that a function $x_0 \in V$ is differentially algebraic over $\mathbb{C}$ if it satisfies a differential algebraic equation with coefficients in the field $\mathbb{C}$. A function $x_0 \in V$ is differentially transcendental over $\mathbb{C}$ if it is not differentially algebraic, (see [**8, 3**]).

We are interested in establishing a connection between the differential transcendence of the solution of system (4.2) and the differential transcendence of the nonhomogeneous parts of system (4.1). Let $x_0 \in V$ be a solution of differential equation

$$x^{(n)}(z) + d_1 x^{(n-1)}(z) + \cdots + d_{n-1}x'(z) + d_n x(z) = \varphi(z),$$

for $d_1, d_2, \ldots, d_n \in \mathbb{C}$ and $\varphi \in V$. The function x_0 is differentially transcendental over $\mathbb{C}$ if and only if the function φ is differentially transcendental over $\mathbb{C}$, (see [**7, 10**]). Since all equations of system (4.2) are of this form, our main task is to determine the conditions on which a differentially transcendental component of the nonhomogeneous term of system (4.1) does not appear in the nonhomogeneous parts of system (4.2).

Let $\varphi_1(z)$ be the only differentially transcendental component of the nonhomogeneous term $\vec{\varphi}(z) = [\varphi_1(z) \ldots \varphi_n(z)]^T$ of system (4.1). Then

$$\sum_{k=1}^{n} \sum_{j=0}^{k-1} d_j \varphi_{k-j}^{(n-k)}(z)$$

is a differentially transcendental function over the field $\mathbb{C}$, because the function $\varphi_1(z)$ appears in the sum in the form

$$\varphi_1^{(n-1)}(z) + d_1 \varphi_1^{(n-2)}(z) + \cdots + d_{n-1}\varphi_1(z).$$

Therefore the first coordinate $x_{01}(z)$ of the solution of system (4.2) is differentially transcendental over $\mathbb{C}$. The function $\varphi_1(z)$ appears in the sums

$$\sum_{k=1}^{n+1-i} \sum_{j=0}^{k-1} d_j \varphi_{i-1+k-j}^{(n-k)}(z) \; - \sum_{k=n+2-i}^{n} \sum_{j=k}^{n} d_j \varphi_{i-1+k-j}^{(n-k)}(z),$$

for $1 < i \leqslant n$, in the form $-d_n \varphi_1^{(i-2)}(z)$, so the sums are differentially algebraic over $\mathbb{C}$ if and only if $d_n = 0$. Hence, the coordinates $x_{02}(z), \ldots, x_{0n}(z)$ of the solution of system (4.2) are differentially algebraic over $\mathbb{C}$ if and only if $d_n = 0$.

From now on let $\varphi_m(z)$, $1 < m < n$, be the only differentially transcendental component of the nonhomogeneous term $\vec{\varphi}(z)$. The nonhomogeneous part

$$\sum_{k=1}^{n+1-i} \sum_{j=0}^{k-1} d_j \varphi_{i-1+k-j}^{(n-k)}(z) \; - \sum_{k=n+2-i}^{n} \sum_{j=k}^{n} d_j \varphi_{i-1+k-j}^{(n-k)}(z)$$

of the i-th equation of system (4.2), for $1 \leqslant i \leqslant m$, contains the function $\varphi_m(z)$ in the form of a differential polynomial

$$\varphi_m^{(n-m+i-1)}(z) + d_1\varphi_m^{(n-m+i-2)}(z) + \cdots + d_{n-m}\varphi_m^{(i-1)}(z).$$

Therefore these nonhomogeneous parts are differentially transcendental functions over $\mathbb{C}$ and consequently the coordinates $x_{01}(z), \ldots, x_{0m}(z)$ of the solution of system (4.2) are differentially transcendental over $\mathbb{C}$. The function $\varphi_m(z)$ appears in the nonhomogeneous part of the i-th equation of system (4.2), for $m+1 \leqslant i \leqslant n$, in the form of a differential polynomial

$$-(d_{n-m+1}\varphi_m^{(i-2)}(z) + d_{n-m+2}\varphi_m^{(i-3)}(z) + \cdots + d_n\varphi_m^{(i-1-m)}(z)).$$

Hence, these nonhomogeneous parts are differentially algebraic over $\mathbb{C}$ if and only if $d_{n-m+1} = d_{n-m+2} = \cdots = d_n = 0$, $m+1 \leqslant i \leqslant n$. Thus, the coordinates $x_{0\,m+1}(z), \ldots, x_{0\,n}(z)$ of the solution of system (4.2) are differentially algebraic over $\mathbb{C}$ if and only if $d_{n-m+1} = d_{n-m+2} = \cdots = d_n = 0$.

If $\varphi_n(z)$ is the only differentially transcendental component of the nonhomogeneous term $\vec{\varphi}(z)$, then all coordinates of the solution of system (4.2) are differentially transcendental over $\mathbb{C}$, because $\varphi_n(z)$ appears in each nonhomogeneous part of system (4.2).

An important point to note here is that if more than one component of the nonhomogeneous term of system (4.1) is differentially transcendental over $\mathbb{C}$ we need to examine their differential independence, (see [**9**]).

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