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UNKNOTTING NUMBERS OF ALTERNATING KNOT AND LINK FAMILIES

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Abstract. After proving a theorem about the general formulae for the signature of alternating knot and link families, we distinguished all families of knots obtained from generating alternating knots with at most 10 crossings and alternating links with at most 9 crossings, for which the unknotting (unlinking) number can be confirmed by using the general formulae for signatures.

1. Introduction

About the Conway notation of knots the reader can consult the seminal paper by Conway [**1**], where this notation is introduced, the paper by Caudron [**2**], and books [**3, 4**]. In particular, drawings of all knots up to $n = 10$ crossings according to the Conway notation, where every knot is represented by a single diagram, are given in Appendix C of the book "Knots and links" by Rolfsen [**3**]. For all computations we used the program "LinKnot" [**4**].

The question of unknotting and unlinking numbers, or Gordian numbers is one of the most difficult in knot theory [**6, 7, 8, 9, 10**]. General formulae for unknotting number are known only for some special classes of knots. For example, according to the famous Milnor conjecture [**11**] proved by Kronheimer and Mrowka [**12, 13**], the unknotting number of a torus knot $[p, q]$ is $u = \frac{1}{2}(p-1)(q-1)$.

Unknotting numbers computed according to the Bernhard–Jablan conjecture [**3, 5**] coincide with all unknotting numbers ($n \leqslant 10$) from the book *A Survey of Knot Theory* by Kawauchi (Appendix F) [**8**] and Kawauchi's table updated with reference to recent unknotting number results from *Table of Knot Invariants* [**14**], if in all ambiguous cases (2 or 3) we take the number 3. The results obtained by Bernhard–Jablan conjecture (BJ-conjecture) are also confirmed for all two-component links whose unlinking numbers were computed by Kohn [**9, 10**]. The complete list of BJ-unknotting numbers for knots with $n = 11$ and $n = 12$ crossings computed using the program *LinKnot*, is included in the *Table of Knot Invariants* by Livingston and Cha [**14**]. The recent results by Owens [**15**] and Nakanishi [**16**] confirmed the unknotting number $u = 3$ for the knot 9_{35}, and the

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unknotting number $u = 2$ for the knots 10_{83}, 10_{97}, 10_{105}, 10_{108}, 10_{109}, and 10_{121} computed according to the BJ-conjecture. Usually, unknotting numbers are confirmed for particular knots and links, but they here are for the first time confirmed for families of knots and links.

In Section 2 we prove a theorem on signature enabling computation of general formulae for the signature of alternating knot families given by their Conway symbols. These general formulae enabled us to recognize the families of knots obtained from alternating generating knots with at most $n = 10$ crossings for which unknotting numbers are determined by signatures computed in Section 3.

2. Signature and alternating knot families

Definition 2.1. Let S denote the set of numbers in the unreduced[1] Conway symbol $C(L)$ of a link L. Given $C(L)$ and an arbitrary (nonempty) subset $\tilde{S} = \{a_1, a_2, \ldots, a_m\}$ of S, the family $F_{\tilde{S}}(L)$ of knots or links derived from L is constructed by substituting each $a_i \in \tilde{S}$, $a_i \neq 1$ in $C(L)$ by $\mathrm{sgn}(a)(|a| + n)$, for $n \in N^+$.

For even integers $n \geqslant 0$ this construction preserves the number of components, i.e., we obtain (sub)families of links with the same number of components. If all parameters in a Conway symbol of a knot or link are 1,2, or 3, such a link is called *generating*.

Murasugi [**18**] defined *signature* σ_K of a knot K as the signature of the matrix $S_K + {S_K}^T$, where ${S_K}^T$ is the transposed matrix of S_K, and S_K is the Seifert matrix of the knot K.

For alternating knots, the signature can be computed by using a combinatorial formula derived by Traczyk [**19**]. We will use this formula, proved by Przytycki, in the following form, taken from [**20**]*Theorem 7.8, Part (2):

Theorem 2.1. *If D is a reduced alternating diagram of an oriented knot, then*

$$\sigma_D = -\frac{1}{2}\, w + \frac{1}{2}\,(W - B) = -\frac{1}{2}\, w + \frac{1}{2}\,(|D_{s+}| - |D_{s-}|),$$

where w is the writhe of D, W is the number of white regions in the checkerboard coloring of D, which is for alternating minimal diagrams equal to the number of cycles $|D_{s+}|$ in the state $s+$, and B is the number of black regions in the checkerboard coloring of D equal to the number of the cycles $|D_{s-}|$ in the state $s-$.

Introducing orientation of a knot, every n-twist (chain of digons) becomes *parallel* or *anti-parallel*. For signs of crossings and checkerboard coloring we use the conventions shown in Fig. 1.

Lemma 2.1. *By replacing n-twist ($n \geqslant 2$) by $(n + 2)$-twist in the Conway symbol of an alternating knot K, the signature changes by -2 if the replacement is made in a parallel twist with positive crossings, the signature changes by $+2$ if the replacement is made in a parallel twist with negative crossings, and remains unchanged if the replacement is made in an anti-parallel twist.*

[1] The Conway notation is called *unreduced* if 1's denoting elementary tangles in vertices are not omitted in symbols of polyhedral links.

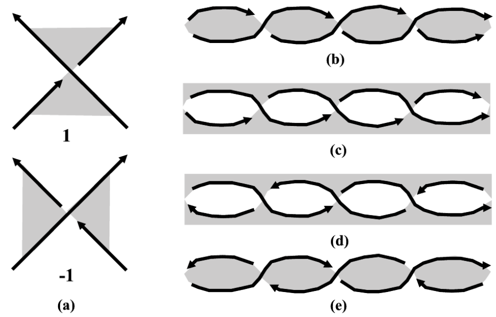

FIGURE 1. (a) Positive crossing and negative crossing (b) parallel positive twist; (c) parallel negative twist; (d) antiparallel positive twist; (e) antiparallel negative twist.

PROOF. According to the preceding theorem:

(1) by adding a full twist in a parallel positive n-twist the writhe changes by $+2$, the number of the white regions W remains unchanged, the number of black regions B increases by $+2$, and the signature changes by -2;
(2) by adding a full twist in a parallel negative n-twist the writhe changes by -2, the number of white regions W increases by 2, the number of black regions B remains unchanged, and the signature increases by 2;
(3) by adding a full twist in an anti-parallel positive n-twist the writhe changes by $+2$, the number of white regions W increases by 2, the number of black regions B remains unchanged, and the signature remains unchanged;
(4) by adding a full twist in an anti-parallel negative n-twist the writhe changes by -2, the number of white regions W remains unchanged, the number of black regions B increases by 2, and the signature remains unchanged. □

THEOREM 2.2. *The signature σ_K of an alternating knot K given by its Conway symbol is*

$$\sigma_K = \sum_P -2\left[\frac{n_i}{2}\right] c_i + 2c_0,$$

where the sum is taken over all parallel twists n_i, $c_i \in \{1, -1\}$ is the sign of crossings belonging to a parallel twist n_i, and $2c_0$ is an integer constant which can be computed from the signature of the generating knot.

The proof of this theorem follows directly from Lemma 2.1, claiming that only additions of twists in parallel twists in a Conway symbol result in the change of signature, and that by every such addition, the signature changes by $-2c_i$. Notice that the condition that we are making twist replacements in the standard Conway symbols, i.e., Conway symbols with the maximal twists, is essential for computation of general formulae for the signature of alternating knot families.

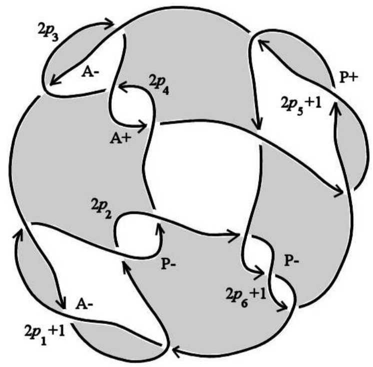

FIGURE 2. Knot family $(2p_1+1)\,(2p_2),(2p_3)\,(2p_4),(2p_5+1)\,1,(2p_6+1)$ beginning with knot $3\,2,2\,2,3\,1,3$.

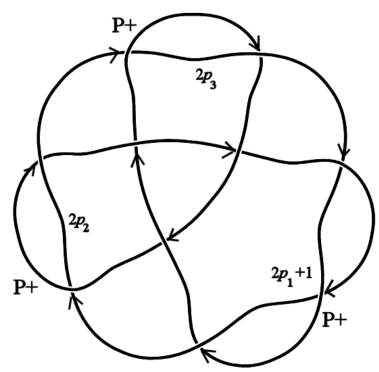

FIGURE 3. Knot family $(2p_1+1):(2p_2):(2p_3)$ beginning with knot $3:2:2$.

EXAMPLE 2.1. For the family of Montesinos knots with the Conway symbol of the form $(2p_1+1)\,(2p_2),(2p_3)\,(2p_4),(2p_5+1)\,1,(2p_6+1)$ (Fig. 2), beginning with the generating knot $3\,2,2\,2,3\,1,3$, the parallel twists with negative signs are $2p_2$ and $2p_6+1$, the parallel twist with positive signs is $2p_5+1$, and the remaining twists are anti-parallel. Hence, the signature is $\sigma=-2p_2+2p_5-2p_6+2c_0$. Since the writhe of the generating knot $G=3\,2,2\,2,3\,1,3$ is $w=-4$ and its checkerboard coloring has $W=9$ white and $B=9$ black regions, its signature is 2. Evaluating the formula $\sigma=2p_2-2p_5 2p_6+2c_0$ for $\sigma_G=$, $p_2=1$, $p_5=1$, and $p_6=1$, we obtain $c_0=0$. Hence, the general formula for the signature of knots belonging to the family $(2p_1+1)\,(2p_2),(2p_3)\,(2p_4),(2p_5+1)\,1,(2p_6+1)$ is $2p_2-2p_5+2p_6$.

EXAMPLE 2.2. For the family of polyhedral knots with the Conway symbol of the form $(2p_1+1):(2p_2):(2p_3)$ ($p_1\geqslant 1$, $p_2\geqslant 1$, $p_3\geqslant 1$), beginning with the knot $3:2:2$ (Fig. 3), all twists are parallel twists with positive crossings, and the

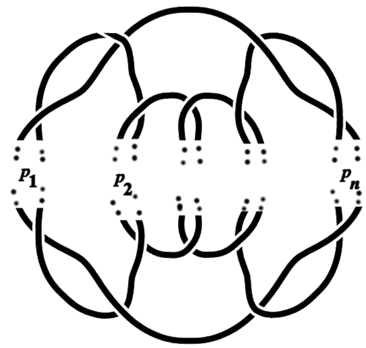

FIGURE 4. Pretzel knot $p_1, p_2, \dots, p_n$.

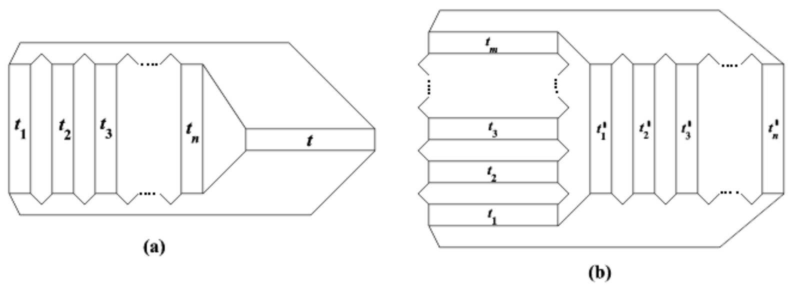

FIGURE 5. (a) Knot $t_1, \dots t_n + t$; (b) knot $(t_1, \dots, t_m)\,(t'_1, \dots, t'_n)$.

formula for the signature is $-2p_1 - 2p_2 - 2p_3 - 2$, i.e., $c_0 = 2$. The constant c_0 is computed from the signature of the generating knot $3:2:2$ which is equal to -4.

EXAMPLE 2.3. Let us consider pretzel knots and links (Fig. 4) given by a Conway symbol $p_1, \dots, p_n$ ($n \geqslant 3$). We obtain knots if all p_i ($i = 1, \dots, n$) are odd and n is an odd number, or if one twist is even, and all the others are odd. If all twists are odd and n is an odd number, all twists are anti-parallel, and the signature is $\sigma_K = n - 1$ for every such knot. If $n = 3$, for the pretzel knots of the form $(2p_1{+}1), (2p_2{+}1), (2q)$, the twists $2p_1{+}1$ and $2p_2{+}1$ are parallel with positive crossings, the twist $2q$ is antiparallel, and the signature is $\sigma_K = 2p_1 + 2p_2$. For $n \geqslant 4$, for the pretzel knots consisting of an even number of odd twists and one even twist, $2p_1 + 1, \dots, 2p_{2k} + 1, 2q$, all odd twists are parallel with positive crossings, the even twist $2q$ is anti-parallel, and the signature is $\sigma_K = 2p_1 + 2p_2 + \dots + 2p_{2k+1}$. For $n \geqslant 4$, for pretzel knots consisting of an odd number of odd twists and one even twist, $2p_1 + 1, \dots, 2p_{2k+1} + 1, 2q$, all twists are parallel with positive crossings, and the signature is $\sigma_K = 2p_1 + 2p_2 + \dots + 2p_{2k+1} + 2q$. Hence, for this class of pretzel knots we simply conclude that their unknotting number is given by the formula $u_K = p_1 + p_2 + \dots + p_{2k+1} + q$.

EXAMPLE 2.4. Let us consider knots of the form $t_1, \dots t_n + t$ $(n \geqslant 3)$, where t_i and t are twists (Fig. 5a). If the twists of an odd length are denoted by p, and twists of an even length by q, we have six possible cases:

(1) if the tangle $t_1, \dots, t_n$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$, and the tangle t is an odd twist p, the signature is given by the formula $2k + 2\left[\frac{p}{2}\right]$
(2) if the tangle $t_1, \dots, t_n$ consists of $2k+1$ odd twists $p_1, \dots, p_{2k}$, and the tangle t is an even twist q, the signature is given by the formula $2k + q$
(3) if the tangle $t_1, \dots, t_n$ consists of $2k+1$ odd twists $p_1, \dots, p_{2k+1}$ and an even twist q_1, and the tangle t is an odd twist p, the signature is given by the formula $\sum_{i=1}^{2k+1} 2\left[\frac{p_i}{2}\right]$
(4) if the tangle $t_1, \dots, t_n$ consists of $2k+1$ odd twists $p_1, \dots, p_{2k+1}$ and an even twist q_1, and the tangle t is an even twist q, the signature is given by the formula $\sum_{i=1}^{2k+1} 2\left[\frac{p_i}{2}\right] + q_1$
(5) if the tangle $t_1, \dots, t_n$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$ and an even twist q_1, and the tangle t is an odd twist p, the signature is given by the formula $\sum_{i=1}^{2k} 2\left[\frac{p_i}{2}\right] + q_1$
(6) if the tangle $t_1, \dots, t_n$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$ and an even twist q_1, and the tangle t is an even twist q, the signature is given by the formula $\sum_{i=1}^{2k} 2\left[\frac{p_i}{2}\right]$.

EXAMPLE 2.5. As a more complex example, we provide general formulae for the signature of knots of the type $(t_1, \dots, t_m)\,(t'_1, \dots, t_n)$ $(m \geqslant 2,\ n \geqslant 2)$, where twists are denoted by t_i or t'_i (Fig. 5b). If the twists of an odd length are denoted by p, and twists of an even length by q, we have seven possible cases:

(1) if the first tangle $t_1, \dots, t_m$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$, and the second tangle $t'_1, \dots, t'_n$ consists of $2r+1$ odd twists $p'_1, \dots, p'_{2r+1}$, the signature is given by the formula

$$\sum_{i=1}^{2r+1} 2\left[\frac{p'_i}{2}\right] + 2k$$

(2) if the first tangle $t_1, \dots, t_m$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$, and the second tangle $t'_1, \dots, t'_n$ consists of $2r$ odd twists $p'_1, \dots, p'_{2r}$, the signature is given by the formula

$$\sum_{i=1}^{2k} 2\left[\frac{p_i}{2}\right] - \sum_{i=1}^{2r} 2\left[\frac{p'_i}{2}\right]$$

(3) if the first tangle $t_1, \dots, t_m$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$ and one even twist q_1, and the second tangle $t'_1, \dots, t'_n$ consists of $2r+1$ odd twists $p'_1, \dots, p'_{2r+1}$, the signature is given by the formula

$$\sum_{i=1}^{2k} 2\left[\frac{p_i}{2}\right] + q_1 + 2r$$

(4) if the first tangle $t_1, \dots, t_m$ consists of $2k+1$ odd twists $p_1, \dots, p_{2k+1}$ and one even twist q_1, and the second tangle $t'_1, \dots, t'_n$ consists of $2r+1$ odd twists $p'_1, \dots, p'_{2r+1}$, the signature is given by the formula

$$\sum_{i=1}^{2k+1} 2\left[\frac{p_i}{2}\right] + 2r$$

(5) if the first tangle $t_1, \dots, t_m$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$ and one even twist q_1, and the second tangle $t'_1, \dots, t'_n$ consists of $2r$ odd twists $p'_1, \dots, p'_{2r}$ and one even twist q'_1, the signature is given by the formula

$$\sum_{i=1}^{2k} 2\left[\frac{p_i}{2}\right] - \sum_{i=1}^{2r} 2\left[\frac{p'_i}{2}\right]$$

(6) if the first tangle $t_1, \dots, t_m$ consists of $2k$ odd twists $p_1, \dots, p_{2k}$ and one even twist q_1, and the second tangle $t'_1, \dots, t'_n$ consists of $2r+1$ odd twists $p'_1, \dots, p'_{2r+1}$ and one even twist q'_1, the signature is given by the formula

$$\sum_{i=1}^{2k} 2\left[\frac{p_i}{2}\right] - \sum_{i=1}^{2r+1} 2\left[\frac{p'_i}{2}\right] - q'_1$$

(7) if the first tangle $t_1, \dots, t_m$ consists of $2k+1$ odd twists $p_1, \dots, p_{2k+1}$ and one even twist q_1, and the second tangle $t'_1, \dots, t'_n$ consists of $2r+1$ odd twists $p'_1, \dots, p'_{2r+1}$ and one even twist q'_1, the signature is given by the formula

$$\sum_{i=1}^{2k+1} 2\left[\frac{p_i}{2}\right] - \sum_{i=1}^{2r+1} 2\left[\frac{p'_i}{2}\right] + q_1 - q'_1.$$

3. Unknotting numbers of knot families

Murasugi [**18**] proved the lower bound for the unknotting number of knots, $u(K) \geqslant \frac{1}{2}|\sigma_K|$. Using this criterion, for many (sub)families of knots we can confirm that their BJ-unknotting numbers, i.e., unknotting numbers computed according to the Bernhard–Jablan Conjecture [**4, 5**] represent the actual unknotting numbers of these (sub)families. In the following tables are given the lists of (sub)families with this property obtained from alternating knots with at most $n = 10$ crossings and alternating links with at most $n = 9$ crossings, where in the first column the first knot or link belonging to the family is given, in the second its Conway symbol, in the third the general Conway symbol, in the fourth the general formula for the signature, in the fifth the unknotting (unlinking) number confirmed by the signature, and in the sixth the conditions for this unknotting (unlinking) number[2].

[2]Conditions for unknotting numbers are determined from the experimental results obtained for knots and links up to $n = 20$ crossings.

K	Con	Fam	σ	u	$Cond$
3_1	3	$(2p_1+1)$	$2p_1$	p_1	
4_1	2 2	$(2p_1)\,(2p_2)$	0		
5_2	3 2	$(2p_1+1)\,(2p_2)$	$2p_2$	p_2	
6_2	3 1 2	$(2p_1+1)\,1\,(2p_2)$	$2p_1$	p_1	$p_1 \geqslant p_2$
6_3	2 1 1 2	$(2p_1)\,1\,1\,(2p_2)$	$2p_1-2p_2$	$\|p_1-p_2\|$	$p_1 \neq p_2$
7_4	3 1 3	$(2p_1+1)\,1\,(2p_2+1)$	2		
7_5	3 2 2	$(2p_1+1)\,(2p_2)\,(2p_3)$	$2p_1+2p_3$	p_1+p_3	
7_6	2 2 1 2	$(2p_1)\,(2p_2)\,1\,(2p_3)$	$2p_3$	p_3	$p_2 \leqslant p_3$
7_7	2 1 1 1 2	$(2p_1)\,1\,1\,1\,(2p_2)$	0		
8_5	3, 3, 2	$(2p_1+1),(2p_2+1),(2p_3)$	$2p_1+2p_2$	p_1+p_2	$p_1 \geqslant p_3$ or $p_2 \geqslant p_3$
8_6	3 3 2	$(2p_1+1)\,(2p_2+1)\,(2p_3)$	$2p_1$		
8_8	2 3 1 2	$(2p_1)\,(2p_2+1)\,1\,(2p_3)$	$2p_1-2p_3$	p_3-p_1	$p_3-p_1>p_2$
8_9	3 1 1 3	$(2p_1+1)\,1\,1\,(2p_2+1)$	$2p_1-2p_2$	$\|p_1-p_2\|$	$p_1 \neq p_2$
8_{10}	3, 2 1, 2	$(2p_1+1),(2p_2)\,1,(2p_3)$	$2p_1-2p_2+2p_3$	$p_1-p_2+p_3$	$p_3>p_2$
8_{11}	3 2 1 2	$(2p_1+1)\,2p_2)\,1\,(2p_3)$	$2p_2$	p_2	$p_2 \geqslant p_3$
8_{12}	2 2 2 2	$(2p_1)\,(2p_2)\,(2p_3)\,(2p_4)$	0		
8_{13}	3 1 1 1 2	$(2p_1+1)\,1\,1\,1\,(2p_2)$	$2p_2-2$	p_2-1	$p_2-1>p_1$
8_{14}	2 2 1 1 2	$(2p_1)\,(2p_2)\,1\,1\,(2p_3)$	$2p_1$	p_1	$p_2 \leqslant p_3$
8_{15}	2 1, 2 1, 2	$(2p_1)\,1,(2p_2)\,1,(2p_3)$	$2p_1+2p_2$	p_1+p_2	
8_{16}	.2.2 0	$.(2p_1).(2p_2)\,0$	$2p_1+2p_2-2$	p_1+p_2-1	
8_{17}	.2.2	$.(2p_1).(2p_2)$	$2p_1-2p_2$	$\|p_1-p_2\|$	$p_1=1, p_2>1$ or $p_2=1, p_1>1$

K	Con	Fam	σ	u	$Cond$
9_{10}	3 3 3	$(2p_1+1)\,(2p_2+1)\,(2p_3+1)$	$2p_2+2$		
9_{13}	3 2 1 3	$(2p_1)\,(2p_2)\,1\,(2p_3+1)$	$2p_1+2$		
9_{15}	2 3 2 2	$(2p_1)\,(2p_2+1)\,(2p_3)\,(2p_4)$	$2p_1$		
9_{16}	3, 3, 2+	$(2p_1+1),(2p_2+1),(2p_3)+$	$2p_1+2p_2+2p_3$	$p_1+p_2+p_3$	
9_{17}	2 1 3 1 2	$(2p_1)\,1\,(2p_2+1)\,1\,(2p_3)$	$-2p_2$	p_2	$p_1+p_3 \leqslant p_2$
9_{18}	3 2 2 2	$(2p_1+1)\,(2p_2)\,(2p_3)\,(2p_4)$	$2p_2+2p_4$	p_2+p_4	
9_{19}	2 3 1 1 2	$(2p_1)\,(2p_2+1)\,1\,1\,(2p_3)$	0		
9_{20}	3 1 2 1 2	$(2p_1+1)\,1\,(2p_2)\,1\,(2p_3)$	$2p_1+2p_3$	p_1+p_3	$p_1+p_3 \geqslant p_2$
9_{21}	3 1 1 2 2	$(2p_1+1)\,1\,1\,(2p_2)\,(2p_3)$	2		
9_{22}	2 1 1, 3, 2	$(2p_1)\,1\,1,(2p_2+1),(2p_3)$	$-2p_2$	p_2	$p_1+p_3-1 \leqslant p_2$
9_{23}	2 2 1 2 2	$(2p_1)\,(2p_2)\,1\,(2p_3)\,(2p_4)$	$2p_1+2p_4$	p_1+p_4	
9_{24}	2 1, 3, 2+	$(2p_1)\,1,(2p_2+1),(2p_3)+$	$2p_1-2p_2$		
9_{25}	2 2, 2 1, 2	$(2p_1)\,(2p_2),(2p_3)\,1,(2p_4)$	$-2p_3$		
9_{26}	3 1 1 1 1 2	$(2p_1+1)\,1\,1\,1\,1\,(2p_2)$	$2p_1$	p_1	
9_{27}	2 1 2 1 1 2	$(2p_1)\,1\,(2p_2)\,1\,1\,(2p_3)$	$2p_3-2p_2$	p_3-p_2	$p_2<p_3$
9_{28}	2 1, 2 1, 2+	$(2p_1)\,1,(2p_2)\,1,(2p_3)+$	$2p_1+2p_2-2p_3$	$p_1+p_2-p_3$	$p_3 \leqslant p_1$ or $p_3 \leqslant p_2$
9_{29}	.2.2 0.2	$.(2p_1).(2p_2)\,0.(2p_3)$	$-2p_2$		
9_{30}	2 1 1, 2 1, 2	$(2p_1)\,1\,1,(2p_2)\,1,(2p_3)$	$2p_2-2p_3$	p_3-p_2	$p_1+p_2 \leqslant p_3$
9_{31}	2 1 1 1 1 1 2	$(2p_1)\,1\,1\,1\,1\,1\,(2p_2)$	$2p_1-2p_2-2$	p_1+p_2-1	$p_1+p_2>2$
9_{32}	.2 1.2 0	$.(2p_1)\,1.(2p_2)\,0$	$2p_2$		
9_{33}	.2 1.2	$.(2p_1)\,1.(2p_2)$	$-2p_2+2$	p_2-1	$p_2-p_1 \geqslant 2$
9_{34}	$8^*2\,0$	$8^8(2p_1)\,0$	0		
9_{35}	3, 3, 3	$(2p_1+1),(2p_2+1),(2p_3+1)$	2		
9_{36}	2 2, 3, 2	$(2p_1)\,(2p_2),(2p_3)\,1,(2p_4)$	$2p_3+2p_4$	p_3+p_4	$p_2 \leqslant p_4$
9_{37}	2 1, 2 1, 3	$(2p_1)\,1,(2p_2)\,1,(2p_3+1)$	0		
9_{38}	.2.2.2	$.(2p_1).(2p_2).(2p_3)$	$2p_2+2$		
9_{39}	2 : 2 : 2 0	$(2p_1):(2p_2):(2p_3)\,0$	2		
9_{41}	2 0 : 2 0 : 2 0	$(2p_1)\,0:(2p_2)\,0:(2p_3)\,0$	0		

K	Con	Fam	σ	u	$Cond$
10_{22}	3 3 1 3	$(2p_1+1)\,(2p_2+1)\,1\,(2p_1+1)$	$2p_1-2p_3$	p_3-p_1	$p_3-p_1>p_2$
10_{23}	3 3 1 1 2	$(2p_1+1)\,(2p_2+1)\,1\,1,(2p_3)$	$2p_2-2p_3+2$	p_2-p_3+1	$p_3\leqslant p_2$
10_{24}	3 2 3 2	$(2p_1+1)\,(2p_2)\,(2p_3+1)\,(2p_4)$	$2p_2$		
10_{25}	3 2 2 1 2	$(2p_1+1)\,(2p_2)\,(2p_3)\,1\,(2p_4)$	$2p_1+2p_3$	p_1+p_3	$p_4\leqslant p_3$
10_{26}	3 2 1 1 3	$(2p_1+1)\,(2p_2)\,1\,1\,(2p_3+1)$	$2p_2-2p_3$	$\lvert p_2-p_3\rvert$	$p_3-p_2>p_1$ or $p_2>p_3$
10_{27}	3 2 1 1 1 2	$(2p_1+1)\,(2p_2)\,1\,1\,1(2p_3)$	$2p_1-2p_3+2$	$\lvert p_1-p_3+1\rvert$	$p_3\leqslant p_1, p_2=2$ or $p_3-p_1>p_2+1$
10_{28}	3 1 3 1 2	$(2p_1+1)\,1\,(2p_2+1)\,1\,(2p_3)$	$-2p_3+2$	p_3-1	$p_1+p_2+1<p_3$
10_{29}	3 1 2 2 2	$(2p_1+1)\,1\,(2p_2)\,(2p_3)\,(2p_4)$	$2p_1$		
10_{30}	3 1 2 1 1 2	$(2p_1+1)\,1\,(2p_2)\,1\,1\,(2p_3)$	2		
10_{31}	3 1 1 3 2	$(2p_1+1)\,1\,1\,(2p_2+1)\,(2p_3)$	$-2p_3+2$		
10_{32}	3 1 1 1 2 2	$(2p_1+1)\,1\,1\,1\,(2p_2)\,(2p_3)$	$2p_1-2p_3$	$\lvert p_1-p_3\rvert$	$p_1>p_2+p_3$ or $p_2=2, p_3>p_1$
10_{33}	3 1 1 1 1 3	$(2p_1+1)\,1\,1\,1\,1\,(2p_2+1)$	0		
10_{37}	2 3 3 2	$(2p_1)\,(2p_2+1)\,(2p_3+1)\,(2p_4)$	$2p_1-2p_4$		
10_{38}	2 3 1 2 2	$(2p_1)\,(2p_2+1)\,1\,(2p_3)\,(2p_4)$	$-2p_4$		
10_{39}	2 2 3 1 2	$(2p_1)\,(2p_2)\,(2p_3+1)\,1\,(2p_4)$	$2p_1+2p_3$	p_1+p_3	$p_1+p_2+p_3\leqslant p_4$
10_{40}	2 2 2 1 1 2	$(2p_1)\,(2p_2)\,(2p_3)\,1\,1\,(2p_4)$	$2p_1+2p_3-2p_4$	$\lvert p_1+p_3-p_4\rvert$	$p_4<p_3$ or $p_4>p_1+p_2+p_3$
10_{41}	2 2 1 2 1 2	$(2p_1)\,(2p_2)\,1\,(2p_3)\,1\,(2p_4)$	$2p_3$	p_3	$p_2+p_4\leqslant p_3$
10_{42}	2 2 1 1 1 1 2	$(2p_1)\,(2p_2)\,1\,1\,1\,1\,(2p_3)$	$-2p_3+2$		
10_{43}	2 1 2 2 1 2	$(2p_1)\,1\,(2p_2)\,(2p_3)\,1\,(2p_4)$	$2p_1-2p_4$		
10_{44}	2 1 2 1 1 1 2	$(2p_1)\,1\,(2p_2)\,1\,1\,1\,(2p_3)$	$2p_1$	p_1	$p_2+p_3\leqslant p_1+1$
10_{45}	2 1 1 1 1 1 1 2	$(2p_1)\,1\,1\,1\,1\,1\,1\,(2p_2)$	0		
10_{50}	3 2, 3, 2	$(2p_1+1)\,(2p_2),(2p_3+1),(2p_4)$	$2p_2+2p_3$	p_2+p_3	$p_4\leqslant p_2$
10_{51}	3 2, 2 1, 2	$(2p_1+1)\,(2p_2),(2p_3)\,1,(2p_4)$	$2p_2+2p_4-2p_3$		
10_{52}	3 1 1, 3, 2	$(2p_1+1)\,1\,1,(2p_2+1),(2p_3)$	$-2p_2-2p_3+2$	p_2+p_3-1	$p_3-p_1\geqslant 2$
10_{53}	3 1 1, 2 1, 2	$(2p_1+1)\,1\,1,(2p_2)\,1,(2p_3)$	$2p_2+2$		
10_{54}	2 3, 3, 2	$(2p_1)\,(2p_2+1),(2p_3+1),(2p_4)$	$2p_1-2p_3-2p_4$	$p_3+p_4-p_1$	$p_4>p_1+p_2$
10_{55}	2 3, 2 1, 2	$(2p_1)\,(2p_2+1),(2p_3)\,1,(2p_4)$	$2p_1+2p_3$	p_1+p_3	
10_{56}	2 2 1, 3, 2	$(2p_1)\,(2p_2)\,1,(2p_3+1),(2p_4)$	$2p_1+2p_3$	p_1+p_3	$p_4\leqslant p_1+p_2$
10_{57}	2 2 1, 2 1, 2	$(2p_1)\,(2p_2)\,1,(2p_3)\,1,(2p_4)$	$2p_1+2p_4-2p_3$	$p_1+p_4-p_3$	$p_3<p_4$
10_{58}	2 1 1, 2 1 1, 2	$(2p_1)\,1\,1,(2p_2)\,1\,1,(2p_3)$	0		
10_{59}	2 2, 2 1 1, 2	$(2p_1)\,(2p_2),(2p_3)\,1\,1,(2p_4)$	$2p_4$	p_4	$p_2+p_3-1\leqslant p_4$
10_{60}	2 1 1, 2 1 1, 2	$(2p_1)\,1\,1,(2p_2)\,1\,1,(2p_3)$	0		
10_{64}	3 1, 3, 3	$(2p_1+1)\,1,(2p_2+1),(2p_3+1)$	$2p_1-2p_2-2p_3$	$p_2+p_3-p_1$	$\max(p_2,p_3)>p_1$

K	Con	Fam	σ	u	$Cond$
10_{65}	$3\,1, 3, 2\,1$	$(2p_1+1)\,1, (2p_2+1), (2p_3)\,1$	$2p_2-2p_3+2$		
10_{66}	$3\,1, 2\,1, 2\,1$	$(2p_1+1)\,1, (2p_2)\,1, (2p_3)\,1$	$2p_1+2p_2+2p_3$	$p_1+p_2+p_3$	
10_{68}	$2\,1\,1, 3, 3$	$(2p_1)\,1\,1, (2p_2+1), (2p_3+1)$	$2p_1-2$		
10_{67}	$2\,2, 3, 2\,1$	$(2p_1)\,(2p_2), (2p_3+1), (2p_4)\,1$	$2p_1$		
10_{69}	$2\,1\,1, 2\,1, 2\,1$	$(2p_1)\,1\,1, (2p_2)\,1, (2p_3)\,1$	$2p_1$		
10_{70}	$2\,2, 3, 2+$	$(2p_1)\,(2p_2), (2p_3+1), (2p_4)+$	$2p_3$		
10_{71}	$2\,2, 2\,1, 2+$	$(2p_1)\,(2p_2), (2p_3)\,1, (2p_4)+$	$2p_4-2p_3$		
10_{72}	$2\,1\,1, 3, 2+$	$(2p_1)\,1\,1, (2p_2+1), (2p_3)+$	$-2p_2-2p_3$	p_2+p_3	$p_1-1 \leqslant p_2+p_3$
10_{73}	$2\,1\,1, 2\,1, 2+$	$(2p_1)\,1\,1, (2p_2)\,1, (2p_3)+$	$2p_2$	p_2	$p_3=1$
10_{74}	$3, 3, 2\,1+$	$(2p_1+1), (2p_2+1), (2p_3)\,1+$	2		
10_{75}	$2\,1, 2\,1, 2\,1+$	$(2p_1)\,1, (2p_2)\,1, (2p_3)\,1+$	0		
10_{76}	$3, 3, 2+2$	$(2p_1+1), (2p_2+1), (2p_3)+(2p_4)$	$2p_1+2p_2$		
10_{77}	$3, 2\,1, 2+2$	$(2p_1+1), (2p_2)\,1, (2p_3)+(2p_4)$	$2p_1+2p_3-2p_2$		
10_{78}	$2\,1, 2\,1, 2+2$	$(2p_1)\,1, (2p_2)\,1, (2p_3)+(2p_4)$	$2p_1+2p_2$	p_1+p_2	$p_4 \leqslant p_1+p_2$
10_{79}	$(3,2)\,(3,2)$	$((2p_1+1),(2p_2))\,((2p_3+1),(2p_4))$	$2p_1+2p_2-2p_3-2p_4$		
10_{80}	$(3,2)\,(2\,1,2)$	$((2p_1+1),(2p_2))\,((2p_3)\,1,(2p_4))$	$2p_1+2p_2+2p_3$	$p_1+p_2+p_3$	
10_{81}	$(2\,1,2)\,(2\,1,2)$	$((2p_1)\,1,(2p_2))\,((2p_3)\,1,(2p_4))$	$2p_1-2p_3$		
10_{83}	$.3\,1.2$	$.(2p_1+1)\,1.(2p_2)$	$-2p_2+2$	p_2-1	$p_2>p_1+1$
10_{84}	$.2\,2.2$	$.(2p_1)\,(2p_2).(2p_3)$	$2p_1+2p_3$	p_1+p_3	$p_2=1$ or $p_3 \geqslant 2$
10_{86}	$.3\,1.2\,0$	$.(2p_1+1)\,1.(2p_2)\,0$	$2p_2$		
10_{87}	$.2\,2.2\,0$	$.(2p_1)\,(2p_2).(2p_3)\,0$	$2p_1-2p_3$		
10_{88}	$.2\,1.2\,1$	$.(2p_1)\,1.(2p_2)\,1$	0		
10_{89}	$.2\,1.2\,1\,0$	$.(2p_1)\,1.(2p_2)\,1\,0$	2		
10_{90}	$.3.2.2$	$.(2p_1+1).(2p_2).(2p_3)$	$2p_2-2p_3$	p_3-p_2	$p_3>p_1+p_2$
10_{91}	$.3.2.2\,0$	$.(2p_1+1).(2p_2).(2p_3)\,0$	$2p_1-2p_2-2p_3+2$	$p_2+p_3-p_1-1$	$p_1 \leqslant p_3, p_2>1$
10_{92}	$.2\,1.2.2\,0$	$.(2p_1)\,1.(2p_2).(2p_3)\,0$	$2p_1+2p_2$	p_1+p_2	$p_3-1 \leqslant p_1$
10_{93}	$.3.2\,0.2$	$.(2p_1+1).(2p_2)\,0.(2p_3)$	$-2p_2-2p_3$	p_2+p_3	$p_1<p_2$ or $p_3>p_1+1$
10_{94}	$.3\,0.2.2$	$.(2p_1+1)\,0.(2p_2).(2p_3)$	$2p_1+2p_2-2p_3$	$p_1+p_2-p_3$	$p_1-p_3 \geqslant 1$ or $p_2-p_3 \geqslant 1$
10_{95}	$.2\,1\,0.2.2$	$(2p_1)\,1\,0.(2p_2).(2p_3)$	$2p_1-2p_2-2$	p_2-p_1+1	$p_1=p_3=1$
10_{96}	$.2.2\,1.2$	$.(2p_1).(2p_2)\,1.(2p_3)$	0		
10_{97}	$.2.2\,1\,0.2$	$.(2p_1).(2p_2)\,1\,0.(2p_3)$	2		
10_{98}	$.2.2.2.2\,0$	$.(2p_1).(2p_2).(2p_3).(2p_4)\,0$	$2p_1+2p_3$	p_1+p_3	$p_4 \leqslant p_1$ or $p_4 \leqslant p_3$
10_{99}	$.2.2.2\,0.2\,0$	$.(2p_1).(2p_2).(2p_3)\,0.(2p_4)\,0$	$2p_1+2p_4-2p_2-2p_3$		
10_{100}	$3:2:2$	$(2p_1+1):(2p_2):(2p_3)$	$2p_1+2p_2+2p_3-2$	$p_1+p_2+p_3-1$	$p_2>1$ or $p_3>1$
10_{101}	$2\,1:2:2$	$(2p_1)\,1:(2p_2):(2p_3)$	$2p_1+2$		
10_{102}	$3:2:2\,0$	$(2p_1+1):(2p_2):(2p-3)\,0$	$2p_3-2p_2$	p_2-p_3	$p_3=1, p_2-p_1>1$
10_{103}	$3\,0:2:2$	$(2p_1+1)\,0:(2p_2):(2p_3)$	$2p_2+2p_3-2$		
10_{104}	$3:2\,0:2\,0$	$(2p_1+1):(2p_2)\,0:(2p_3)\,0$	$2p_1-2p_2-2p_3+2$	$p_1-p_2-p_3+1$	$p_1>p_2, p_3=1$ or $p_1>p_3, p_2=1$

K	Con	Fam	σ	u	$Cond$
10_{105}	$2\,1:2\,0:2\,0$	$(2p_1)\,1:(2p_2)\,0:(2p_3)\,0$	$2p_1$	p_1	$p_1>p_2+p_3$
10_{106}	$3\,0:2:2\,0$	$(2p_1+1)\,0:(2p_2):(2p_3)\,0$	$2p_1+2p_3-2p_2$	$p_1+p_3-p_2$	$p_2+1\leqslant p_1$
10_{107}	$2\,1\,0:2:2\,0$	$(2p_1)\,1\,0:(2p_2):(2p_3)\,0$	$2p_1-2$		
10_{108}	$3\,0:2\,0:2\,0$	$(2p_1+1)\,0:(2p_2)\,0:(2p_3)\,0$	$-2p_2-2p_3+2$	p_2+p_3-1	$p_2>p_1+2$ or $p_3>p_1+2$
10_{109}	$2.2.2.2$	$(2p_1).(2p_2).(2p_3).(2p_4)$	$2p_1+2p_3-2p_2-2p_4$	$p_1+p_3-p_2-p_4$	$p_1\geqslant p_2+p_4$
10_{110}	$2.2.2.2\,0$	$(2p_1).(2p_2).(2p_3).(2p_4)\,0$	$-2p_4$	p_4	$p_4\geqslant p_1+p_3$
10_{111}	$2.2.2\,0.2$	$(2p_1).(2p_2).(2p_3)\,0.(2p_4)$	$2p_2+2p_3$	p_2+p_3	$p_2+p_3\geqslant p_4\geqslant p_1$
10_{112}	8^*3	$8^*(2p_1+1)$	$2p_1$	p_1	$p_1\geqslant 2$
10_{113}	$8^*2\,1$	$8^*(2p_1)\,1$	$2p_1$	p_1	$p_1\geqslant 2$
10_{114}	$8^*3\,0$	$8^*(2p_1+1)\,0$	0		
10_{115}	$8^*2\,0.2\,0$	$8^*(2p_1)\,0.(2p_2)\,0$	0		
10_{116}	$8^*2:2$	$8^*(2p_1):(2p_2)$	$2p_1+2p_2-2$	p_1+p_2-1	$p_1\geqslant 2$ or $p_2\geqslant 2$
10_{117}	$8^*2:2\,0$	$8^*(2p_1):(2p_2)\,0$	$2p_2$		
10_{118}	$8^*2:.2$	$8^*(2p_1):.(2p_2)\,0$	$2p_1-2p_2$	$\lvert p_1-p_2\rvert$	$p_1\geqslant 2, p_2=1$ $p_2\geqslant 2, p_1=1$ or $\lvert p_1-p_2\rvert\geqslant 2$
10_{119}	$8^*2:.2\,0$	$8^*(2p_1):.(2p_2)\,0$	$-2p_2+2$	p_2-1	$p_2-p_1\geqslant 2$
10_{120}	$8^*2\,0::2\,0$	$8^*(2p_1)\,0::(2p_2)\,0$	4		
10_{121}	$9^*2\,0$	$9^8(2p_1)\,0$	2		
10_{122}	$9^*.2\,0$	$9^*.(2p_1)\,0$	0		

In the following table we provide the same results for link families obtained from generating links with at most $n=9$ crossings.

K	Con	Fam	σ	u	$Cond$
2^2_1	2	$(2p_1)$	$-2p_1+1$	p_1	
5^2_1	$2\,1\,2$	$(2p_1)\,1\,(2p_2)$	$-2p_1+1$	p_1	$p_1>p_2$
6^2_2	$3\,3$	$(2p_1+1)\,(2p_2+1)$	$-2p_1-1$		
6^2_3	$2\,2\,2$	$(2p_1)\,(2p_2)\,(2p_3)$	$-2p_1-2p_3+1$	p_1+p_3	
7^2_2	$3\,1\,1\,2$	$(2p_1+1)\,1\,1\,(2p_2)$	$-2p_1+2p_2$	$\lvert p_1-p_2\rvert$	
7^2_3	$2\,3\,2$	$(2p_1)\,(2p_2+1)\,(2p_3)$	$-2p_1+1$		
7^2_4	$3,2,2$	$(2p_1+1),(2p_2),(2p_3)$	$-2p_1-2p_3+1$	p_1+p_3	$p_2=1$
7^2_5	$2\,1,2,2$	$(2p_1)\,1,(2p_2),(2p_3)$	$2p_2+2p_3-2p_1+1$		
7^2_6	$.2$	$.(2p_1)$	$-2p_1+1$		
8^2_4	$3\,2\,3$	$(2p_1+1)\,(2p_2)\,(2p_3+1)$	$-2p_1-2p_3-1$		
8^2_5	$3\,1\,2\,2$	$(2p_1+1)\,1\,(2p_2)\,(2p_3)$	$-2p_1-1$		
8^2_7	$2\,1\,2\,1\,2$	$(2p_1)\,1\,(2p_2)\,1\,(2p_3)$	$-2p_1-2p_3+1$	p_1+p_3	$p_1+p_3\geqslant p_2$
8^2_9	$2\,2,2,2$	$(2p_1)\,(2p_2),(2p_3),(2p_4)$	$-2p_3-2p_4+1$		
8^2_{10}	$2\,1\,1,2,2$	$(2p_1)\,1\,1,(2p_2),(2p_3)$	$2p_3-1$	p_3	$p_1=p_2=1$
8^2_{11}	$3,2,2+$	$(2p_1+1),(2p_2),(2p_3)+$	$-2p_1+1$		
8^2_{12}	$2\,1,2,2+$	$(2p_1)\,1,(2p_2),(2p_3)+$	$-2p_1+2p_2-1$	p_1-p_2+1	$p_2=1$

K	Con	Fam	σ	u	$Cond$
8^2_{13}	$.2\,1$	$.(2p_1)\,1$	-1		
8^2_{14}	$.2:2$	$.(2p_1):(2p_2)$	1		
9^2_6	$3\,3\,1\,2$	$(2p_1+1)\,(2p_2+1)\,1\,(2p_3)$	$-2p_1+2p_3-1$	p_3-p_1	$p_3 \geqslant p_1+p_2$
9^2_7	$3\,2\,1\,1\,2$	$(2p_1+1)\,(2p_2)\,1\,1\,(2p_3)$	$-2p_1+1$	p_1+1	$p_2=1, p_3 \leqslant p_1+2$
9^2_8	$3\,1\,3\,2$	$(2p_1+1)\,1\,(2p_2+1)\,(2p_3)$	$-2p_1+2p_3-1$	p_1-1	$p_1 \geqslant p_2+p_3$
9^2_9	$3\,1\,1\,1\,3$	$(2P-1+1)\,1\,1\,1\,(2p_2+1)$	$-2p_1+1$	p_1	$p_1-p_2 \geqslant 1$
9^2_{11}	$2\,2\,2\,1\,2$	$(2p_1)\,(2p_2)\,(2p_3)\,1\,(2p_4)$	$-2p_1-2p_3+1$	p_1+p_3	$p_3 \geqslant p_4$
9^2_{12}	$2\,2\,1\,1\,1\,2$	$(2p_1)\,(2p_2)\,1\,1\,1\,(2p_3)$	$-2p_1+2p_3-1$	p_3-p_1 p_1-p_3+1	$p_3>p_1+p_2$ $p_2=1, p_1>p_3$
9^2_{15}	$3\,2,2,2$	$(2p_1+1)\,(2p_2),(2p_3),(2p_4)$	$-2p_2-2p_4+1$		
9^2_{16}	$3\,1\,1,2,2$	$(2p_1+1)\,1\,1,(2p_2),(2p_3)$	$2p_2+2p_3-3$		
9_{17}	$2\,3,2,2$	$(2p_1)\,(2p_2)\,1,(2p_3),(2p_4)$	$-2p_2+2p_3+2p_4-1$		
9^2_{18}	$2\,2\,1,2,2$	$(2p_1)\,(2p_2)\,1,(2p_3),(2p_4)$	$-2p_1-2p_4+1$	p_1+p_4	$p_3=1$
9^2_{21}	$3\,1,3,2$	$(2p_1+1)\,1,(2p_2+1),(2p_3)$	$-2p_1+2p_2+2p_3-1$	$p_2+p_3-p_1$	$p_3>p_1$
9^2_{22}	$3\,1,2\,1,2$	$(2p_1+1)\,1,(2p_2)\,1,(2p_3)$	$-2p_1-2p_2-1$		
9^2_{23}	$3,3,2\,1$	$(2p_1+1),(2p_2+1)\,(2p_3)\,1$	$-2p_1-2p_2+2p_3-1$	$p_1+p_2-p_3+1$	$p_1 \geqslant p_3$ or $p_2 \geqslant p_3$
9^2_{24}	$2\,1,2\,1,2\,1$	$(2p_1)\,1,(2p_2)\,1,(2p_3)\,1$	$-2p_1-2p_2-2p_3-1$	$p_1+p_2+p_3$	
9^2_{25}	$2\,2,2,2+$	$(2p_1)\,(2p_2),(2p_3),(2p_4)+$	$-2p_2+1$		
9^2_{26}	$2\,1\,1,2,2+$	$(2p_1)\,1\,1,(2p_2),(2p_3)+$	-1		
9^2_{27}	$3,2,2+2$	$(2p_1+1),(2p_2),(2p_3)+(2p_4)$	$-2p_1-2p_3+1$		
9^2_{28}	$2\,1,2,2,2+2$	$(2p_1)\,1,(2p_2),(2p_3)+(2p_4)$	$-2p_1+2p_2+2p_3-1$		
9^2_{29}	$(3,2)\,(2,2)$	$(2p_1+1),(2p_2))\,((2p_3),(2p_4))$	$2p_1+2p_2-2p_3-2p_4-1$		
9^2_{30}	$(2\,1,2)\,(2,2)$	$((2p_1)\,1,(2p_2))\,((2p_3),(2p_4))$	$-2p_1-2p_3-2p_4+1$	$p_1+p_3+p_4$	
9^2_{32}	$.3\,1$	$.(2p_1+1)\,1$	-1		
9^2_{33}	$.2\,2$	$.(2p_1)\,(2p_2)$	$-2p_1+1$		
9^2_{34}	$.3.2$	$.(2p_1+1).(2p_2)$	$-2p_1+2p_2-1$	p_1	$p_1>1, p_2=1$
9^2_{35}	$.3.2\,0$	$.(2p_1+1).(2p_2)\,0$	$-2p_1-2p_2+1$	p_1+p_2	
9^2_{36}	$.3:2$	$.(2p_1+1):(2p_2)$	$-2p_1+2p_2-1$	p_1-p_2+1	$p_1>1, p_2=1$
9^2_{37}	$.3:2\,0$	$.(2p_1+1):(2p_2)\,0$	$-2p_1+2p_2-1$		
9^2_{38}	$.2\,1:2\,0$	$.(2p_1)\,1:(2p_2)\,0$	$-2p_1+3$		
9^2_{39}	$.2.2.2\,0$	$.(2p_1).(2p_2).(2p_3)\,0$	$-2p_1+2p_2+2p_3-1$	$p_2+p_3-p_1$	$p_3>p_1$
9^2_{40}	$2:2:2$	$(2p_1):(2p_2):(2p_3)$	$-2p_1-2p_2-2p_3+3$		
9^2_{41}	$2:2\,0:2\,0$	$(2p_1):(2p_2):(2p_3)\,0$	$-2p_1+2p_2+2p_3-1$	$p_2+p_3-p_1$	$p_2>p_1$ or $p_3>p_1$
9^2_{42}	8^*2	$8^*(2p_1)$	$-2p_1+1$	p_1	$p_1 \geqslant 2$
6^3_1	$2,2,2$	$(2p_1),(2p_2),(2p_3)$	$-2p_1+2p_2-2p_3$		
7^3_1	$2,2,2+$	$(2p_1),(2p_2),(2p_3)+$	$-2p_1+2$		
8^3_2	$3\,1,2,2$	$(2p_1+1)\,1,(2p_2),(2p_3)$	$-2p_1+2p_2+2p_3-2$		
8^3_3	$2,2,2+2$	$(2p_1),(2p_2),(2p_3)+(2p_4)$	$-2p_1-2p_3+2$		
8^3_4	$(2,2)\,(2,2)$	$((2p_1),(2p_2))\,((2p_3),(2p_4))$	$-2p_1-2p_2+2p_3+2p_4$		
8^3_5	$.3$	$.(2p_1+1)$	$-2p_1$		
8^3_6	$.2:2\,0$	$.(2p_1):(2p_2)\,0$	$-2p_1+2p_2$		
9^3_1	$2\,1\,2,2,2$	$(2p_1)\,1\,(2p_2),(2p_3),(2p_4)$	$-2p_1-2p_3-2p_4+2$		
9^3_2	$2\,1\,1\,1,2,2$	$(2p_1)\,1\,1\,1,(2p_2),(2p_3)$	$-2p_1+2p_3$		
9^3_3	$3,2,2,2$	$(2p_1+1),(2p_2),(2p_3),(2p_4)$	$-2p_1-2p_4+2$		
9^3_4	$2\,1,2,2,2$	$(2p_1)\,1,(2p_2),(2p_3),(2p_4)$	$-2p_1+2p_2+2p_4-2$		
9^3_6	$3\,1,2,2$	$(2p_1+1)\,1,(2p_2),(2p_3)+$	$-2p_1+2p_2-2$		

9^3_7	$2,2,2+3$	$(2p_1),(2p_2),(2p_3)+(2p_4+1)$	$-2p_1+2$		
9^3_8	$(2,2+)\,(2,2)$	$((2p_1),(2p_2)+)\,((2p_3),(2p_4))$	$-2p_1+2p_3+2p_4$		
9^3_9	$(2,2)\,1\,(2,2)$	$((2p_1),(2p_2)\,1\,((2p_3),(2p_4))$	$-2p_1-2p_4+2$		
9^3_{10}	$.2\,1\,1$	$.(2p_1)\,1\,1$	$-2p_1$		
9^3_{11}	$.2\,1:2$	$(-2p_1)\,1:2$	$-2p_1+2$		
9^3_{12}	$.(2,2)$	$.((2p_1),(2p_2))$	$-2p_1+2$		
8^4_1	$2,2,2,2$	$(2p_1),(2p_2),(2p_3),(2p_4)$	$-2p_1-2p_4+3$		

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