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ON THE CONVERSE OF WEYL'S CONFORMAL AND PROJECTIVE THEOREMS

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Abstract. This note investigates the possibility of converses of the Weyl theorems that two conformally related metrics on a manifold have the same Weyl conformal tensor and that two projectively related connections on a manifold have the same Weyl projective tensor. It shows that, in all relevant cases, counterexamples to each of Weyl's theorems exist except for his conformal theorem in the 4-dimensional, positive definite case, where the converse actually holds. This (conformal) 4-dimensional problem is then solved completely for the other possible signatures.

1. Introduction

Let M denote a (smooth, connected, Hausdorff) manifold of dimension n admitting a (smooth) metric g of arbitrary signature whose Levi-Civita connection is denoted ∇. The curvature tensor from ∇ is denoted Riem with components $R^a{}_{bcd}$, its associated Ricci tensor is denoted Ricc with components $R_{ab} \equiv R^c{}_{acb}$ and the Ricci scalar is $R \equiv R_{ab}\, g^{ab}$. For $n \geqslant 3$, the Weyl *conformal tensor* associated with g and ∇ is the type $(1,3)$ tensor denoted by C with components $C^a{}_{bcd}$ given by

$$C^a{}_{bcd} = R^a{}_{bcd} + \frac{1}{n-2}(\delta^a{}_d\, R_{bc} - \delta^a{}_c\, R_{bd} + g_{bc}\, R^a{}_d - g_{bd}\, R^a{}_c) + \frac{R}{(n-1)(n-2)}(\delta^a{}_c\, g_{bd} - \delta^a{}_d\, g_{bc}) \tag{1.1}$$

For $n \geqslant 2$, the Weyl *projective tensor* associated with ∇ is the type $(1,3)$ tensor denoted W with components $W^a{}_{bcd}$ given by

$$W^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{n-1}(\delta^a{}_c R_{bd} - \delta^a{}_d R_{bc}) \tag{1.2}$$

If $n = 3$, C is identically zero on M whilst if $n \geqslant 4$, Weyl [**2**] showed that if g' is another metric on M *conformally related to* g (so that $g' = \phi g$ for some nowhere zero function $\phi : M \to \mathbb{R}$) with Levi-Civita connection ∇', the Weyl tensor C' associated with g' and ∇' equals C. If $n = 2$, C is not defined and W is identically zero on M and for $n \geqslant 3$ Weyl [**2**] showed that if g' is another metric of arbitrary

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signature on M then, with the above notation, if ∇ and ∇' are *projectively related* (that is, the *unparameterized* geodesics of ∇ and ∇' coincide) the projective tensors W' and W are equal.

The purpose of this paper is to investigate the possibility of converses to these theorems of Weyl. It will be seen that there is no *general* converse to either theorem but that one exists for the conformal theorem when $n = 4$ and g is of positive definite signature. The situation in the conformal case when $n = 4$ and g is of either Lorentz signature $(-,+,+,+)$ or neutral signature $(+,+,-,-)$ will also be fully described.

Throughout, T_mM will denote the tangent space to M at $m \in M$ and $\Lambda_m M$ the collection of all 2-forms (bivectors) at m. An abuse of notation will be permitted which, because of the existence of a metric, makes no distinction between the tensor types $(0,2)$, $(1,1)$ $(2,0)$ for members of $\Lambda_m M$. (Only metric connections will be considered here.) If $F \in \Lambda_m M$, F has *even* (matrix) rank and, in the case when $\dim M = 4$, if the rank of F equals 2, F is called *simple* and if its rank is 4, it is called *non-simple*. If F is simple, it may be written in component form as $F^{ab} = p^a q^b - q^a p^b$ for $p, q \in T_mM$ and the 2-dimensional subspace (2-space) of T_mM spanned by p and q is unique and called the *blade* of F. Sometimes F (or its blade) is denoted simply $p \wedge q$.

2. The Weyl Conformal Theorem in the Lorentz Signature Case

Regarding a possible converse to the Weyl *conformal* theorem one asks the following question (in the notation of section 1 and which removes the obvious barrier to such a converse if either $n = 3$ or if $n \geqslant 4$ and C vanishes over some non-empty open subset of M). Suppose $n \geqslant 4$ and C and C' are equal and nowhere-zero on some open, dense subset of M (and hence equal on M). Are g and g' conformally related on M? It was shown in [**5**] that the answer is negative if $n \geqslant 5$ or if $n = 4$ *and* g (or g') is of Lorentz or neutral signature, but that the answer is positive if $n = 4$ and g (or g') has positive definite signature. [It is noted here that in the work of [**5**] use was made of product metrics and the conditions under which the Weyl tensor "products" in a well defined sense. In the appendix of that paper, some confusion is unfortunately introduced but which is easily corrected and does not affect the rest of that paper.]

Henceforth attention will be concentrated on the cases when $\dim M = 4$ and g is of Lorentz or neutral signature. First consider the case when g has Lorentz signature. For $m \in M$ the Weyl tensor $C(m)$ has been classified by Petrov [**1**] into its various (Petrov) types **I**, **D**, **II**, **N**, **III** and **O** with type **O** reserved for the case $C(m) = 0$. The Petrov type **N** may be considered, algebraically, the most degenerate "null" case (and physically, in the general theory of relativity, are sometimes interpreted as being associated with an idealised type of pure radiation field). The algebraic case here for "null-ness" stems from the following argument. Consider the following linear map $f_C : \Lambda_m M \to \Lambda_m M$ at m defined by $C(m)$ and given by $f_C : F^{ab} \to C^a{}_{bcd} F^{cd}$ (the latter term being sometimes shortened to CF) for $F \in \Lambda_m M$ [**1**, **4**]. This will be referred to as the *Weyl function* at m (cf. the curvature function in [**4**]) [It is noted here that f_C is defined purely by

$C(m)$. However, if $g(m)$ is given this map may be written in the equivalent form $F^{ab} \to C^{ab}{}_{cd}F^{cd}$.] It turns out that, for this signature, the rank of (either of) these maps is, for $C(m) \neq 0$, an *even* integer.

In the Lorentz case, it is convenient to replace the Weyl tensor and map f_C by an associated equivalent complex tensor and map, the latter being a linear map $\mathbb{C}^3 \to \mathbb{C}^3$ and then to classify C according to the Jordan form (Segre type over $\mathbb{C}$) of this latter map at each $m \in M$. The resulting possible types (bearing in mind the tracefree condition on C, the latter now assumed not zero at the chosen m) are the Segre types $\{111\}$, $\{1(11)\}$, $\{21\}$, $\{(21)\}$ and $\{3\}$ (and for which the latter two types have only zero eigenvalues) and which correspond to the types **I**, **D**, **II**, **N** and **III**, respectively, at m. The rank of f_C then attains its least non-zero value $(=2)$ if and only if $C(m)$ is of Petrov type **N** at m and then the range space, $\mathrm{rg} f_C$, of f_C is spanned by a (Hodge) dual pair of simple, null bivectors at m (and hence they possess a common unique principal null direction at m).

Again suppose that $\dim M = 4$ and that g is a metric of Lorentz signature on M and with Weyl tensor C. Suppose also that the (necessarily closed) subset of points of M at which the Weyl tensor C is of Petrov type **O** or **N** (equivalently the subset of points of M at which the rank of $f_C \leqslant 2$) has empty interior in the manifold topology on M. Then if g' is any metric on M of arbitrary signature and whose Weyl tensor C' equals C on M, g and g' are conformally related on M [**4**]. However if, for example and in the above notation, C is of Petrov type **N** at each point of M and $C' = C$ on M, g and g' need not be conformally related (but g' is necessarily of Lorentz signature on M) [**4**]. In fact, quite generally, if $C' = C$ on M and if there exists $m \in M$ such that $C(m) \neq 0$ (so that (M, g) is not conformally flat), it can easily be checked that g' also has Lorentz signature at m and hence, since M is connected and thus the signatures of g and g' are constant on M, g and g' *necessarily have the same (Lorentz) signature* (up to an overall sign; this will always be implicitly assumed in the definition of signature).

Before discussing the case of neutral signature, it is useful, for later comparison purposes, to make a few remarks. For $\dim M = 4$, $F \in \Lambda_m M$ is simple if and only if $\overset{*}{F}$ is simple. In the Lorentz case a *simple* bivector is either *spacelike*, *timelike* or *null* and (introducing the inner product $\langle\, ,\rangle$ on $\Lambda_m M$ defined by $\langle F, G\rangle \equiv F_{ab}G^{ab}$) these arise if and only if, respectively, $\langle F, F\rangle$ is positive, negative or zero. Again, in the case of Lorentz signature, if $F \in \Lambda_m M$, the Hodge dual satisfies $\overset{**}{F} = -F$ and the Weyl tensor satisfies the equivalent conditions ${}^*C = C^* \Leftrightarrow {}^*C^* = -C$.

3. The Weyl Conformal Theorem in the Neutral Signature Case

Now let $\dim M = 4$ with g a metric on M of neutral signature $(+,+,-,-)$. For this signature and $F \in \Lambda_m M$, $\overset{**}{F} = F$ and the Weyl tensor satisfies the equivalent conditions ${}^*C = C^* \Leftrightarrow {}^*C^* = C$. Define the ± 1 eigenspaces, $\overset{+}{S}_m$ and $\overset{-}{S}_m$, of the linear duality map $F \to \overset{*}{F}$ by $\overset{+}{S}_m = \{F \in \Lambda_m M : \overset{*}{F} = F\}$ and $\overset{-}{S}_m = \{F \in \Lambda_m M : \overset{*}{F} = -F\}$ and, for convenience, let $\tilde{S}_m = \overset{+}{S}_m \cup \overset{-}{S}_m$ [**5**,**12**]. Since each $F \in \Lambda_m M$ can be written in exactly one way as the sum of a member of $\overset{+}{S}_m$ and a member of $\overset{-}{S}_m$ one has the vector space sum $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. Now, under matrix commutation, denoted [ ], $\Lambda_m M$ is a Lie algebra and $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are 3-dimensional subalgebras of it. Also, if $F \in \overset{+}{S}_m$ and $G \in \overset{-}{S}_m$, $[F, G] = 0$ (and $< F, G >= 0$) and so $\Lambda_m M$ is the Lie product of $\overset{+}{S}_m$ and $\overset{-}{S}_m$. [It is remarked that a similar decomposition holds also in the positive definite case [**5**, **11**] but not in the Lorentz case since, there, $\overset{+}{S}_m = \overset{-}{S}_m = \{0\}$. However, a "complex equivalent" holds in this latter case (see, e.g. [**3**, **4**, **8**]).] Since g is of neutral signature, it can be checked that $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are, to within isomorphism, the Lie algebra of bivectors in a 3-dimensional Lorentz space under [ ] and hence $\overset{+}{S}_m \approx \overset{-}{S}_m \approx o(1, 2)$ (see, e.g. [**12**]). Also it is easily checked that $(f_C \overset{*}{F}) = (\overset{*}{CF}) = {}^*CF = C^*F = C\overset{*}{F}$ and so f_C maps $\overset{+}{S}_m$ into itself and $\overset{-}{S}_m$ into itself. A simple member $F \in \Lambda_m M$ may be spacelike, timelike or null, as in the Lorentz case but an additional possibility arises for neutral signature because a (simple) F may be *totally null*, that is, $F = p \wedge q$ for p and q null *and* orthogonal. It is also important for this signature to note that if $F \in \tilde{S}_m$ with $F \neq 0$, the statements that (i) F is simple, (ii) F is totally null and (iii) $\langle F, F \rangle = 0$ are equivalent. [In comparison, it is noted that in the positive definite case all non-zero members of $\tilde{S}_m$ are non-simple [**5**, **11**].]

In the above discussion of Lorentz signature the special case when $C(m)$ was of Petrov type **N** ($\Leftrightarrow$ rank$f_C = 2$ at $m \Leftrightarrow$ the range space of f_C at m was 2-dimensional and consisted entirely of simple null bivectors) was rather important in considering the converse of Weyl's theorem (and there is no equivalent of this in the positive definite case). In the case of neutral signature, since f_C maps $\overset{+}{S}_m$ into itself and $\overset{-}{S}_m$ into itself, one may classify the Weyl tensor at m by classifying the separate actions of f_C on $\overset{+}{S}_m$ and $\overset{-}{S}_m$. Since $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are now 3-dimensional vector spaces with Lorentz signature these separate actions, if non-trivial, may each be shown to have one of the Jordan–Segre types $\{111\}$ (over $\mathbb{C}$), $\{111\}$ (over $\mathbb{R}$), or $\{21\}$ or $\{3\}$ (each over $\mathbb{R}$), together with their possible degeneracies. [It is remarked that in the positive definite case a similar classification of C may be performed but now $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are 3-dimensional vector spaces with positive definite signature and so their only algebraic types are of the form $\{111\}$ over $\mathbb{R}$, together with degeneracies.] Thus, for the case of neutral signature, one might conjecture that there are two special situations which qualify for "equivalents" of the "null" situation in the Lorentz case. These are the cases when the separate actions of f_C on $\overset{+}{S}_m$ and $\overset{-}{S}_m$ either each have Segre type $\{(21)\}$ with zero eigenvalue or one has this type and the other is trivial.

The second of these special cases can be shown to be equivalent to the situation when rank$f_C = 1$ at m (and it is noted that this rank is impossible in the

$(+,+,+,+)$ and $(+,+,+,-)$ cases [**4, 5**]). This follows since f_C maps each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ into themselves, and so if rank$f_C = 1$, rgf_C at m is a subset of $\tilde{S}_m$. Thus if rank $f_C = 1$ the Weyl tensor at m satisfies $C_{abcd} = \alpha F_{ab}F_{cd}$ with $\alpha \in \mathbb{R}$ and $F \in \tilde{S}_m$. Using square brackets to denote skew-symmetrisation over the indices enclosed by them, the identity $C_{a[bcd]} = 0 \Rightarrow F_{a[b}F_{cd]} = 0$ and this implies that F is simple and hence, from a remark above, totally null. It can then be checked that if, e.g., $F \in \overset{+}{S}_m$, the action of f_C on $\overset{+}{S}_m$ has only zero eigenvalues and two independent eigenvectors and hence is of type $\{(21)\}$ with zero eigenvalue and that f_C acts trivially on $\overset{-}{S}_m$. Conversely, if the action of f_C on $\overset{+}{S}_m$ is of type $\{(21)\}$ with zero eigenvalue and on $\overset{-}{S}_m$ is trivial, a consideration of Jordan-Segre types shows that the rank of f_C is 1.

The first of the above special cases is equivalent to the situation when rgf_C is 2-dimensional and is spanned by $F \in \overset{+}{S}_m$ and $G \in \overset{-}{S}_m$ (and it then follows that F and G are necessarily totally null). To see this suppose that rgf_C is spanned by $F \in \overset{+}{S}_m$ and $G \in \overset{-}{S}_m$. If $\langle F, F\rangle$ and $\langle G, G\rangle$ are non-zero and of the same sign, one may choose two independent, *simple*, linear combinations, A and B, of F and G whose blades are mutually orthogonal and intersect only trivially, to span rgf_C [**12**] and then one may write, at m

$$C_{abcd} = \alpha A_{ab}A_{cd} + \beta B_{ab}B_{cd} + \gamma(A_{ab}B_{cd} + B_{ab}A_{cd}) \tag{3.1}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$. It can be checked [**12**] that the identity $C^c{}_{acb} \equiv 0 \Rightarrow \alpha = \beta = 0$ and so $\gamma \neq 0$ at m. Then the condition $C_{a[bcd]} = 0$ at m yields $A_{a[b}B_{cd]} + B_{a[b}A_{cd]} = 0$ at m. Given that A and B are simple this last condition on A and B can be checked to be equivalent to the blades of A and B having a non-trivial intersection. Thus one achieves a contradiction. In the case when $\langle F, F\rangle$ and $\langle G, G\rangle$ are non-zero and of opposite signs, say $\langle F, F\rangle$ positive and $\langle G, G\rangle$ negative, then [**12**] one may choose a pseudo-orthonormal basis x, y, s, t with x and y spacelike and s and t timelike in which, after a scaling of F and G, if necessary, $F = x\wedge y + s\wedge t$ and $G = x\wedge t + y\wedge s$. Then C takes the form (3.1) with $A = F$ and $B = G$ and, since f_C maps each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ to itself, $\gamma = 0$. The condition $C^c{}_{acb} = 0$ then implies that $\alpha = \beta$ and the condition $C_{a[bcd]} = 0$ gives $F_{a[b}F_{cd]} + G_{a[b}G_{cd]} = 0$. A contraction of this last condition with F^{ab} then gives the contradiction that $F = 0$. If $\langle F, F\rangle \neq 0 = \langle G, G\rangle$ (the opposite case is similar) then (3.1) holds with $A = F$ and $B = G$ (and, as before, $\gamma = 0$). Since in this case G is totally null, $G^c{}_aG_{cb} = 0$, and so the condition $C^c{}_{acb} = 0$ implies that $\alpha = 0$. The contradiction that rgf_C is 1-dimensional at m is obtained. Thus F and G satisfy $\langle F, F\rangle = \langle G, G\rangle = 0$ and so each is totally null. The tensor C then takes the form (3.1) with $A = F$, $B = G$ and since $F \in \overset{+}{S}_m$ and $G \in \overset{-}{S}_m$ it can be shown [**12**] that their (totally null) blades intersect in a common null direction at m. As before one finds $\gamma = 0$ and it then follows that the action of f_C on $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is of type $\{(21)\}$ with zero eigenvalue. The converse is clear.

The range of f_C thus contains totally null members F and G with the remainder being *null* and with the above mentioned common null direction as their principal null direction. It will be seen later that each of these two special cases can actually occur. One now has the following theorem.

THEOREM 1. *Let* $\dim M = 4$ *and let* g *be a metric on* M *of neutral signature. Consider the (necessarily closed) subset* U *of points of* M *at which the associated Weyl conformal tensor* C *is either zero or satisfies one of the two special conditions described above. Suppose* U *has empty interior in* M. *Then if* g' *is another metric on* M *of arbitrary signature and whose Weyl tensor* C' *equals* C *on* M, g *and* g' *are conformally related.*

PROOF. It is first noted that $M \setminus U$ is open in M because of continuity and the fact that $\langle H, H\rangle = 0$ for all H in $\mathrm{rg} f_C$ at m if and only if $m \in U$ since $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are each 3-dimensional and of Lorentz signature. So let $m \in M \setminus U$. Then we can assume that $\dim \mathrm{rg} f_C \geqslant 2$ at m and that this range contains two independent members F and G of $\overset{+}{S}_m$ (or of $\overset{-}{S}_m$, the proof being similar in this case). Since $(\overset{+}{S}_m, \langle\rangle)$ is (with $\langle\rangle$ taken with respect to g) a 3-dimensional vector space of signature $(-,-,+)$ one can by taking linear combinations, if necessary, ensure that $\langle F, F\rangle$ and $\langle G, G\rangle$ are both negative. Then a g-orthonormal basis x, y, s, t may be chosen at m, together with its naturally related null basis, l, n, L, N, where $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ $\sqrt{2}N = y - s$, so that F is proportional to $x \wedge t - y \wedge s$ and hence to $l \wedge n - L \wedge N$ [**12**]. Now retain g as the "original" metric and raise and lower all indices using g (so that, e.g., $F^a{}_b = g_{cb}F^{ac}$). Then since $F^a{}_b$ is in the range of f_C at m and $C' = C$, $F^a{}_b$ is in the range of $f_{C'}$ at m and one has from the algebraic symmetries of C'

$$g'_{ac}F^c{}_b + g'_{bc}F^c{}_a = 0 \tag{3.2}$$

Now extend F to a basis $(l \wedge n - L \wedge N,\ l \wedge N,\ n \wedge L)$ for $\overset{+}{S}_m$. Then G is a linear combination of these basis members and also satisfies (3.2). It follows that

$$g'_{ac}(\alpha A^c{}_b + \beta B^c{}_b) + g'_{bc}(\alpha A^c{}_a + \beta B^c{}_a) = 0, \qquad A = l \wedge N,\ B = n \wedge L \tag{3.3}$$

for $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 \neq 0$. Now (3.2) with $F = l \wedge n - L \wedge N$ implies that, at m, the 2-spaces $l \wedge N$ and $n \wedge L$ are invariant 2-spaces for g' with respect to g [**12**]. This invariance for $l \wedge N$ gives

$$g'_{ab}l^b = al_a + bN_a \quad g'_{ab}N^b = cl_a + dN_a \tag{3.4}$$

for $a, b, c, d \in \mathbb{R}$ and so $g'(l, l) = g'(N, N) = g'(l, N) = 0$. Similarly, using the 2-space $n \wedge L$, one finds $g'(n, n) = g'(L, L) = g'(n, L) = 0$. It follows that, at m,

$$g'_{ab} = \mu l_{(a}n_{b)} + \nu L_{(a}N_{b)} + \rho l_{(a}L_{b)} + \sigma n_{(a}N_{b)} \tag{3.5}$$

for $\mu, \nu, \rho, \sigma \in \mathbb{R}$ and where round brackets denote symmetrisation of the enclosed indices. Now consider (3.3). If $\alpha \neq 0 \neq \beta$, contractions of (3.3) with $l^a l^b$, $n^a n^b$ and $l^a N^b$ show that $g'(l, L) = g'(n, N) = 0$ and $g'(l, n) = g'(L, N)$. Using these in (3.5) gives $\rho = \sigma = 0$, $\mu = \nu$. The same results follow if $\alpha = 0 \neq \beta$ (contract with $l^a l^b$,

$l^a N^b$ and $N^a N^b$) and if $\alpha \neq 0 = \beta$ (contract with $n^a n^b$, $n^a L^b$ and $L^a L^b$). Thus from (3.5), $g'_{ab} = \mu(l_{(a} n_{b)} + L_{(a} N_{b)}) = \mu g_{ab}$ at m.

Thus g' and g are conformally related on $M \setminus U$ with smooth conformal function since g and g' are smooth. If at some $m' \in U$, g' and g are not conformally related then since g is not positive definite, g' and g do not share their zeros at m'. So there exists $k \in T_{m'}M$ such that $g(k,k) = 0 \neq g'(k,k)$. By smoothly extending k to a smooth g-null vector field on some open neighbourhood V of m' (and noting that, since $M \setminus U$ is open and dense in M, $V \cap (M \setminus U)$ is not empty) V may be chosen so that $g'(k,k)$ never vanishes on V and hence g and g' are not conformally related on V. This contradicts the fact that g and g' are conformally related on $M \setminus U$ and completes the proof. □

That this theorem is "best possible" can be seen from the following example. Let M be a connected open subset of $\mathbb{R}^4$ and consider the metric g of neutral signature on M given in terms of a global coordinate system u, v, x, y by

$$ds^2 = H(u,x,y)du^2 + 2dudv + dx^2 - dy^2 \tag{3.6}$$

where H is some function on M and with H and M chosen, as they can be, so that the associated Weyl tensor C is nowhere zero on M. This metric together with the metric g' on M obtained by replacing H in (3.6) by $H' = H + \psi(u) + \rho(u)x + \sigma(u)y$ for appropriate functions ψ, ρ and σ and which is not conformally related to g have the same conformal tensor C on M. The range space, $\mathrm{rg} f_C$ at m is in general 2-dimensional, being of the type described in the second of the special cases above. If one starts with $H = f(u)e^{x+y}$ in (3.6), the range space, $\mathrm{rg} f_C$ at m is 1-dimensional, being of the type described in the first of the special cases above.

It is remarked here that the equality of the Weyl tensors C and C' for the metrics g and g' in the positive definite and Lorentz cases, as described above, certainly implies that g and g' have the same signature. As a consequence this must also be true in the case of neutral signature since this is the only other possible signature. This suggests a consistency check on theorem 1. To do this suppose that g and g' have the same Weyl tensor and that g has neutral signature. One proves that g' also has neutral signature. This is trivially true if g and g' are conformally related. Otherwise, the range of the map f_C associated with g (and also with g' since $C = C'$) is one of the special cases above and is hence either 1-dimensional and spanned by a g-totally null bivector of the form $l \wedge N$ or 2-dimensional and spanned by g-totally null bivectors of the form $l \wedge N$ and $l \wedge L$. In each of these cases g' satisfies (3.2) with $F = l \wedge N$ and so, from the comments following (3.3), $l \wedge N$ is an invariant 2-space of g' with respect to g. From this information, one can, for each of these cases, easily write down an expression for g' in terms of g and a null basis containing l, N (and L). Hence (see after (3.4)), l and N are each g'-null and are g'-orthogonal. These relations are only possible if g' has neutral signature and the check is complete. It is also clear that, since M is connected and hence the signatures of g and g' are constant on M, the result that g and g' have the *same signature* relies only on their common Weyl tensor being non-zero at some $m \in M$, that is, on the pair (M, g) not being conformally flat (cf. the Lorentz case discussed earlier).

A curiosity arises from this analysis. In the above notation, let $\dim M = 4$ and let g be a metric on M of arbitrary signature and with Weyl conformal tensor C. Let V be the subset of points of M on which the equation $C^a{}_{bcd}k^d = 0$ has a solution for $0 \neq k \in T_mM$ (with V including those points where C vanishes). Let g' be another metric of arbitrary signature on M whose Weyl tensor also equals C on M. Then if V has empty interior in M, g and g' are conformally related. The proof (briefly) follows from the fact that if such a solution k of the above equation exists at $m \in M$, the range of the map f_C at m, if not trivial, must consist entirely of simple bivectors. If $C(m) \neq 0$ this can never happen for signature case $(+,+,+,+)$ since f_C maps $\tilde{S}_m$ into itself and no non-zero member of $\tilde{S}_m$ is simple. Thus V is simply the set of points at which $C(m) = 0$ and is thus closed in M. For signature $(+,+,+,-)$, the theory of the Petrov types [**3**, **4**, **7**, **14**] shows that the subset V consists of precisely those points where either $C(m) = 0$ or $C(m)$ is of Petrov type **N** and is again closed (section 2). For signature $(+,+,-,-)$, if $m \in V$ and if $\mathrm{rg}(f_C) \geqslant 3$, there exists $F, G \in \mathrm{rg} f_C$ with F and G both in $\overset{+}{S}_m$ (or $\overset{-}{S}_m$; this case is similar). Then $\mathrm{rg}(f_C)$ contains a member $P \in \overset{+}{S}_m$ satisfying $\langle P, P\rangle \neq 0$ and thus P is not simple. Thus for $m \in V$, $\mathrm{rg} f_C \leqslant 2$ with $\mathrm{rg} f_C$, if not trivial, being spanned by one or both of $F \in \overset{+}{S}_m$ and $G \in \overset{-}{S}_m$ and with F and G simple and hence totally null. Thus $C(m)$ is of one of the two special cases discussed above. Conversely, if $C(m)$ is either zero or one of these two special cases, it is clear that there are non-trivial solutions for $k \in T_mM$ to $C^a{}_{bcd}k^d = 0$ and so V consists of precisely those points where either $C(m) = 0$ or $C(m)$ is of one of the two special cases discussed above (and is closed, from theorem 1). From the preceding two paragraphs one has the following theorem.

THEOREM 2. *Let* $\dim M = 4$*, let* g *be a metric on* M *of arbitrary signature and which is not conformally flat and let* g' *be another metric on* M *of arbitrary signature and whose Weyl tensor* C' *equals the Weyl tensor* C *of* g *on* M*. Then* g *and* g' *have the same signature on* M*. If, in addition, the (necessarily closed) subset of points of* M *at which the equation* $C^a{}_{bcd}k^d = 0$ *has a non-trivial solution for* $k \in T_mM$ *has empty interior in* M*,* g *and* g' *are conformally related on* M*.*

[The first conclusion in theorem 2 fails if $\dim M \geqslant 5$ because for these cases it can be checked that one may always choose M and two metrics on M with the same (not identically zero) Weyl conformal tensors but whose signatures are different. This is, perhaps, most simply achieved by using a technique involving product manifolds [**5**]. One takes the metric product of two manifolds of dimensions $n_1 \geqslant 4$ and $n_2 \geqslant 1$ admitting metrics g and g', respectively, of arbitrary signatures and which are Ricci flat and with g having a Weyl conformal tensor (which equals its curvature tensor Riem) which is nowhere zero. The (metric) product manifold is then Ricci flat with nowhere zero Weyl conformal tensor (and which equals its curvature tensor). Then by replacing g and/or g' by $-g$ and/or $-g'$ two different product metrics may be put on the product manifold which have the same nowhere zero Weyl tensor but different signatures.]

4. The Weyl Projective Tensor

Now suppose that $\dim M = n \geqslant 3$ with g a metric on M of arbitrary signature. The Weyl projective tensor W in (1.2) satisfies the following conditions; (i) $W^a{}_{acd} = 0$, (ii) $W^a{}_{bad} = 0$, (iii) $W^a{}_{bcd} = -W^a{}_{bdc}$ and (iv) $W^a{}_{[bcd]} = 0$. If Riem is eliminated between (1.1) and (1.2), one finds

$$W_{abcd} = W_{(ab)cd} + W_{[ab]cd} = P_{abcd} + E_{abcd} + C_{abcd}$$

where

$$P_{abcd} = W_{(ab)cd} = \frac{1}{2(n-1)}\big(g_{ad}\widetilde{R}_{bc} - g_{ac}\widetilde{R}_{bd} + g_{bd}\widetilde{R}_{ac} - g_{bc}\widetilde{R}_{ad}\big),$$

$$E_{abcd} = \frac{n}{2(n-1)(n-2)}\big(g_{ac}\widetilde{R}_{bd} - g_{ad}\widetilde{R}_{bc} + g_{bd}\widetilde{R}_{ac} - g_{bc}\widetilde{R}_{ad}\big)$$

with $C \equiv 0$ if $n = 3$. Here, $\widetilde{\mathrm{Ricc}}$, with components $\tilde{R}_{ab} \equiv R_{ab} - \frac{R}{n}g_{ab}$, denotes the *trace-free Ricci tensor*. [The tensor E differs from that sometimes used in general relativity theory by a factor $\frac{2}{3}$ (see, e.g., [**4**, **7**])]. The tensors P and E satisfy the conditions $P_{abcd} = P_{bacd}$, $P_{abcd} = -P_{abdc}$, $P^a{}_{acd} = 0$, $P_{a[bcd]} = 0$, $E_{abcd} = -E_{bacd} = -E_{abdc} = E_{cdab}$ and $E_{a[bcd]} = 0$ together with the conditions

$$E^a{}_{bad} = -P^a{}_{bad} = \frac{n}{2(n-1)}\tilde{R}_{bd}$$

whilst, for $m \in M$, $E(m) = 0 \Leftrightarrow P(m) = 0 \Leftrightarrow \widetilde{\mathrm{Ricc}} = 0 \Leftrightarrow$ the Einstein space condition holds at m. *Thus the failure of the skew-symmetric condition on the first two indices of W at m is equivalent to the failure of the Einstein space condition at m; $W_{(ab)cd}(m) = 0 \Leftrightarrow P(m) = 0$* [**6**]. It has been brought to the author's attention that this result is known; see e.g. [**15**]. The tensor W also has the properties that (a) $W(m) = 0 \Leftrightarrow$ the constant curvature condition for Riem holds at m, (b) $W(m) = \mathrm{Riem}(m) \Leftrightarrow$ the Ricci-flat condition $\mathrm{Ricc}(m) = 0$ holds at m and (c) $W(m) = C(m) \Leftrightarrow$ the Einstein space condition $\widetilde{\mathrm{Ricc}}(m) = 0$ holds at m.

5. The Weyl Projective Theorem

In this section the lack of a converse to the Weyl projective theorem will be established. Let M be a (smooth, connected, Hausdorff) manifold of dimension $n \geqslant 3$ admitting metrics g and g' of arbitrary signature and associated Levi-Civita connections ∇ and ∇'. It will be shown that one can always find examples of g and g' such that the associated Weyl projective tensors are equal but with ∇ and ∇' not projectively related. Let N be a connected, open subset of $\mathbb{R}^{n'}$ ($n' \geqslant 2$) admitting a metric h of arbitrary signature. Consider the $n(= n'+1)$ dimensional manifold $M = I \times N$ (where I is some open interval of $\mathbb{R}$) with product metric g given on a global chart (t, x^α) with x^α representing a global chart on N and $t \equiv x^0$ representing a global chart on I, by

(5.1) $$ds^2 = dt^2 + h_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, 2, \ldots, n')$$

The global smooth vector field $k \equiv \partial/\partial t$ satisfies $\nabla k = 0$ where ∇ is the Levi-Civita connection for the metric g on M. Then Riem for g satisfies $R^a{}_{bcd}k^d = 0$, from the Ricci identity, and so the only non-vanishing components of Riem in this coordinate system are (a subset of) $R^\alpha{}_{\beta\gamma\delta}$ with $(\alpha, \beta, \gamma, \delta = 1, 2, \ldots, n')$, these latter components being equal to the curvature components $\overline{R}^\alpha{}_{\beta\gamma\delta}$ of the curvature tensor $\overline{\text{Riem}}$ associated with the Levi-Civita connection of h. Now let $\Lambda_p N$ denote the vector space of 2-forms at $p \in N$ and consider the linear map (the curvature map in (N, h) [**4**]) $\Lambda_p N \to \Lambda_p N$ given by $F^{\alpha\beta} \to \overline{R}^{\alpha\beta}{}_{\gamma\delta}F^{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta}F^{\gamma\delta}$. It will be assumed that the rank of this map is maximum $(= \frac{1}{2}n'(n'-1))$ at each $p \in N$. [Briefly, this can be achieved by choosing h to be of non-zero constant curvature or even some small perturbation of such a metric.] It is then easily checked that (M, g) satisfies lemma 1(ii) in [**9**] or lemma 1(part 2) in [**10**]. (This requires a suitable modification for arbitrary dimension and signature and is facilitated by noting that, in the notation of [**9**,**10**], the kernel of the analogous curvature map (M, g) consists of simple bivectors of the form $k \wedge p$ for all $p \in T_m M$ and that the range of the curvature map is spanned by all simple bivectors whose blades are orthogonal to k.) It then follows from the general techniques of these references that any metric g' on M, projectively related to g, satisfies $\nabla' = \nabla$ (see, for example the holonomy type R_{10} or R_{13} case of theorem 4 in [**9**]). However, if $r : I \to \mathbb{R}$ is a smooth positive function with nowhere-zero derivative on I, the metric g' obtained from g in (5.1) by replacing dt^2 by $r(t)dt^2$ does not have Levi-Civita connection ∇ and so is not projectively related to g but is easily checked to have the same tensor Riem and hence the same tensor Ricc and so it has the same Weyl projective tensor W as g. Since the signature of h was arbitrary this example shows that the converse of Weyl's projective theorem fails for M of any dimension $n \geqslant 3$ and for g of any signature.

Another example (which only covers some of the possibilities for the dimension and signature mentioned but is of a different and less trivial nature) is as follows. It is first noted that if M_1 and M_2 are manifolds of dimensions $n_1 \geqslant 1$ and $n_2 \geqslant 1$, respectively, and g and g' are metrics of arbitrary signature on M_1 and h and h' are metrics of arbitrary signature on M_2 then if $(M_1 \times M_2, g \otimes h)$ is projectively related to $(M_1 \times M_2, g' \otimes h')$ it is easily checked that (M_1, g) is projectively related to (M_1, g') and (M_2, h) is projectively related to (M_2, h'). Now consider the metric (3.6) with the sign of the dy^2 changed so that it becomes of Lorentz signature, with the restriction $u > 0$ and with M and H chosen, as they can be, so that the metric represents a vacuum (that is, a Ricci-flat) plane wave in general relativity theory with nowhere vanishing tensor Riem. Now define another metric g' on M by $g' = \phi g$ with $\phi = u^{-2}$. Then g' is also a vacuum plane wave and has an identical Weyl *conformal* tensor to that of g (since they are conformally related). It follows from (1.1) that g and g' have identical curvature tensors and from (1.2) that they have identical Weyl projective tensors. However, since g and g' are conformally related with a conformal factor that is not constant on M they are *not* projectively related (see, e.g. [**13**]). This counterexample applies to dimension 4 with Lorentz signature. If one now takes the above metrics g and g' on the manifold M_1

(= M above), chooses a manifold M_2 of arbitrary dimension $n_2 \geqslant 1$ admitting a Ricci-flat metric h and takes $h = h'$, the product metrics $g \otimes h$ and $g' \otimes h$ on $M_1 \times M_2$ have the same Riemann tensors (since g and g' do), are each Ricci-flat and hence have the same Weyl projective tensors. But, by an above remark, they are not projectively related (since g and g' are not) and so give counterexamples of the type required for any dimension $\geqslant 5$ and metric of any *strictly indefinite* signature (that is, any signature having at least one plus and one minus sign in its Sylvester canonical form).

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